Problem Sheet 1 (with solutions to sections A and C)

Section A

QUESTION 1. **Error Estimates for the Contraction Mapping Theorem.** Let (X, d) be a complete metric space and let $T\colon X \to X$ be a contractive map with constant $\kappa < 1$. Given $x_0 \in X$ consider the sequence $x_{n+1} = Tx_n$, and $x = \lim_{n\to\infty} x_n$. Show that

(1) $d(x_n, x_{n+m}) \leq \frac{\kappa^n}{1-\kappa} d(x_1, x_0)$

(2) $d(x_n, x) \leq \frac{\kappa^n}{1-\kappa} d(x_1, x_0)$

(3) $d(x_{n+1}, x) \leq \frac{\kappa}{1-\kappa} d(x_{n+1}, x_n)$

(4) $d(x_{n+1}, x) \leq \kappa d(x_n, x)$

Solution

(1) By definition of the sequence and by iterating the contraction estimate, we obtain

$$d(x_n, x_{n+1}) = d(T^n x_1, T^n x_0) \leq \kappa^n d(x_1, x_0).$$

Thus, by iterating the estimate above with the triangle inequality, and using the formula for the sum of a geometric serie we get

$$d(x_{n+m}, x_n) \leq \sum_{k=0}^{m-1} d(x_{n+k+1}, x_{n+k}) \leq \sum_{k\geq n} \kappa^n d(x_1, x_0) = \frac{\kappa^n}{1-\kappa} d(x_1, x_0).$$

(2) Enough to take the limit for $m \to \infty$ in point (1)

(3) Either repeat the argument in (1) but with

$$d(x_{n+m+1}, x_{n+m}) \leq \kappa^m d(x_{n+1}, x_n)$$

or apply (1) to the new iterates $(\tilde{x}_n)$ starting from $\tilde{x}_0 = x_n$: this gives

$$d(x_{n+1}, x_{n+1+m}) \leq \frac{\kappa}{1-\kappa} d(x_{n+1}, x_n),$$

and at this point it is enough to pass to the limit as $m \to \infty$.

(4) Recalling that $x_{n+1} = Tx_n$ and that $Tx = x$, the contraction property directly gives:

$$d(x_{n+1}, x) = d(Tx_n, Tx) \leq \kappa d(x_n, x).$$

□

QUESTION 2. **Revisions on Banach Spaces.** Which of the following spaces are Banach spaces? Please justify your answer.

(1) $C_c(\mathbb{R}) = \{u \in C(\mathbb{R})\colon \text{supp}(u) \subset\subset \mathbb{R}\}$ equipped with the supremum norm $\|u\|_{sup} := \sup_{x\in\mathbb{R}}|u(x)|$.

(2) $C_V(\mathbb{R}) = \{u \in C(\mathbb{R})\colon u(x) \to 0 \text{ for } |x| \to \infty\}$ with the supremum norm $\|u\|_{sup}$.

(3) $C_b(\mathbb{R}) := \{u \in C(\mathbb{R})\colon u \text{ bounded}\}$ equipped with $\|u\| := \sup_{x\in\mathbb{R}} \frac{2+\sin(x)}{3+\cos(x)}|u(x)|$

[You may use that $(C_b(\mathbb{R}), \|\cdot\|_{sup})$ is a Banach space]

Solution

(1) $C_c(\mathbb{R})$ is not a Banach space as it is *not complete.*
E.g. Let $\varphi \in C_c(\mathbb{R})$ be a cut-off function, identically equal to 1 on $[-1/2, 1/2]$ and with supp$\varphi \subset [-1, 1]$. Let $f(x) = \frac{1}{1+x^2}$ and let $f_n(x) := \varphi(x/n) \cdot f(x)$ and notice that $f_n \in C_c(\mathbb{R})$ and that $f_n \to f$ with respect to $\|\cdot\|_{\text{sup}}$. Thus, f_n is a Cauchy sequence with respect to $\|\cdot\|_{\text{sup}}$. However $f \notin C_c(\mathbb{R})$ so f_n is not a convergent sequence in $C_c(\mathbb{R})$.

(2) $C_V(\mathbb{R})$ is a Banach space. Indeed:

- It is a normed vector space as a subspace of a Banach space.
- Is is complete: Let $(f_n) \subset C_V(\mathbb{R})$ be a Cauchy sequence with respect to $\|\cdot\|_{\text{sup}}$. By completeness of $(C_b(\mathbb{R}), \|\cdot\|_{\text{sup}})$ we know that there exists $f \in C_b(\mathbb{R})$ such that $f_n \to f$ wrt $\|\cdot\|_{\text{sup}}$. It is then enough to show that $f \in C_V(\mathbb{R})$. Let's prove it. Fix $\varepsilon > 0$. From the uniform convergence, there exists $N > 0$ such that $|f_n(x) - f(x)| \leq \varepsilon$ for all $x \in \mathbb{R}$. Since $f_n \in C_V(\mathbb{R})$ then there exists $K > 0$ such that $|f_n(x)| \leq 2\varepsilon$ for all $|x| \geq K$; but then $|f(x)| \leq 2\varepsilon$ for all $|x| \geq K$.

(3) is a Banach space. Indeed $\|\cdot\|$ is a norm which is equivalent to $\|\cdot\|_{\text{sup}}$, as

$$\frac{1}{4}\|u\|_{\text{sup}} \leq \| \leq \|\frac{3}{2}\|u\|_{\text{sup}}.$$

Now, since $(C_b(\mathbb{R}), \|\cdot\|_{\text{sup}})$ is a Banach space, it follows that also $(C_b(\mathbb{R}), \|\cdot\|)$ is a Banach space (as equivalent norms give the same Cauchy sequences and the same convergent sequences).

□

QUESTION 3. **Revision on Gronwall Lemma.** Let $f\colon [t_0, t_0+c] \to [0, \infty)$ be a continuous function such that there exists two non-negative constants α and β such that

$$f(t) \leq \alpha + \beta \int_{t_0}^{t} f(s)\, ds \quad \text{for all } t \in [t_0, t_0 + c].$$

Show that

$$f(t) \leq \alpha \exp \beta(t - t_0)$$

for all $t_0 \leq t \leq t_0 + c$.

Solution You can find it the lectures notes of Differential Equations 1. Anyway, let's recall it here. Let $F(t) = \int_{t_0}^{t} f(s)\, ds$. Then

$$F'(t) \leq \alpha + \beta F(t),$$

which gives:

$$\frac{d}{dt}\left(F(t)\exp(-\beta\, t)\right) \leq \alpha\, \exp(-\beta\, t).$$

Now integrate from t_0 to t and obtain:

$$F(t) \leq \exp(\beta\, t)\left(\exp(-\beta\, t) - \exp(-\beta\, t_0)\right)\frac{\alpha}{\beta} = \exp(\beta\,(t - t_0)(1 - \exp(-\beta\,(t - t_0)))\frac{\alpha}{\beta}.$$

So

$$f(t) \leq \alpha + \alpha\, \exp(\beta\,(t - t_0)) - \alpha = \alpha\, \exp(\beta\,(t - t_0).$$

□

Section C

QUESTION 7. **Uniqueness of Solutions to ODEs.** Let H be a real Hilbert space endowed with the scalar product $(\cdot,\cdot)$.

(a) Show that the initial value problem for $y\colon \mathbb{R}\to H$, given by

$$y'(t) = f(t, y(t)) \text{ for } t > 0, \quad y(0) = y_0, \tag{1}$$

has at most one continuously differentiable solution on the interval $[0,T]$, provided that $f\colon \mathbb{R}\times H \to H$ is continuous and satisfies for some $L > 0$

$$(f(t,y) - f(t,z), y - z) \leq L\|y - z\|^2 \text{ for all } y, z \in H. \tag{2}$$

[Hint: Use the product rule $\frac{d}{dt}(y(t), z(t)) = (y'(t), z(t)) + (z'(t), y(t))$ for functions $y, x\colon R \to H$ and Gronwall's Lemma.]

(b) Give furthermore an example of a function f for which (2) is satisfied but for which the Lipschitz-condition of Picard's theorem does not hold.

Solution Part (a) Let y_1, y_2 be solutions of the Initial Value Problem 1. Let $g(t) := \|y_1(t) - y_2(t)\|^2$. Then

$$\begin{aligned}
\frac{d}{dt}g(t) &= 2\big(y_1'(t) - y_2'(t), y_1(t) - y_2(t)\big) = 2\big(f(t, y_1(t)) - f(t, y_2(t)), y_1(t) - y_2(t)\big)\\
&\leq 2\|f(t, y_1(t)) - f(t, y_2(t))\|\, \|y_1(t) - y_2(t)\|\\
&\leq 2L\|y_1(t) - y_2(t)\|^2\\
&= 2L\, g(t).
\end{aligned}$$

So

$$\frac{d}{dt}\left(e^{-2Lt}g(t)\right) \leq 0,$$

yielding that

$$g(t) \leq e^{2Lt}g(0) = 0.$$

We conclude that $g(\cdot) \equiv 0$ and thus $y_1 \equiv y_2$.

Part (b). Let $H = \mathbb{R}$, $f(x) = x^{-\frac{1}{3}}$ (or any decreasing, non-Lipschitz function), so that $(f(x) - f(y), (x - y)) \leq 0$.
Note that we only get uniqueness for $t > 0$. With such an f the solution on $(-T, T)$ could not be unique, as the solution on $(-T, 0)$ could not be unique (e.g. look for Peano's brush on the web). □

QUESTION 8. **Equivalence between Retraction Principle and Brouwer's FPT.** Let B be the closed unit ball in $\mathbb{R}^n$. Using Brouwer's Fixed Point Theorem, show that there does not exist a retraction r from B to ∂B, i.e. a map $r : B \to \partial B$ such that r restricted to ∂B is the identity map.

Hint: by contradiction, consider the map $g(x) = -r(x)$.

Solution If $r : B \to \partial B$ is a retraction, then

$$g(x) = -r(x) \tag{3}$$

is continuous and maps B to $\partial B \subset B$. By Brouwer's Fixed point theorem there exists $x_0 \in B$ such that

$$g(x_0) = x_0. \tag{4}$$

Since $g(x_0) \in \partial B$ we infer that $x_0 \in \partial B$. Since r restricted to ∂B is the identity map, we must have

$$x_0 = r(x_0)\,. \tag{5}$$

The combination of (3), (4) and (5) gives a contradiction. □

QUESTION 9. **Application of Brouwer's FPT.** Given a map $f \in C(\mathbb{R}^n : \mathbb{R}^n)$ such that $|f(x)| \leq a + b|x|$, with $a \geq 0$ and and $0 < b < 1$, show that f has a fixed point.

Solution Choose $R > 0$ such that $|a| + bR \leq R$, i.e. $R \geq \frac{|a|}{1-b}$. Then $f : \overline{B_R(0)} \to \overline{B_R(0)}$ is continuous and has a fixed point by Brouwer's FPT. □