Problem Sheet 3

Section A

QUESTION 1. **Proof of the weak Maximum Principle.**

Let $b \in L^\infty(\Omega, \mathbb{R}^n)$. Give a weak formulation of the condition

$$(\star) \qquad \Delta u + b \cdot \nabla u \le 0$$

that is well defined for functions $u \in H^1(\Omega)$.

Then show that there exists a number $c_1 > 0$ so that if $u \in H^1(\Omega)$ satisfies the weak form of $(\star)$ for some b with $\|b\|_{L^\infty} \le c_1$ and $u \ge 0$ on $\partial\Omega$, then $u \ge 0$.

Hint: You may use that $u^- = -\min(u, 0) \in H_0^1(\Omega)$ with $\nabla u^- = -\nabla u \cdot \chi_{\{u<0\}}$ a.e.

Solution. Seen Δu as an element of $\left(H_0^1(\Omega)\right)^*$ by

$$\langle \Delta u, v\rangle = -\int_\Omega \nabla u \cdot \nabla v \, dx, \quad \forall v \in H_0^1(\Omega),$$

a weak formulation of $(\star)$ is given by

$$(1) \qquad \int_\Omega -\nabla u \cdot \nabla v + b \cdot \nabla u \, v \, dx \le 0 \quad \forall v \in H_0^1(\Omega),\ v \ge 0 \text{ a.e. },$$

which is well defined since $b \in L^\infty$. Note that, since we have an inequality, we restrict to non-negative test functions.

Choose now as test function $v = u^-$. Then

$$\begin{aligned} 0 &\ge \int_\Omega -\nabla u \cdot \nabla(u^-) + b \cdot \nabla u \, u^- \, dx \qquad \text{(Now use the hint)} \\ &= \int_\Omega |\nabla u^-|^2 \cdot \nabla(u^-) - b \cdot \nabla u^- \, u^- \, dx \\ &\ge \|\nabla u^-\|_{L^2}^2 - \|b\|_{L^\infty} \|\nabla u^-|\|_{L^2} \|u^-|\|_{L^2} \\ &\ge \|\nabla u^-\|_{L^2}^2 \left(1 - c_1 \, C_{Poinc}\right), \end{aligned}$$

where C_{Poinc} is the Poincaré constant of Ω. It follows that, for $c_1 \le 1/C_{Poinc}$ we must have $\|\nabla u^-\|_{L^2} = 0$, which in turn implies (by Poincaré inequality) $\|u^-\|_{L^2} = 0$, which yields $u^- = 0$ a.e. . □

Section C

QUESTION 5. H^2 **regularity** Show that for all $\phi \in C_0^\infty(\Omega)$ with $\Omega \subset R^n$ we have

$$\int_\Omega (\Delta\phi)^2 dx = \sum_{i,j=1}^n \int_\Omega (\partial_{ij}\phi)^2 \, dx.$$

Show that this equality is also true in $H_0^2(\Omega)$, and that the map $v \to \|\Delta v\|_{L^2(\Omega)}$ is a norm on $H_0^2(\Omega)$ equivalent to the usual $H^2(\Omega)$ norm.

Solution. Let $\phi \in C_0^\infty(\Omega)$ with $\Omega \subset R^n$. Integrating by parts twice, we get

$$\int_\Omega (\Delta\phi)^2 dx = \int_\Omega \left(\sum_{i=1}^n (\partial_i^2 \phi)\right)\left(\sum_{j=1}^n (\partial_j^2 \phi)\right) dx = \sum_{i,j=1}^n \int_\Omega (\partial_{ij}^2\phi)(\partial_{ij}^2\phi) dx = \sum_{i,j=1}^n \int_\Omega (\partial_{ij}\phi)^2 \, dx.$$

Notice that both the maps $u \mapsto \int_\Omega (\Delta u)^2 dx$ and $u \mapsto \sum_{i,j=1}^n \int_\Omega (\partial_{ij}\phi)^2 \, dx$ are continuous from $H_0^2(\Omega)$ to $\mathbb{R}$. Since they coincide on the dense subset $C_0^\infty(\Omega) \subset H_0^2(\Omega)$, they must coincide on the whole $H_0^2(\Omega)$.

By iterating the Poincaré inequality twice, we have that $\left(\sum_{i,j=1}^n \int_\Omega (\partial_{ij} u)^2 \, dx\right)^{1/2}$ defines a norm on $H_0^2(\Omega)$ equivalent to the standard H^2 norm restricted to $H_0^2(\Omega)$. The claim then follows by the identity $\left(\sum_{i,j=1}^n \int_\Omega (\partial_{ij} u)^2 \, dx\right)^{1/2} = \|\Delta u\|_{L^2(\Omega)}$ proved above for every $u \in H_0^2(\Omega)$. □