Chapter 4
Affine schemes
§ Structure sheaf

Reminder: A ring, $S \subset A$ multiplicatively closed subset, $0 \notin S$.

$$S^{-1}A := \{(a,s) \mid s \in S,\ a \in A\}/\sim$$

where $(a,s) \sim (a',s')$ iff $\exists s''$: $s''(as' - a's) = 0$ in A.

Ex: 1) $S = \{1, f, f^2, \dots\}$, denoted A_f

2) $S = A \setminus \mathfrak{p}$, $\mathfrak{p}$ prime ideal, denoted $A_{\mathfrak{p}}$.

def.-Thm: The structure sheaf $\mathcal{O}_{\operatorname{Spec} R}$ is the sheaf of rings on $\operatorname{Spec} R$ s.t.:

1) $\mathcal{O}_{\operatorname{Spec} R}(D(f)) = R_f \quad \forall f \in R$

2) $\mathcal{O}_{\operatorname{Spec} R, x} = R_{\mathfrak{p}_x} \quad \forall x \in \operatorname{Spec} R.$

similar to varieties

stalks

Moral reason:

1) $D(f) = \{x \in X \mid f(x) \neq 0\} \rightsquigarrow$
$\mathcal{O}_X(D(f)) = R_f$
we allow to invert powers of f because they do not vanish on $D(f)$

2) $\mathcal{O}_{X,x} = \left\{ (U,f) \mid \begin{matrix} x \in U \subseteq X \text{ open} \\ f \in \mathcal{O}_X(U) \end{matrix} \right\} / \sim \quad \rightsquigarrow$

$\mathcal{O}_{X,x} = R_{p_x}$

germs of functions encode local behaviour around x, hence we allow to invert all functions that don't vanish at x, i.e. $R - p_x$.

Ex. 1) $X = \operatorname{Spec} \mathbb{Z}$

$\mathcal{O}_X(D(p)) = \mathcal{O}_X(\operatorname{Spec}\mathbb{Z} - (p)) = \mathbb{Z}[\frac{1}{p}] = \{\frac{m}{p^n}\}$

$\mathcal{O}_{X,(p)} = \mathbb{Z}_{(p)} = \{\frac{m}{\ell}, p \nmid \ell\}$

$\mathcal{O}_{X,(0)} = \mathbb{Z}_{(0)} = \mathbb{Q}$

2) $X = \operatorname{Spec} D$, D a dvr, $m = (t)$, $K := \operatorname{Frac} D$

Denote $X = \{\underset{\text{closed pt}}{x}, \underset{\text{generic pt}}{\eta}\}$.

We have:

$\mathcal{O}_X(\emptyset) = 0$; $\mathcal{O}_X(X) = D$; $\mathcal{O}_X(\eta) = D_t = K$

$\mathcal{O}_{X,x} = D_{(t)} = D$; $\mathcal{O}_{X,\eta} = D_{(0)} = K$.

Proof:

(I) define $\mathcal{O}$ as a presheaf on $\{D(f)\}_{f\in R}$ given by $\mathcal{O}(D(f)) = R_f$.

Since different $f \in R$ may give the same $D(f)$, you define $\mathcal{O}(D(f)) := S_{D(f)}^{-1} R$,

where $S_{D(f)} := \{s \in R \mid s \notin p \quad \forall p \in D(f)\}$ ⇐ "saturation of $\{f^n\}_n$", depends only on $D(f)$, not f

and check that $R_f \xrightarrow{\sim} S_{D(f)}^{-1} R$ is an isom.

The restriction maps are localizations:

$D(g) \subseteq D(f) \implies S_{D(f)} \subseteq S_{D(g)} \rightsquigarrow S_{D(f)}^{-1} R \xrightarrow{\rho} S_{D(g)}^{-1} R.$

(II) check that $\mathcal{O}$ satisfies sheaf conditions on the open sets $\{D(f)\}_{f\in R}$ ← basis, this is called being a sheaf on a basis (instead of any opens $\mathcal{U}$, take only open sets from the basis).

The sheaf conditions on $\{D(f)\}$ can be algebraically reformulated as follows.

Let $D(f) = \bigcup_{i\in I} D(f_i)$ be any open cover.

There are localization maps

$\rho_i : R_f \to R_{f_i}, \quad \rho_{ij} : R_{f_i} \to R_{f_i \cdot f_j}.$

Then $\mathcal{O}$ being a sheaf on $\{D(f)\}$ is equivalent to the following sequence being exact:

$$0 \to R_f \xrightarrow{\alpha} \prod_i R_{f_i} \xrightarrow{\beta} \prod_{i,j} R_{f_i \cdot f_j},$$

„sheaf exact sequence"

where $\alpha(a) = p_i(a)$ and $\beta((a_i))_{i,j} = (p_{ij}(a_i) - p_{ji}(a_j))$.

That means, by definition, that:

- α is injective (Locality)

„sections agree locally $\Rightarrow$ agree globally"

- $\operatorname{Ker} \beta = \operatorname{Im} \alpha$ (gluing)

„sections agreeing on overlaps can be glued"

Locality Want: $\alpha, \beta \in R_f$, $\alpha|_{R_{f_i}} = \beta|_{R_{f_i}}$ $\forall i \Rightarrow \alpha = \beta$.

By replacing X, R with $D(f)$ and R_f can assume $f = 1$, $R_f = R$, $D(f) = X$.

$$\alpha - \beta = 0 \in R_{f_i} \Rightarrow f_i^{N_i}(\alpha - \beta) = 0 \text{ for some } N_i \in \mathbb{N}$$

N_i depends on i, but $\operatorname{Spec} R$ is quasi-compact, so we can pick a finite subcover by $D(f_i)$ and let $N := \max N_i$.

We get: $f_i^N(\alpha - \beta) = 0$ $\forall i$

$\Rightarrow \underbrace{\langle \text{all } f_i^N \rangle}_{= R}(\alpha - \beta) = 0 \Rightarrow 1 \cdot (\alpha - \beta) = 0 \Rightarrow \alpha = \beta$.

because $\operatorname{Spec} R = \bigcup D(f_i) = \bigcup D(f_i^N)$

Gluing We have $s_i \in R_{f_i}$ s.t. $s_i|_{R_{f_if_j}} = s_j|_{R_{f_if_j}}$.

Want: find $s \in R_f = R$ (can assume $f = 1$)

s.t. $s|_{R_{f_i}} = s_i \quad \forall i$.

Can assume $X = \operatorname{Spec} R = \bigcup_{i=1}^{n} D(f_i)$ finite cover,

$s_i = \dfrac{g_i}{f_i^{\cancel{h_i}}}$ and we can assume $h_i = 1$

because $D(f_i) = D(f_i^{h_i})$.

$s_i = s_j$ in $R_{f_if_j}$ $\Rightarrow (f_i f_j)^N (f_j g_i - f_i g_j) = 0$.

for some $N \in \mathbb{N}$

(pick same big N for all

pairs (i,j) — finitely many)

Rewrite: $\underbrace{(f_j^{N+1})}_{b_j}\underbrace{(f_i^N \cdot g_i)}_{a_i} - \underbrace{(f_i^{N+1})}_{b_i}\underbrace{(f_j^N g_j)}_{a_j} = 0$

Notice $s_i = \dfrac{a_i}{b_i}$, $D(f_i) = D(b_i)$ so

we can assume $N = 0$ and $f_j g_i = g_j f_i$.

$\operatorname{Spec} R = \bigcup_{i=1}^{n} D(f_i) \Rightarrow 1 = \sum r_i f_i \Rightarrow$

$1 \cdot g_j = (\sum r_i f_i g_j) = \sum r_i f_j g_i = f_j \cdot \sum r_i g_i$

$\Rightarrow s_j = \dfrac{g_j}{f_j} = \dfrac{\sum r_i g_i}{1} \in R_{f_j} \quad \forall j \Rightarrow$ we globalized

$s_j \in R_{f_j}$ to $\sum r_i g_i \in R = \mathcal{O}_X(X)$.

III. Define $\mathcal{O}_{\operatorname{Spec} R}$ to be the unique sheaf extending $\mathcal{O}$ from the basis $\{D(f)\}_{f\in R}$:

it's a general construction:

sheaf on a basis ⇝ sheaf on the whole space

$$\mathcal{O}_{\operatorname{Spec} R}(U) := \lim_{D(f)\subseteq U} \mathcal{O}(D(f)) = \lim_{D(f)\subseteq U} R_f :=$$

$$\Big\{ (s_f) \in \prod_{D(f)\subseteq U} R_f \;\Big|\; s_f|_{D(g)} = s_g \quad \forall\; D(g) \subseteq D(f) \subseteq U \Big\}$$

„compatible families of local sections on basic opens"

In more detail: lecture notes by Alex Ritter
(e.g. uniqueness)

Intuition:

„lim generalizes $\cap$, colim generalizes $\cup$"
to the situation when $R_f \to R_g$ are not injective
(lim and colim are defined via universal properties)

(IV). We compute the stalks by definition:

$$\mathcal{O}_{\operatorname{Spec} R, x} = \operatorname*{colim}_{U \ni x} \underbrace{\mathcal{O}_{\operatorname{Spec} R}(U)}_{\coprod_{U \ni x;\, f \in \mathcal{O}(U)} \{(U, f)\}/\sim} = \operatorname*{colim}_{D(f) \ni x} \mathcal{O}(D(f)) = \operatorname*{colim}_{f \notin \mathfrak{p}_x} R_f = R_{\mathfrak{p}_x}.$$

Rem. $\forall\, U \subseteq \operatorname{Spec} R$ open,

$\mathcal{O}_{\operatorname{Spec} R}(U)$ is an R-algebra.

Indeed, for $a \in R$ we define

$[a] : R_f \xrightarrow{\cdot a} R_f$ on $D(f)$,

and that induces an R-module structure $\forall U$ which gives a map of sheaves

$[a] : \mathcal{O}_{\operatorname{Spec} R} \to \mathcal{O}_{\operatorname{Spec} R}$.

§Affine schemes

Unlike varieties: we need ringed spaces in a more general context, when $\mathcal{O}_X$ can't be thought of as some k-valued functions for a fixed field k.

def. A ringed space is a pair $(X, \mathcal{O}_X)$ where X is a top. space and $\mathcal{O}_X$ a sheaf of rings on X.

A morphism of ringed spaces is a pair $(f, f^\#)$ where $f: X \to Y$ is continuous and $f^\#: \mathcal{O}_Y \to f_* \mathcal{O}_X$ is a map of sheaves of rings on Y, or equivalently, $f^\# : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$.

That means, $\forall\, U \subseteq Y$ open we have extra data of a ring hom

$$f^\#(U) : \mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}U),$$

s.t. for $V \subseteq U$ ring homs $f^\#(-)$ are compatible with restrictions ρ_{UV}:

$$\begin{array}{ccc} \mathcal{O}_Y(U) & \xrightarrow{f^\#(U)} & \mathcal{O}_X(f^{-1}U) \\ \rho_{UV}\downarrow & & \downarrow \rho_{f^{-1}U, f^{-1}V} \\ \mathcal{O}_Y(V) & \xrightarrow{f^\#(V)} & \mathcal{O}_X(f^{-1}V) \end{array}$$

is commutative.

In particular, $(f, f_\#)$ is an isom if f is a homeo and $\forall\, U \subseteq Y$ open $f^\#(U) : \mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}U)$ is an isom.

Rem. 1) $f^\#$ generalizes pullback of regular functions on k-varieties:

$h : U \to k \quad \mapsto \quad h \circ f : f^{-1}U \to k$, $U \subseteq Y$

but we need extra data of $f^\#$ because, unlike with k-varieties, we don't have such pullback maps for free.

2) $\forall\, x \in X$, $y = f(x)$ there's an induced map $f^\#_x : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ that sends (s, V), $y \in V \subseteq Y$ open to $(f^\# s, f^{-1}V)$, where $x \in f^{-1}V \subseteq X$ open; the map respects $\sim$ because $f^\#$ commutes with ρ_{UV}.

Since the definition of $f^\#$ is more general, we have too much freedom on the choice of $f^\#$, so we'll introduce a restriction.

def. A **locally-ringed space** is a ringed space $(X, \mathcal{O}_X)$ such that $\forall$ point $x \in X$ the stalk $\mathcal{O}_{X,x}$ is a local ring.

A morphism of locally-ringed spaces is a morphism of ringed spaces s.t. $\forall x \in X$, $y = f(x)$ the induced map

$$f_x^\# : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$$

is a **local hom**, i.e. $f_x^\#(m_y) \subseteq m_x$, or equivalently, $(f_x^\#)^{-1}(m_x) = m_y$.

For k-varieties, this condition was automatically satisfied, because

$$m_y = \{ f \in \mathcal{O}_{Y,y} \mid f(y) = 0 \} \Rightarrow f^* m_y \subseteq m_x.$$

Main Ex. $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ is a locally ringed space.

Rem. $f_x^\#$ local $\Rightarrow$ induced hom on residue fields:

$$\kappa(f(x)) = \mathcal{O}_{Y,f(x)}/m_{f(x)} \hookrightarrow \mathcal{O}_{X,x}/m_x = \kappa(x)$$ field extension

Prop. A ring hom $\varphi : R \to S$ induces a map of locally ringed spaces $\operatorname{Spec} \varphi = (\varphi^*, \varphi^\#) : (\operatorname{Spec} S, \mathcal{O}_{\operatorname{Spec} S}) \longrightarrow (\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ that satisfies:

1) on distinguished opens, $\forall f \in R$

$$R_f = \mathcal{O}_{\operatorname{Spec} R}\, D(f) \xrightarrow[\varphi^\#(D(f))]{} \mathcal{O}_{\operatorname{Spec} S}\, D(\varphi(f)) = S_{\varphi(f)}$$

is the localization of φ at f: $\frac{a}{f^n} \mapsto \frac{\varphi(a)}{\varphi(f)^n}$

2) on stalks, $\forall p \in \operatorname{Spec} S$ the induced map

$$\varphi^\#: R_{\varphi^{-1}(p)} \to S_p$$

is the localization of φ.

Proof sketch:

- define $\varphi^\#$ on $D(f)$ as in 1)
- check compatibility with ρ_{uv}
- compute $\varphi^\#$ on stalks as in 2)

def. An affine scheme is a locally ringed space $(X, \mathcal{O}_X)$ isomorphic to $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$.

Affine schemes form a subcategory AffSch of locally ringed spaces.

We get a functor $\operatorname{Spec}: \mathrm{Ring}^{op} \to \mathrm{AffSch}$.

We also have a functor in the other direction, which sends $(X, \mathcal{O}_X)$ to $\mathcal{O}_X(X) =: \Gamma(X, \mathcal{O}_X)$

global sections

and $f: (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ to $f^\#(Y): \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$.

Thm. The functor

$$\text{Spec}: \text{Ring}^{op} \xrightarrow{\sim} \text{AffSch}$$

is an equivalence, with inverse functor Γ.

In particular, $f : \text{Spec}\, S \to \text{Spec}\, R$ is an isom of locally ringed spaces iff $f^{\#} : R \xrightarrow{\sim} S$ is an isom.

Proof: enough to show that for $X = \text{Spec}\, S$, $Y = \text{Spec}\, R$,
$f : X \to Y \in \text{AffSch} \rightsquigarrow \text{Spec}(\Gamma(f)) = f$.
Let $\varphi := \Gamma(f) = f^{\#}(Y) : R \to S$; let $x \in X \leftrightsquigarrow q \subset S$, $f(x) \leftrightsquigarrow p \subset R$.
Want: $f = \text{Spec}\, \varphi$ & $f^{\#} = (\text{Spec}\, \varphi)^{\#}$.
We have a commutative diagram:

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[d] \arrow[dr, phantom, "\leftarrow \text{localizations} \rightarrow"] & S \arrow[d] \\ R_p \arrow[r, "f^{\#}_x"'] & S_q \end{tikzcd}$$

Hence $\varphi(R - p) \subseteq S - q$, so $\varphi^{-1}(q) \subseteq p$.
However $f^{\#}_x$ is local, so $\varphi^{-1}(q) \supseteq p$.
$\Rightarrow \varphi^{-1}(q) = p$,
so $\text{Spec}\, \varphi = f$ as maps of top. spaces.

From the diagram we also get that $\forall\, x$ the stalk map $f^{\#}_x$ equals the localization of φ, i.e. $(\text{Spec}\, \varphi)^{\#}_x$, because the map
$R \xrightarrow{\varphi} S \to S_q$ factors uniquely through R_p.
(Similarly $f^{\#}(D(h)) : R_h \to S_{\varphi(h)}$ is the localization of φ $\forall h \in R$.)
Hence maps of sheaves $\text{Spec}\, \varphi^{\#}$ and $f^{\#}$ coincide.

Examples of some rings / affine schemes, which naturally occur in alg geom:

1) varieties $k[x_1 \ldots, x_m]/I$

2) hypersurfaces $\mathbb{Z}[x_1 \ldots, x_m]/(f)$

3) invariant rings R^G or $k[x_1 \ldots x_m]^G$
   - these give quotients of varieties under group action

4) Artinian rings: $k[t]/t^2$ or $k[t,s]/(t^a, s^b)$
„thickenings", appear as deformations

5) monoid rings: $\mathbb{Z}[x,y]$, $\mathbb{Z}[\underbrace{x^{\pm 1}}, y^{\pm 1}]$ etc

fix the allowed powers, take a free abelian group on them