

# C4.1 Further Functional Analysis

Sheet 0 — MT 2022

## Initial Sheet

This problem sheet is not for handing in. It is intended for revision and consolidation (during week 0 and the beginning of Week 1 of MT) of some important concepts in Functional Analysis.

1. Let  $X$  be a normed vector space. Prove that  $X$  is a Banach space if and only if every absolutely convergent series with terms in  $X$  converges to a limit in  $X$ .
2. Given an example of Banach spaces  $X, Y$  and a bounded linear operator  $T : X \rightarrow Y$  such that  $\text{Ran } T$  is not closed in  $Y$ .
3. Let  $X_n$ ,  $n \geq 1$ , be normed vector spaces. Consider the vector space  $X$  of sequences  $(x_n)_{n=1}^{\infty}$  such that  $x_n \in X_n$ ,  $n \geq 1$ , and  $\sum_{n=1}^{\infty} \|x_n\| < \infty$ , endowed with the norm

$$\|x\| = \sum_{n=1}^{\infty} \|x_n\|, \quad x = (x_n)_{n=1}^{\infty} \in X.$$

- (a) Prove that if  $X_n$  is complete for each  $n \geq 1$  then so is  $X$ .
- (b) Let  $X_n^*$  denote the dual space of  $X_n$ ,  $n \geq 1$ . Show that the dual space  $X^*$  of  $X$  is isometrically isomorphic to the vector space  $Y$  of all sequences  $(f_n)_{n=1}^{\infty}$  such that  $f_n \in X_n^*$ ,  $n \geq 1$ , and  $\sup_{n \geq 1} \|f_n\| < \infty$ , endowed with the norm given by  $\|f\| = \sup_{n \geq 1} \|f_n\|$ ,  $f = (f_n)_{n=1}^{\infty} \in Y$ .

[Think about the proof that the dual space of  $\ell^1$  is isometrically isomorphic to  $\ell^\infty$ . If you've not seen this result in your earlier courses, this problem will probably be hard, and I'd encourage you instead to spend time considering finding out about dual spaces of  $\ell^p$  for  $1 \leq p < \infty$  and for  $c_0$ .]

4. Let  $X$  be a Banach space.

- (a) What does it mean to say that an operator  $T \in \mathcal{B}(X)$  is *invertible*?
- (b) Suppose that  $T \in \mathcal{B}(X)$  and that  $\|T\| < 1$ . Show that  $I - T$  is invertible.
- (c) Let  $S, T \in \mathcal{B}(X)$  and suppose that  $T$  is invertible and that  $\|S\| < \|T^{-1}\|^{-1}$ . Prove that  $S + T$  is invertible and that

$$(S + T)^{-1} = \sum_{n=1}^{\infty} (-1)^n (T^{-1}S)^n T^{-1},$$

where the series converges in the norm of  $\mathcal{B}(X)$ .

- (d) Deduce that the set of invertible operators is an open subset of  $\mathcal{B}(X)$  and that the spectrum

$$\sigma(T) = \{\lambda \in \mathbb{F} : \lambda - T \text{ is not invertible}\}$$

of any operator  $T \in \mathcal{B}(X)$  is a compact subset of the field  $\mathbb{F}$ .

- (e) Given a non-empty compact subset  $K$  of  $\mathbb{F}$ , show that there exist a Banach space  $X$  and  $T \in \mathcal{B}(X)$  such that  $\sigma(T) = K$ . What can you say if  $K$  is empty? [Does it make a difference whether  $\mathbb{F}$  is  $\mathbb{C}$  or  $\mathbb{R}$ ?]

5. Let  $X$  be a Banach space,  $Y$  a normed vector space and let  $T \in \mathcal{B}(X, Y)$ .

- (a) Suppose there exist  $\varepsilon \in (0, 1)$  and  $M > 0$  such that  $\text{dist}(y, T(B_X^\circ(M))) < \varepsilon$  for all  $y \in B_Y^\circ$ . Prove that  $B_Y^\circ \subseteq T(B_X^\circ(M(1 - \varepsilon)^{-1}))$ . [Take  $y_1 = y \in B_Y^\circ$  and take  $x_1 \in B_X^\circ(M)$  with  $\|Tx_1 - y_1\| < \varepsilon$ . Now take  $y_2 = Tx_1 - y_1$ . How well can you approximate  $y_2$  by something in the range of  $T$ ?]
- (b) Deduce that if  $T(B_X^\circ(M))$  contains a dense subset of  $B_Y^\circ$  then  $B_Y^\circ \subseteq T(B_X^\circ(M))$ .

[In this course the notation  $B_X^\circ(r)$  is used for the open unit ball in  $X$  of radius  $r$ . This result is the successive approximation lemma - if you get stuck, you might have a look at the proof of the open mapping theorem from B.4.2.]