C8.2: Stochastic analysis and PDEs
Part C Solutions to Problem sheet 1

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A. *Resolvent for Brownian motion ...* First note

$$R_\lambda f(x) = \int_0^\infty T_t f(x)dt = \int_0^\infty \int_{-\infty}^\infty (2\pi t)^{-0.5} \exp(-(x-y)^2/(2t))f(y)dydt$$

An application of Fubini shows this reduces to show that

$$\int_0^\infty (2\pi t)^{0.5} \exp(-\lambda t - (x-y)^2/2t)dt = r_\lambda(x,y)$$

The expression depends only on $|x-y|$ so we can set $y=0$ wlog. Hence, substituting $y=0$ and $t = xs^2/\gamma$ gives

$$\gamma^2 t + x^2/t = \gamma x s^2 + \gamma x/s^2 = \gamma x(s-1/s)^2 + 2\gamma x$$

Using this, the integral becomes

$$I = \frac{2\sqrt{x}}{2\pi\gamma} e^{-\gamma x} \int_0^\infty \exp(-0.5\gamma x(s-1/s)^2)dx$$

Now observe that $u(s) = s - 1/s$ is one-to-one from $(0,\infty)$ to $(-\infty,\infty)$ and $s(u) = u + s(-u)$ so that in particular $s'(u) + s'(-u) = 1$. Thus

$$I = \frac{2\sqrt{x}}{2\pi\gamma} \exp(-\gamma x) \int_0^\infty \exp(-0.5\gamma x u^2)du = \frac{1}{\gamma}\exp(-\gamma x)$$

B. *Suppose that X is a continuous time Markov process on a discrete state space (so can be characterised by a Q-matrix). Let $f_{ij}(t)$ denote the density of the first hitting time of state j if the chain starts in state i. Use the Markov property to find an integral equation which expresses the transition densities $p_{ij}(t)$ of the chain as a convolution of f_{ij} and p_{jj} and hence find an expression for the Laplace transform of the first hitting densities in terms of the resolvent of the chain.*

Conditioning on the first hitting time of j, by the Partition Theorem we obtain

$$p_{ij}(t) = \int_0^t f_{ij}(s)p_{jj}(t-s)ds.$$

Using the convolution theorem for Laplace transforms and rearranging,

$$\hat{f}_{ij} = \frac{\hat{p}_{ij}}{\hat{p}_{jj}} = \frac{r_{ij}(\lambda)}{r_{jj}(\lambda)}.$$

C. *If X is a Feller process and f a non-negative function, check that*

$$Y_t^\lambda = e^{-\lambda t} R_\lambda f(X_t) i, \qquad t \geq 0,$$

defines a supermartingale (with respect to distribution of X and the natural filtration), where R_λ is the resolvent corresponding to X.

Let $\mathcal{F}_t$ denote the natural filtration.

$$\begin{aligned}
\mathbb{E}\left[Y_{t+h}^\lambda \middle| \mathcal{F}_t \right] &= \mathbb{E}\left[e^{-\lambda(t+h)} R_\lambda f(X_{t+h}^\lambda \middle| \mathcal{F}_t \right] \\
&= e^{-\lambda(t+h)} T_h R_\lambda f(X_t) \\
&= e^{-\lambda(t+h)} R_\lambda T_h f(X_t) \\
&= e^{-\lambda(t+h)} \int_0^\infty e^{-\lambda s} T_{s+h} f(X_t) \\
&= e^{-\lambda t} \int_h^\infty e^{-\lambda s} T_s f(X_t) \\
&\leq Y_t^\lambda,
\end{aligned}$$

as required.

[This result provides a large supply of continuous supermartingales for any given Feller process.]

D. *The* Cauchy process, *X, is the real-valued process for which $X_{s+t} - X_s$ is distributed as a Cauchy random variable with density*

$$\frac{1}{\pi} \frac{t}{t^2 + x^2},$$

and increments corresponding to disjoint time intervals are independent.

Suppose that ϕ is an odd *function, which is twice continuously differentiable with compact support and for which $\phi'(0) = 1$. Let $T(t)$ denote the expectation semigroup of X, that is $T(t)f(x) = \mathbb{E}[f(X_t|X_0 = x]$. Suppose that f is twice continuously differentiable. Show that*

$$\frac{T(t)f(x) - f(x)}{t} = \frac{1}{\pi} \int_{-\infty}^\infty \frac{f(x+y) - f(x) - f'(x)\phi(y)}{t^2 + y^2} dy,$$

and hence find an expression for the infinitesimal generator of $T(t)$.

Unlike the case of Brownian motion, the generator of the Cauchy process is not a local *operator. (An operator A is local if $Af(x)$ depends on the values of f only in an infinitesimal neighbourhood of x.) The Cauchy process does not have continuous paths, while Brownian motion does. In general, continuity of paths corresponds to locality of A.*

Since ϕ is odd and $1/(t^2 + y^2)$ is even, their product is odd and so, in particular, integrates to zero and so the first claim is immediate. [The whole point of subtracting this zero term is that it will allow us to take the limit as $t \downarrow 0$.] Now

$$f(x+y) - f(x) - f'(x)\phi(y) = f'(x)\big(y - \phi(y)\big) + \mathcal{O}(y^2),$$

and the error is uniform in x (since f is twice continuously differentiable with compact support). Moreover, since ϕ is continuous and odd, $\phi(0) = 0$ and we have assumed that $\phi'(0) = 1$, so expanding ϕ around zero gives $y - \phi(y)$ is also $\mathcal{O}(y^2)$ and so taking the limit as $t \downarrow 0$ yields a well-defined expression:

$$Af(x) = \frac{1}{\pi} \int_{-\infty}^\infty \frac{f(x+y) - f(x) - f'(x)\phi(y)}{y^2} dy.$$