Geometric Group Theory

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Some inspirational quotations

George Polya: “Where should I start? Start from the statement of the problem. ... What can I do? Visualize the problem as a whole as clearly and as vividly as you can. ... What can I gain by doing so? You should understand the problem, familiarize yourself with it, impress its purpose on your mind.”

Th. Bröcker and K. Jänich, “Introduction to differential topology” (p.25) “Having thus refreshed ourselves in the oasis of a proof, we now turn again into the desert of definitions.”

Actions on simplicial trees

General theorem: G is free if and only if G acts freely by isometries on a simplicial tree T. The $(\Rightarrow)$ direction is given by the theorem below.

Theorem

$\hat{\Gamma}(S, G)$ is a simplicial tree on which G acts freely $\iff S = X \sqcup X^{-1}$, $G = F(X)$.

For the $(\Leftarrow)$ direction, the following lemma is a key step.

Lemma

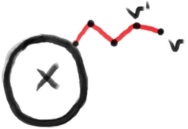
There exists $X \subseteq T$, X a tree, such that X contains exactly one vertex from each orbit.

Actions on simplicial trees

Lemma

There exists $X \subseteq T$, X a tree, such that X contains exactly one vertex from each orbit.

Proof: Take X maximal such that X intersects each orbit $G \cdot v$ in at most one point (X exists by Zorn's lemma). Assume there exists some v such that $Gv \cap X = \emptyset$. Take v at minimal distance from X. If $d(v, X) = 1$, then we can add it to X - contradiction. So assume $d(v, X) \geq 2$.



By minimality, $gv' \in X$ for some $g \in G$. Therefore $d(gv, X) = 1$ and so we can add gv to X - contradiction. $\square$

Actions on simplicial trees

Lemma

There exists $X \subseteq T$, X a tree, such that X contains exactly one vertex from each orbit.

Theorem

G is free if and only if G acts freely by isometries on a simplicial tree T.

Actions on simplicial trees

Proof.

$(\Leftarrow)$: A 'tiling' of $V(T)$:
If $gX \cap X \neq \emptyset$ then there exists $v \in X$ such that $gv = v$ and so $g = 1$ by the freeness of the action. Hence if $g_1 \neq g_2$ then $g_1 X \cap g_2 X = \emptyset$.

Choose an orientation E^+ on the edges of T that is G-invariant. Let

$$S = \{g \in G : \exists e \in E^+, o(e) \in X, t(e) \in g(X)\}$$

We will prove that $G = F(S)$.

$\{g_1, g_2\}$ is an edge of $\hat{\Gamma}(S \cup S^{-1}, G)$ if and only if there exists an edge of T with one endpoint in $g_1 X$ and the other in $g_2 X$.

$\hat{\Gamma}(S \cup S^{-1}, G)$ is connected because T is. It is simplicial because it is a Cayley graph. And if $\hat{\Gamma}(S \cup S^{-1}, G)$ contains a cycle then so does T. So $G = F(S)$. □

Actions on simplicial trees

Theorem

G is free if and only if G acts freely by isometries on a simplicial tree T.

Corollary

Subgroups of free groups are free.

In order to study groups having actions on simplicial trees that are not free, we need the notion of amalgam.

Amalgams

Let A, B be groups with two isomorphic subgroups: i.e. there exist injective homomorphisms $\alpha : H \to A$, $\beta : H \to B$.

The amalgam of A and B over H is the "largest" group containing copies of A and B identified along H such that no other relation is imposed and such that it is generated by the copies of A and B.

We will define the amalgam by its universal property.

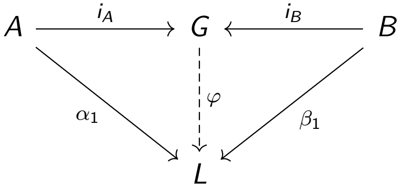
Amalgams

Notation: $\alpha(h) = h \in A$; $\beta(h) = \bar{h} \in B$.

Definition

G is the amalgamated product of A and B over H (written $G = A *_H B$) if

- there exist homomorphisms $i_A : A \to G$, $i_B : B \to G$ with $i_A(h) = i_B(\bar{h})$ for all $h \in H$;
- $\forall$ group L and $\forall$ homomorphisms $\alpha_1 : A \to L$, $\beta_1 : B \to L$ satisfying $\alpha_1(h) = \beta_1(\bar{h})$ for all $h \in H$, there exists a unique homomorphism $\varphi : G \to L$ such that $\alpha_1 = \varphi \circ i_A$ and $\beta_1 = \varphi \circ i_B$:



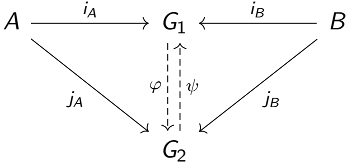
Amalgams

Remarks

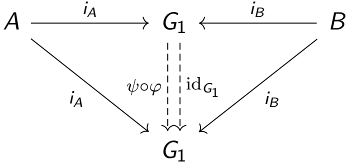
1. *The construction depends on the homomorphisms $\alpha : H \hookrightarrow A$, $\beta : H \hookrightarrow B$ but the notation is simplified.*
2. *It is not clear from the definition whether i_A and i_B are injective. However, this turns out to be the case.*

Uniqueness of the amalgam

Uniqueness of the amalgam: Suppose G_1 and G_2 are both amalgams of A, B over H. Then we have a commutative diagram



This implies that $\mathrm{id}_{G_1} : G_1 \to G_1$ and $\psi \circ \varphi : G_1 \to G_1$ both make the following diagram commute



And so $\psi \circ \varphi = \mathrm{id}_{G_1}$ by uniqueness of the induced homomorphism. Similarly $\varphi \circ \psi = \mathrm{id}_{G_2}$.

Existence of the amalgam

Existence of the amalgam:

Let $A = \langle S_1 | R_1 \rangle$, $B = \langle S_2 | R_2 \rangle$. WLOG $S_1 \cap S_2 = \emptyset$. Then

$$A *_H B = \langle S_1 \cup S_2 | R_1 \cup R_2 \cup \{h = \bar{h} : h \in H\} \rangle$$

Proof: Check that it satisfies the universal property (exercise).

Remarks

- *A and B generate $A *_H B$.*
- *i_A and i_B are injective.*

When $H = \{1\}$, the amalgam does not depend on α, β and it is called the free product of A and B, denoted by $A * B$.

Example

$F_2 = \mathbb{Z} * \mathbb{Z}$ *since if* $\mathbb{Z} = \langle a | \ \rangle$, $\mathbb{Z} = \langle b | \ \rangle$, *then* $\mathbb{Z} * \mathbb{Z} = \langle a, b | \ \rangle = F_2$.

Amalgams

We would like to describe the elements of $A *_H B$ by words.

Simplified notation: we identify H with $\alpha(H)$ and $\beta(H)$, and we identify A with $i_A(A)$, B with $i_B(B)$.

Let A_1 be a set of right coset representatives of H in A, and similarly let B_1 be a set of right coset representatives of H in B, such that $1 \in A_1$, $1 \in B_1$.

Definition

A reduced word of the amalgam $A *_H B$ is a word of the form $(h, s_1, ..., s_n)$, $h \in H$, $s_i \in A_1 \cup B_1$, $s_i \neq 1$, s_i alternating from A_1 to B_1. We associate to this the element $hs_1...s_n$ of $A *_H B$. The length of the reduced word is n.

Theorem

Each $g \in G = A *_H B$ is represented by a unique reduced word.