Chapter 9
Divisors

More details: Hartshorne, Chapter II.6

Moral: codim 1 subschemes are the easiest closed subschemes to study, because height 1 ideals in good cases are principal

Recall: $Z \subseteq X$ closed $\Rightarrow$ codimension of Z in X is $\operatorname{codim}_X Z = \sup\{n \mid Z = Z_0 \subsetneq \dots \subsetneq Z_n \subseteq X$ closed irreducible subsets$\}$

§Weil divisors

Assume X Noetherian, integral, separated and that all $\mathcal{O}_{X,x}$ of dim 1 are dvr's (i.e. X is regular in codim 1, e.g. smooth or normal).

def. A prime divisor on X is a closed integral subscheme of codim 1.

A Weil divisor is an element of

$$\operatorname{Div}(X) := \bigoplus_{\substack{Z \subset X \\ \text{prime divisor}}} \mathbb{Z} \cdot [Z]$$

A Weil divisor is effective if it has positive coeff.

Construction: divisor of a function

$$f \in \underset{\substack{\| \\ \mathcal{O}_{X,\eta}}}{k(X)} \rightsquigarrow \operatorname{div}(f) := \sum_{\substack{Y \subset X \\ \text{prime}}} \operatorname{ord}_Y(f) \cdot [Y],$$

where $\operatorname{ord}_Y(f)$ is the valuation of f in $\mathcal{O}_{X,\eta_Y}$

„sum of zeroes minus poles with multiplicities"

Prop. div(f) is in fact a divisor, i.e. the sum is finite. (use X qcompact)

def. A principal divisor is div(f). They form a subgroup of Div(X) because $\mathrm{div}(f) + \mathrm{div}(g) = \mathrm{div}(f \cdot g)$.

The class group of X is
$$Cl(X) = \mathrm{Div}(X) / \text{principal divisors}$$

- interesting invariant, hard to compute!
- easiest of „Chow groups" of X
- for X = Spec R it's the Cl from number theory

§ Calculations

① $X = \operatorname{Spec} A$, A UFD $\Rightarrow Cl(X) = 0$
Because every prime divisor is principal
[if X is normal, $Cl(X) = 0 \Leftrightarrow$ A is a UFD]

② $X = \mathbb{P}^n_k \Rightarrow Cl(X) = \mathbb{Z}$ generated by $[H]$.
(also true over $\mathbb{Z}$, not k) hyperplane

Proof: • consider degree map
$$\deg : \mathrm{Div}(\mathbb{P}^n_k) \to \mathbb{Z}$$
$$\sum n_y \cdot [y] \mapsto \sum n_y \cdot \deg(y)$$
degree of a hypersurface

Let's extend $\mathrm{div}(-)$ to all functions on $\mathbb{P}^n_k$:

$g \in k[x_0, \dots, x_n]$ homog of deg $d \rightsquigarrow$

$g = g_1^{n_1} \cdot \ldots \cdot g_r^{n_r}$ g_i irred of deg $d_i \rightsquigarrow$

each g_i defines a hypersurface Y_i of deg d_i $\rightarrow$

define $\mathrm{div}(g) := \sum n_i \cdot [Y_i] \in \mathrm{Div}(\mathbb{P}^n_k)$; deg $= d$.

- $k(\mathbb{P}^n_k)$ consists of $\frac{g}{h}$; g, h homog of same degree

$\Rightarrow \mathrm{div}\left(\frac{g}{h}\right) = \mathrm{div}(g) - \mathrm{div}(h)$ has deg 0,

so deg factors as $\deg : \mathcal{Cl}(\mathbb{P}^n_k) \to \mathbb{Z}$

- surjectivity: $d \cdot [H] \mapsto d$, for H, say, $\{x_0 = 0\}$.
- injectivity: let $\deg D = d$ for $D \in \mathrm{Div}(\mathbb{P}^n_k)$,

write $D = D_1 - D_2$ for D_1, D_2 effective of deg d_1, d_2.

Then $D_i = \mathrm{div}(g_i)$ for some homog g_i because:

irred. hypersurface in $\mathbb{P}^n_k$ $\leftrightarrow$ homog prime ideal of height 1 in $k[x_0, \dots, x_n]$

and such an ideal is principal!

Taking powers and products, get any D_i as $\mathrm{div}(g_i)$.

Now $D - d \cdot H = \mathrm{div}(f)$ where $f = g_1 / x_0^d \cdot g_2 \in k(\mathbb{P}^n_k)$ $\Rightarrow D \sim d \cdot H$ in $\mathcal{Cl}$:)

Prop: (easy) 1) $Z \hookrightarrow X$ closed, $U = X - Z \Rightarrow$

$\mathcal{Cl}(X) \twoheadrightarrow \mathcal{Cl}(U)$ given by intersection with U.

2) $\operatorname{codim} Z \geq 2 \Rightarrow$ it's an isom

3) $\operatorname{codim} Z = 1$, Z irred $\Rightarrow$

$$\begin{array}{c} \mathbb{Z} \to \mathcal{Cl}(X) \to \mathcal{Cl}(U) \to 0 \\ 1 \mapsto [Z] \end{array}$$ is exact „excision sequence"

③ **Cor:** $U = \mathbb{P}^n_k - \deg d$ hypersurface $\Rightarrow$

$\mathcal{Cl}(U) \simeq \mathbb{Z}/d\mathbb{Z}$

§ Cartier divisors

Closely related notion: harder to define; easier to compute

Assume X Noetherian, integral, separated

recall: D principal $\Rightarrow D = \mathrm{div}(f)$, $f \in K(X)^\times =: K^\times$

f is defined up to $\mathcal{O}_X^\times(X) \subseteq K^\times$,

invertible functions

so D gives a section of $K^\times/\mathcal{O}^\times$.

<u>def</u>. A **Cartier divisor** on X is a global section of the sheaf $K^\times/\mathcal{O}^\times$: it's given by $X = \bigcup U_i$, $f_i \in K^\times$ s.t.

$$\left.\frac{f_i}{f_j}\right|_{U_i \cap U_j} \in \mathcal{O}^\times(U_i \cap U_j),$$

and we identify Cartier divisors given by refining the open cover and also $(U_i, f_i) \sim (U_i, \beta_i f_i)$ for $\beta_i \in \mathcal{O}^\times(U_i)$.

They form a group $\mathrm{Cartier}(X)$ via $\cdot$ of f_i's.

A Cartier divisor is **principal** if it is given by a rational function $f \in K^\times$: $(U_i, f \cdot \beta_i)$ for $\beta_i \in \mathcal{O}^\times(U_i)$.

$$\mathrm{CaCl}(X) := \mathrm{Cartier}(X) / \text{principal divisors}$$

Cartier to Weil (X integral, Noeth., sep., regular in codim 1):

fix $D = (U_i, f_i)$.

$\forall\, Y \subseteq X$ codim 1 integral $\exists i : \eta_Y \in U_i \rightsquigarrow$ take $\mathrm{val}_{\mathcal{O}_{\eta_Y}}(f_i) =: n_Y$ and define

$$D \in \mathrm{Cart}(X) \mapsto \sum n_Y \cdot [Y] \in \mathrm{Div}(X)$$

(rescaling by invertible function doesn't change!)

Thm. 1) X as above, all local rings are UFD's (X is „factorial", e.g. smooth/k) $\Rightarrow$

$$\mathrm{Cartier}(X) \xrightarrow{\sim} \mathrm{Div}(X),$$

and principal Cartier $\leftrightarrow$ principal Weil.

Key idea: A is UFD $\Leftrightarrow$ height 1 primes are principal

Moral: Cartier divisors are Weil divisors that are locally principal.

Non-Ex: X singular $\Rightarrow$ isom can fail!

$$X = \mathrm{Spec}\, k[x, y, z]/(xy - z^2) \subseteq \mathbb{A}^3_k$$

$$\mathrm{CaCl}(X) = 0 \quad \text{but} \quad \mathrm{Cl}(X) \simeq \mathbb{Z}/2,$$

where $\mathbb{Z}/2$ is generated by $Z = \{y = z = 0\}$.

At $0 \in Z$ we need 2 equations to cut out Z, 1 equation is not enough $\Rightarrow$ not locally principal!

Connection to line Bundles

def. The Picard group of X is

$$\mathrm{Pic}(X) := \{\text{line bdls on } X \text{ up to isom}, \otimes\}$$

invertible by last chapter!

There's a canonical map

$$\mathrm{CaCl}(X) \to \mathrm{Pic}(X)$$

$$D := (U_i, f_i) \mapsto \mathcal{O}(D)^{\mathrm{CaK}} : U_i \mapsto (f_i^{-1})\mathcal{O}(U_i),\ \alpha_{ij} = \frac{f_j}{f_i} \in \mathcal{O}^\times(U_{ij})$$

principal: $(U_i, f) \mapsto f^{-1} \cdot \mathcal{O} \simeq \mathcal{O}$

Claim: $\mathrm{CaCl}(X) \to \mathrm{Pic}(X)$ is an isom. (X integral!)

we'll see later why!

Example: (do check for $n=1$!)

- $H = \{x_0 = 0\} \longleftrightarrow \left(U_i \simeq \mathbb{A}^n, f_i = \frac{x_0}{x_i}\right) \longleftrightarrow \mathcal{O}(1)$
- $m \cdot H \longleftrightarrow \left(U_i \simeq \mathbb{A}^n, f_i = \left(\frac{x_0}{x_i}\right)^m\right) \longleftrightarrow \mathcal{O}(m)$

$$\mathrm{Cl}(\mathbb{P}^n) \simeq \mathrm{CaCl}(\mathbb{P}^n) \simeq \mathrm{Pic}(\mathbb{P}^n)$$

subschemes sheaf sections line bundles

cool comparison!

Next week: use scheme cohomology to interpret Cartier divisors & line bundles!