Hints/Solutions

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1. (a) Apply the mean value theorem to $f(x) = e^x - 1 - x$; (b) use (a); (c) expand e^k. For (d) use (c); and note also that, if $0 \leqslant j < k \leqslant n$, then $\frac{n-j}{k-j} \geqslant \frac{n}{k}$.

2. (a) For k fixed, $f(n) = \Theta(g(n))$ (since $f(n) \leqslant g(n)$ and $f(n) \geqslant g(n)/k^k$ for $n \geqslant k$ by 1(d)). For $k = k(n) \to \infty$, $f(n) = o(g(n))$ (since $f(n) \leqslant g(n)/k!$).

   (b) $f(n) = o(g(n))$ (take logs and note $\log\log n = o(\log n)$). This is just a version of 'exponentials grow faster than powers'.

3. Using part of 1 (d), for the given n,

$$\binom{n}{k} 2^{1-\binom{k}{2}} \leqslant \left(\frac{en}{k}\right)^k 2^{1-\binom{k}{2}} \leqslant \left(\frac{e \cdot e^{-1} k 2^{\frac{k}{2}}}{k}\right)^k 2^{1-\frac{k^2}{2}+\frac{k}{2}} = 2^{1+\frac{k}{2}}$$

   and the last term is $o(n)$ as $k \to \infty$.

4. Colour the edges of K_n independently, each red with probability p and blue otherwise. Let X be the number of red K_ks and Y the number of blue K_ℓs. Find $\mathbb{E}[X+Y]$ and use the fact that $\mathbb{P}(X+Y \leqslant \mathbb{E}[X+Y]) > 0$. N.B. It's not enough to argue that the events $A = \{X \leqslant \mathbb{E}[X]\}$ and $B = \{Y \leqslant \mathbb{E}[Y]\}$ both have positive probability!

5. Pick a 3-colouring of the vertices uniformly at random. Call an edge e *bad* if e gets at most 2 colours. Then $\mathbb{P}(e \text{ is bad}) \leqslant 3\left(\frac{2}{3}\right)^r$, and so the expected number of bad edges is < 1. (The result is ok even if r is 1 or 2, since then H has no edges.)

6. If F is finite, pick uniformly at random a $0,1$ string of length t, where $t \geqslant \max_i c_i$. Let A_i be the event that the initial c_i bits form the ith codeword. Then $\mathbb{P}(A_i) = 2^{-c_i}$. But the events are disjoint, so ...

   F may be infinite; the same argument works using a random infinite sequence. (Or note that its enough to prove the final bound for all finite subsets of F.)

Bonus question (for MFoCS students)

7. Consider the first displayed equation (0.1). Fix a realisation (i.e., an outcome, i.e., a point ω in the probability space Ω we are working in). Let K be the set of i such that A_i holds, and let $k = |K|$.

   Suppose $k \geqslant 1$. LHS is 0. RHS is

   $$\sum_{r=0}^{k}(-1)^r S_r = \sum_{r=0}^{k}(-1)^r \sum_{A\subseteq K,\, |A|=r} 1 = \sum_{r=0}^{k}(-1)^r \binom{k}{r} = (1-1)^k = 0.$$

   Suppose $k = 0$. LHS is 1. RHS is $(-1)^0 S_0 = 1$.

   Thus (0.1) holds, and taking expectations gives (0.2).

   For the alternating inequalities, again consider the RHS in (0.1). Arguing as before, it suffices to check alternating inequalities for $\sum_{r\geqslant 0}(-1)^r\binom{k}{r}$. If $k = 0$, the LHS is 1 and $\sum_{r=0}^{m}$ is 1 for each $m \geqslant 0$. Suppose that $k \geqslant 1$, so the LHS is 0.

   If $m \geqslant k$ then $\sum_{r=0}^{m}(-1)^r\binom{k}{r} = 0$.

   Method 1. Let $0 \leqslant m \leqslant (k+1)/2$. For $r \leqslant (k+1)/2$, $\binom{k}{r}$ increases, and so $\sum_{r=0}^{m}(-1)^r\binom{k}{r}$ is $\geqslant 0$ for m even and $\leqslant 0$ for m odd, as required.

   Let $(k+1)/2 < m < k$. We may use

   $$\sum_{r=0}^{m}(-1)^r\binom{k}{r} = -\sum_{r=m+1}^{k}(-1)^r\binom{k}{r} = -(-1)^k\sum_{s=0}^{k-m-1}(-1)^s\binom{k}{s}$$

   (setting $s = k - r$) to see from the previous case that the alternating inequalities hold for such m.

   Method 2. (The slick way.) Notice that

   $$\sum_{r=0}^{m}(-1)^r\binom{k}{r} = (-1)^m\binom{k-1}{m},$$

   which easily follows from $\binom{k}{r} = \binom{k-1}{r-1} + \binom{k-1}{r}$.

   Either way, we have the alternating inequalities for $\sum_{r\geqslant 0}(-1)^r S_r$ in (0.1), and taking expectations gives the corresponding result for (0.2).

If you find an error please check the website, and if it has not already been corrected, e-mail riordan@maths.ox.ac.uk