Asymptotic notation

In this course we use the (fairly) standard notation below to compare the sizes of two functions of a (usually integer) variable $n \geqslant 1$. Here we assume always that $g(n) > 0$. If necessary, to ensure this we only consider $n \geqslant n_0$ for some suitable n_0.

> $f = O(g)$ means there exists a constant C such that $|f(n)| \leqslant Cg(n)$ for all n (or all $n \geqslant n_0$),
>
> $f = o(g)$ means that $f(n)/g(n) \to 0$ as $n \to \infty$,
>
> $f = \Theta(g)$ means that $f = O(g)$ and $g = O(f)$, so there exist constants c, $C > 0$ such that $cg(n) \leqslant f(n) \leqslant Cg(n)$ for all n,
>
> $f \sim g$ means that $f(n)/g(n) \to 1$ as $n \to \infty$.

Less standard but still common:

> $f = \Omega(g)$ means that $g = O(f)$, i.e., there exists a constant $c > 0$ such that $f(n) \geqslant cg(n)$ for all n.

Note that there is an implicit restriction to values of n such that $g(n)$ is both defined and positive. For example, $f = O(n/\log n)$ means there exists C such that $|f(n)| \leqslant Cn/\log n$ for all $n \geqslant 2$.

More generally, we may compare a function of n with a formula involving $O(\cdot)$ or $o(\cdot)$ notation; then each occurrence refers to a function with the corresponding property. For example,

$$f = n^3 + O(n^2)$$

means there is a function $g(n)$ with $g = O(n^2)$ such that $f(n) = n^3 + g(n)$. In other words, there exists a constant C such that

$$n^3 - Cn^2 \leqslant f(n) \leqslant n^3 + Cn^2.$$

Similarly,

$$f \geqslant (2 - o(1))n^2$$

means there is a function $g(n)$ with $g \to 0$ such that $f(n) \geqslant (2 - g(n))n^2$ for all n, i.e., that $\liminf f(n)/n^2 \geqslant 2$. In other words,

$$\forall \varepsilon > 0 \; \exists n_0 \; \forall n \geqslant n_0 : f(n) \geqslant (2 - \varepsilon)n^2.$$

Note that saying, for example, $f(n) = o(1)$ makes no statement about the sign of f; formally $1 + o(1)$ and $1 - o(1)$ mean the same thing.

Warning: some people/books use $f \ll g$ to mean $f = o(g)$; others use it to mean $f = O(g)$. Some people use $f = \omega(g)$ to mean $g = o(f)$, i.e., $f/g \to \infty$, but the notation $\omega(n)$ is often used in a different way, as the default notation for a function of n that tends to infinity. I will try to avoid these.