Problem Sheet 4

Section A

QUESTION 1. **Monotone operators satisfy (H3)** Let $M \subset M$ satisfy (SA) and let $A : M \to X^*$ be a monotone operator. Using monotonicity first, and then Minty's Lemma, show that A satisfies the assumption (H3), i.e.:

(i) If $(u_n) \subset M$, $u_n \rightharpoonup u$ weakly in X and $A(u_n) \rightharpoonup \xi$ weakly in X^*, then

$$\langle \xi, u \rangle \leq \liminf_{n\to\infty} \langle A(u_n), u_n \rangle . \tag{1}$$

(ii) Equality in (**??**) implies that

$$\langle A(u) - \xi, u - v \rangle \leq 0 , \quad \text{for all } v \in M . \tag{2}$$

Solution. Notice that, thanks to Minty's inequality, (**??**) is equivalent to

$$\langle A(v) - \xi, u - v \rangle \leq 0 , \quad \text{for all } v \in M . \tag{3}$$

Notice that since by assumption X is reflexive, then weak convergence is equivalent to weak* convergence in X^*.

It follows that, if $(u_n) \subset M$, $u_n \rightharpoonup u$ weakly in X and $A(u_n) \rightharpoonup \xi$ weakly in X^*, then $A(u_n) \overset{*}{\rightharpoonup} \xi$ weakly* in X^*. Then

$$\langle A(u_n) - \xi, v \rangle \to 0, \quad \text{for every } v \in X.$$

Moreover, using that $u_n \rightharpoonup u$ weakly in X, we have

$$\langle A(u), u_n - u \rangle \to 0 . \tag{4}$$

Proof of (i). The monotonicity of A gives

$$\langle A(u_n) - A(u), u_n - u \rangle \geq 0 , \quad \text{for all } n \in \mathbb{N}.$$

Thus, using (**??**), we get

$$\liminf_{n\to\infty} \langle A(u_n), u_n - u \rangle = \liminf_{n\to\infty} \Big(\langle A(u_n) - A(u), u_n - u \rangle + \langle A(u), u_n - u \rangle \Big) \geq 0 ,$$

giving

$$\liminf_{n\to\infty} \langle A(u_n), u_n \rangle \geq \limsup_{n\to\infty} \langle A(u_n), u \rangle = \langle \xi, u \rangle .$$

Proof of (ii). We aim to prove that equality in (**??**) implies (**??**). The monotonicity of A gives that

$$\langle A(v) - A(u_n), u_n - v \rangle \leq 0, \quad \text{for all } v \in M, \text{ for all } n \in \mathbb{N}.$$

Expanding and taking the limsup, we obtain

$$\begin{aligned} 0 &\geq \limsup_{n\to\infty} \Big(\langle A(v), u_n \rangle - \langle A(u_n), u_n \rangle - \langle A(v), v \rangle + \langle A(u_n), v \rangle \Big) \\ &= \langle A(v), u \rangle - \langle \xi, u \rangle - \langle A(v), v \rangle + \langle \xi, v \rangle . \\ &= \langle A(v) - \xi, u - v \rangle, \quad \text{for all } v \in M . \end{aligned}$$

□

QUESTION 2. **Monotonicity, Convexity**

Let X be a Banach space and $F\colon X \to \mathbb{R}$ Gâteaux differentiable in every point $u \in X$ with Gâteaux derivative $F'(u)$. Show that

$$F \text{ is convex} \quad \Leftrightarrow \quad F'\colon X \to X^* \text{ is monotone.}$$

Remark:

- A map $G\colon X \to X^*$ is monotone if $\langle G(u) - G(v), u - v\rangle \geq 0$ for all $u, v \in X$ (i.e. hemicontinuity, as in the definition of a monotone operator, is not required).
- A function $F\colon X \to \mathbb{R}$ is convex on X, if $F(tu + (1-t)v) \leq tF(u) + (1-t)F(v)$ for all $t \in [0, 1]$ and $u, v \in X$.
- Recall that a differentiable function $g : I \subset \mathbb{R} \to \mathbb{R}$ is convex on I if g' is monotonically increasing on I. Consider $g(t) := F(tu + (1-t)v)$.

Solution. First of all recall that if F is Gateaux differentiable, then for every $x \in X$ there exists $F'(x) \in X^*$ such that $F'(x)(v) = \partial_v F(u)$.

Proof that F convex $\Rightarrow F' : X \to X^*$ is monotone.

- Since F is convex, then for every $u, v \in X$ the function $t \mapsto g_{u,v}(t) := F(tu + (1-t))v$ is convex.
- Since F is Gateaux differentiable, the directional derivative exists, so the function $g_{u,v}$ is differentiable with $g'_{u,v}(t) = \langle F'(tu + (1-t))v, (u - v)\rangle$.
- The convexity of $g_{u,v}$ implies that $t \mapsto g'_{u,v}(t)$ is non-decreasing.

The combination of the facts above implies that

$$0 \leq g'_{u,v}(1) - g'_{u,v}(0) = \langle F'(u), (u - v)\rangle - \langle F'(v), u - v\rangle\,,$$

i.e. F' is monotone.

Proof that $F' : X \to X^*$ is monotone $\Rightarrow F$ convex.

Let $u, v \in X$ and $g_{u,v}$ be as above. We first show that F' monotone implies that

$$g'_{u,v}(s) - g'_{u,v}(t) \geq 0\,, \qquad \text{for } s \geq t. \tag{5}$$

Denote $u_s := su + (1-s)v$. For $s \geq t$, we have

$$\begin{aligned} g'_{u,v}(s) - g'_{u,v}(t) &= \langle F'(u_s), u - v\rangle - \langle F'\big(u_s + (t-s)(u-v)\big), u - v\rangle \\ &= \frac{1}{s-t}\langle F'(u_s) - F'\big(u_s - (s-t)(u-v)\big), (s-t)(u-v)\rangle \\ &\geq 0\,, \end{aligned}$$

where in the last inequality we used that F' is monotone. This proves the claim (**??**).

The convexity of $t \mapsto g_{u,v}(t)$ follows directly from (**??**). We conclude that

$$F(tu + (1-t)v) = g_{u,v}(t) \leq tg_{u,v}(0) + (1-t)g_{u,v}(1) = tF(u) + (1-t)F(v) \quad \text{for all } u, v \in X.$$

□

Section B

QUESTION 3. **Strongly monotone operator** Let $\Omega = (-1, 1)$ and $X = H^2(\Omega) \cap H^1_0(\Omega)$ endowed with the H^2-norm.

(a) Let $A\colon X \to X^*$ be defined via

$$\langle A(u), v\rangle := \int_\Omega u''v''dx.$$

Show that A is a strongly monotone operator, i.e. hemicontinuous and so that there exists some $c_0 > 0$ with

$$\langle A(u) - A(v), u - v\rangle \geq c_0\|u - v\|^2 \quad \text{for all } u, v \in M.$$

Hint: Use Poincaré's inequality, as well as Poincaré's inequality for functions with mean value zero.

(b) Let now $F_\mu(u) := A(u) + \mu B(u)$ where $B(u)(v) := u(0) \cdot v(0) + \int_\Omega x \cdot v(x)dx$.

Show that $F_\mu : X \to X^*$ is well defined for any $\mu \in \mathbb{R}$ and that there exists a number $\mu_0 > 0$ so that for each μ with $|\mu| \leq \mu_0$ there exists a unique solution of the equation

$$F_\mu(u) = 0.$$

(c) Let now $\mu \geq 0$. Determine a functional $I_\mu : X \to \mathbb{R}$ so that the following holds: $u \in X$ is a solution of $F_\mu(u) = 0$ if and only if u is a minimiser of I_μ on X

Question 4. Consider a domain $\Omega \subset \mathbb{R}^n$ which is smooth and bounded, and $g \in C^2(\mathbb{R}^n)$ such that $g \leq 0$ on $\partial\Omega$. Consider the energy I given by

$$I(v) = \int_\Omega |\Delta v|^2 + fv dx,$$

for some $f \in L^2(\Omega)$.

(1) Find the Euler-Lagrange equation satisfied by the critical points of $I(v)$ and prove that every critical point of I is a minimiser.

(2) Consider the set M given by

$$M := \left\{v \in H^2(\Omega) \cap H_0^1(\Omega) \,|\, v \geq g \ \text{ a.e. on } \Omega\right\}.$$

Show that there exists a unique minimizer of I on M —check carefully that the assumptions of the Theorem(s) you use are satisfied. You may use without proof that for all $u \in H_0^1(\Omega) \cap H^2(\Omega)$

$$\|u\|_{H_0^1(\Omega)} \leq C\|\Delta u\|_{L^2(\Omega)},$$

where the constant C is independent of u.

Question 5. **Three approaches to the same problem**. Consider a domain $\Omega = \{(x, y) \in \mathbb{R}^2 \text{ s.t. } x^2 + y^2 \leq 1\}$ and the equation

$$-\Delta u + u^5 = 1 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

- Show that this equation makes sense in $H_0^1(\Omega)$, that is, it has a legitimate weak variational formulation.
- Using the first part of the course, show that you can formulate it as a fixed point problem of the form $u = T(u)$ where T is a continuous compact map.
- Find a simple subsolution $\underline{u}$ and a simple supersolution $\bar{u}$. Show that the problem can be transformed into

$$-\Delta u + \lambda u = f_\lambda(u)$$

for a constant $\lambda > 0$ chosen so that $f_\lambda(u)$ is increasing when $\underline{u} \leq u \leq \bar{u}$, and use the method of sub and super solutions to show that a solution u can be found by a constructive (iterative) method.
- Using Schauder's FPT and the above show that there exists a solution.
- Use the variational inequality approach to find a solution in $H_0^1(\Omega)$.
- What can you say about uniqueness?

Section C

Instead of more exercises, in the Section C (usually devoted to complimentary material) of this last problem sheet, I encourage you to read some fundamental topics that we did not have time to cover in the lectures and exercise sheets. Of course, the list below is not exhaustive; however it is a good starting point for the enthusiastic students. The corresponding material is not examinable, however it is fundamental if you want to do research in PDEs in your graduate studies.

(1) Hopf's Strong Maximum principle. See for instance Evan's PDE Book Chapter 6.4.2.

(2) Harnack Inequality. See for instance Evan's PDE Book Chapter 6.4.3.

(3) Eigenvalues of Symmetric Elliptic Operators. See for instance Evan's PDE Book Chapter 6.5.1.

(4) In the course we ofter considered minimizers of integral energies. For existence of minimizers via the so-called "Direct method in the calculus of variations" see for instance Evan's PDE Book Chapter 8.2. For a more thorough treatment, see for instance Chapter I of Struwe's book "Variational methods".

(5) For the existence of critical points of min-max type see for instance Evan's PDE Book Chapter 8.5. For a more thorough treatment of min-max type critical points, see for instance Chapter II of Struwe's book "Variational methods".

(6) Regularity for second order elliptic PDEs. The topic is very broad. Some standard references are "Elliptic Partial Diffferential Equations of Second Order" by Gilbarg-Trudinger, "Elliptic Partial Diffferential Equations" by Han-Lin, "Lectures on Elliptic Partial Differential Equations" by Ambrosio-Carlotto-Massaccesi, "Elliptic Regularity Theory-a first course" by Beck.