Chapter 5

Schemes

def. A **scheme** is a locally ringed space $(X, \mathcal{O}_X)$ which is locally isomorphic to an affine scheme, i.e. $X = \bigcup_{i \in I} U_i$ open cover, s.t.

$\forall i \; \exists$ ring R_i: $(U_i, \mathcal{O}_X|_{U_i}) \simeq (\operatorname{Spec} R_i, \mathcal{O}_{\operatorname{Spec} R_i})$.

$x \in X \rightsquigarrow$ the stalk $\mathcal{O}_{X,x}$ is the **local ring at x.**

If $x \in U = \operatorname{Spec} R \subseteq X$ open, then

$$\mathcal{O}_{X,x} = \mathcal{O}_{U,x} = R_p, \quad p = p_x.$$

Residue field at x:

$$m \subset \mathcal{O}_{X,x} \rightsquigarrow \kappa(x) := \mathcal{O}_{X,x}/m\cdot\mathcal{O}_{X,x} = R_p/p\cdot R_p.$$

A **morphism** or a map of schemes is a map $(f, f^\#)$ of locally ringed spaces.

Schemes form a category $\mathrm{Sch} \supset \mathrm{AffSch}$.

$\mathbb{F}$-valued points

Schematic points: $\mathbb{F}$ any field $\rightsquigarrow X(\mathbb{F}) := \{\operatorname{Spec} \mathbb{F} \to X\}$ – set

$x \in X \rightsquigarrow x \in U \hookrightarrow X$ affine open $\rightsquigarrow$

$$\begin{tikzcd} \operatorname{Spec} \underset{R_p/p\cdot R_p}{\underset{\shortparallel}{\kappa(x)}} \arrow[r] \arrow[rr, bend right] & \underset{\operatorname{Spec} R}{\underset{\shortparallel}{U}} \subset & X \end{tikzcd}$$

Thm. $\forall$ X scheme, R ring

$$\mathrm{Maps}_{\mathrm{Sch}}(X, \operatorname{Spec} R) = \mathrm{Maps}_{\mathrm{Ring}}(R, \mathcal{O}_X(X)).$$

Hence giving a map $X \to \operatorname{Spec} R$ is equivalent to giving an R-algebra structure to $\mathcal{O}_X$.

Proof sketch: $X = \operatorname{Spec} S$: proved before

X general: we'll define a map $\textcircled{\leftarrow}$.

Given $R \xrightarrow{\varphi} \mathcal{O}_X(X)$, consider

$\forall x \in X$

$$\begin{tikzcd}
R \arrow[r] \arrow[rr, bend right, "\varphi_x"'] & \mathcal{O}_X(X) \arrow[r] & \mathcal{O}_{X,x} \\
\cup & & \cup \\
\varphi_x^{-1}(m_x) \arrow[rr, dashed] & & m_x
\end{tikzcd}$$

Define

$$\begin{tikzcd}
X \arrow[r, "g"] & \operatorname{Spec} R \\
\cup & \cup \\
x \arrow[r, mapsto] & \varphi_x^{-1}(m_x)
\end{tikzcd}$$

. – prime ideal

The map g is continuous because one can check that $g^{-1}(D(f)) = D(\varphi(f))$.

Further define

$$\begin{tikzcd}
\mathcal{O}_{\operatorname{Spec} R}(D(f)) = R_f \arrow[r, "\varphi_f"] & \mathcal{O}_X(X)_{\varphi(f)} \arrow[r, dashed] & \mathcal{O}_X(D(\varphi(f))) = \mathcal{O}_X(g^{-1}(D(f))) \\
& & g_*\mathcal{O}_X(D(f)) \arrow[u, equal]
\end{tikzcd}$$

$\uparrow$ Because $\varphi(f)$ is invertible in $\mathcal{O}_X(D(\varphi(f)))$

Cor. $\operatorname{Maps}(X, \mathbb{A}^1) \simeq \mathcal{O}_X(X)$. „functions on X"

since $\mathbb{Z}[x] \to \mathcal{O}_X(X)$ is determined by the image of x.

Ex: open subschemes

$(X, \mathcal{O}_X)$ scheme, $U \subseteq X$ open subset $\Rightarrow$

$(U, \mathcal{O}_X|_U)$ is also a scheme.

Because: $\forall\, u \in U$ has a distinguished open $U \supseteq U(x) \ni u$, so $(U(x), \mathcal{O}_U|_{U(x)})$ is an affine scheme.

[closed subschemes
are more complicated – later! :)]

Ex. non-affine scheme

Consider $U := \mathbb{A}^2 - \{(0,0)\} \subset \mathbb{A}^2 = \operatorname{Spec} \mathbb{Z}[x,y]$.

Exercise: $\underset{\substack{\shortparallel \\ \mathbb{Z}[x,y]}}{\mathcal{O}_{\mathbb{A}^2}(\mathbb{A}^2)} \xrightarrow{\sim} \mathcal{O}_U(U)$ is an isom.,

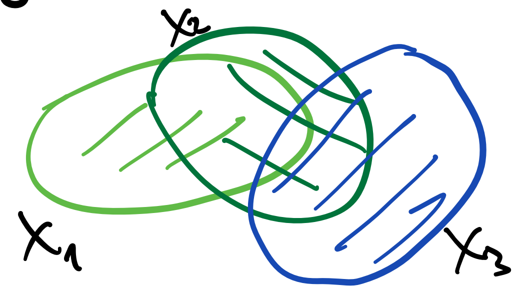
but $U \hookrightarrow \mathbb{A}^2$ is not an isom.

Because $Z(x,y) = \emptyset$ in U
$\neq \emptyset$ in $\mathbb{A}^2$,

hence $U = \mathbb{A}^2 - \{(0,0)\}$ cannot be affine.

Gluing: how to get non-affine schemes

Idea:



X_i – schemes that „agree on intersections"
$\Rightarrow X = \bigcup X_i$ is a scheme

Roughly speaking, on $X_i \cap X_j$ you should specify gluing isoms

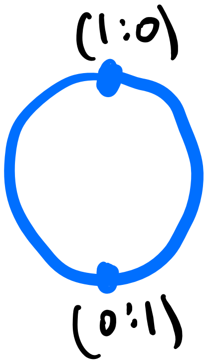
$$\mathcal{O}_{X_i}|_{X_i \cap X_j} \xrightarrow{\sim} \mathcal{O}_{X_j}|_{X_i \cap X_j} .$$

(see exercises!)

Ex.: projective line

two charts

⇝



$\mathbb{P}^1_{\mathbb{Z}}$ ← can be any ring R

$$U_0 = \operatorname{Spec} \mathbb{Z}[u] \simeq \mathbb{A}^1 , \qquad U_1 = \operatorname{Spec} \mathbb{Z}[\underset{\nwarrow \text{variable}}{u^{-1}}] \simeq \mathbb{A}^1_{\mathbb{Z}}$$

$$\cup \qquad\qquad\qquad\qquad \cup$$

$$U_{01} := D(u) = \operatorname{Spec} \mathbb{Z}[u^{\pm 1}] \qquad U_{10} := D(u^{-1}) = \operatorname{Spec} \mathbb{Z}[u^{\pm 1}]$$

We glue U_0 and U_1 along $U_{01} \simeq U_{10}$ and get the projective line!

Indeed, for $a \neq 0$ $\quad [1 : \underset{\wedge}{a}] = [\underset{\wedge}{\tfrac{1}{a}} : 1]$

coordinates in different charts

More generally: projective space $\mathbb{P}^n_{\mathbb{Z}}$; $\mathbb{P}^n_R$ $\forall R$
(see exercises)
$\mathbb{P}^n_{\mathbb{C}}(\mathbb{C})$ is the variety $\mathbb{P}^n$ over $\mathbb{C}$

Rem: $\mathbb{P}^n_R$ can also be defined using „Proj" (Hartshorne, Vakil, ...)

§Integral schemes

def. A scheme $(X, \mathcal{O}_X)$ is reduced if all local rings are reduced (no nilpotents).

Exercise: $\mathcal{O}_{X,x}$ reduced $\forall x \in X \iff$ $\mathcal{O}_X(U)$ reduced $\forall$ (affine) open $U \subseteq X$.

In particular, Spec R is reduced iff R is reduced.

Associated reduced scheme:

$$\operatorname{Spec} R_{red} \hookrightarrow \operatorname{Spec} R, \quad \text{where } R_{red} := R/_{\mathrm{Nil} R}$$

closed immersion

same top. spaces, different structure sheaves!

Ex: $R = k[t]/_{t^n} \rightsquigarrow \operatorname{Spec} R_{red} = \operatorname{Spec} k \hookrightarrow \operatorname{Spec} R$

For any scheme X, one can glue $X_{red} \longrightarrow X$, and it is universal: for a reduced scheme Y any map $Y \to X$ factors through X_{red}.

def. A scheme is integral if it is reduced and irreducible.

Prop: $(X, \mathcal{O}_X)$ is integral iff $\mathcal{O}_X(U)$ is an int. domain $\forall$ (affine) open $U \subseteq X$.

Proof for Spec R:

Spec R integral $\iff$ $\mathrm{Nil} R = (0)$ & $\mathrm{Nil} R$ is prime $\iff$ (0) is prime.

Structure sheaf of an integral scheme

We can think of sections of $\mathcal{O}_X$ as of certain rational functions!

def. X integral scheme, $\eta \in X$ the generic pt. The function field of X is

because X irreducible

$$\kappa(X) := \mathcal{O}_{X,\eta}.$$

It is a field because $\forall$ $\operatorname{Spec} R \subseteq X$ open

$$\mathcal{O}_{X,\eta} = \mathcal{O}_{\operatorname{Spec} R,\eta} = R_{(0)} = \operatorname{Frac} R$$

integral domain

Prop: X integral, $U \subseteq X$ open, $\eta \in X$ generic pt.

1) The canonical map
$$\mathcal{O}_X(U) \to \mathcal{O}_{X,\eta} = \kappa(X) \quad \text{is injective}$$

2) $V \subseteq U$ open $\Rightarrow$ the restriction map
$$\rho_{UV} : \mathcal{O}_X(U) \to \mathcal{O}_X(V) \subseteq \kappa(X) \quad \text{is injective}$$

3) $\forall x \in X$ $\mathcal{O}_{X,x} \subseteq \kappa(X)$, and

$$U \ni x \Rightarrow \mathcal{O}_X(U) \subseteq \mathcal{O}_{X,x}$$

4) $$\mathcal{O}_X(U) = \bigcap_{x \in U} \mathcal{O}_{X,x} \subseteq \kappa(X)$$

If $X = \operatorname{Spec} R$, $$\mathcal{O}_X(U) = \left\{ f \in \kappa(X) \;\middle|\; \forall x \in U \;\; f = \frac{g}{h},\ h(x) \neq 0,\ g,h \in R \right\}$$

can pick a representative, it depends on x

Proof: 1) Let $f \in \mathcal{O}_X(U)$, assume $f(\eta) = 0$. Then $\forall$ affine open $V = \operatorname{Spec} S \subseteq U$ we have $\rho_{UV}(f) = 0$, because S is an integral domain $\Rightarrow S \hookrightarrow \operatorname{Frac} S = k(X)$.

Take $U = \bigcup_{i \in I} V_i$ affine open cover $\Rightarrow$ $\rho_{UV_i}(f) = 0$ $\forall i$ $\Rightarrow f = 0$ because $\mathcal{O}_X$ is a sheaf.

2) The inclusions $\mathcal{O}_X(U) \hookrightarrow k(X)$ are compatible with restriction maps ρ_{UV}:

$$\begin{tikzcd}
\mathcal{O}_X(U) \arrow[rr, "\rho_{UV}"] \arrow[dr, hook] & & \mathcal{O}_X(V) \arrow[dl, hook'] \\
& \mathcal{O}_{X,\eta} &
\end{tikzcd}$$

commutes $\Rightarrow \rho_{UV}$ is injective.

3) we have a canonical map

$$\begin{aligned} \mathcal{O}_{X,x} &\longrightarrow \mathcal{O}_{X,\eta} \\ [U,f] &\longmapsto [U,f] \end{aligned}$$

It's injective: $\mathcal{O}_{X,x} = \mathcal{O}_{V,x}$ for $x \in \overset{\text{open nbhd}}{V} = \operatorname{Spec} A \Rightarrow$

$\mathcal{O}_{X,x} = \mathcal{O}_{V,x} = A_p \hookrightarrow \operatorname{Frac} A = \mathcal{O}_{X,\eta}$ for A an integral domain.

For $U \ni x$ the map $\mathcal{O}_X(U) \hookrightarrow k(X)$ factors through $\mathcal{O}_{X,x} \Rightarrow$

$\mathcal{O}_X(U) \hookrightarrow \mathcal{O}_{X,x}$.

4) by 3), $\mathcal{O}_X(U) \subseteq \bigcap \mathcal{O}_{X,x}$

Let $f \in \bigcap \mathcal{O}_{X,x} \subseteq k(X)$.

Then $\forall x$ $\exists$ open nbhd $x \in V(x) \subseteq U$:
$f \in \mathcal{O}_X(V(x))$.

Since $U = \bigcup_x V(x)$, we can glue a function $f \in \mathcal{O}_X(U)$ because the values agree on all $V(x) \cap V(x')$ since they coincide inside $k(X)$.

The last equality follows because $X = \operatorname{Spec} R \rightsquigarrow \mathcal{O}_{X,x} = R_{\mathfrak{p}_x}$.