# Chapter 10
# Čech cohomology

Goal: singular coh $\rightsquigarrow$ coh of any sheaf — interesting invariants!

## § Definition & examples

$X$ top. space, $\mathcal{F}$ sheaf of ab grps on $X$
$\mathcal{U} = \{U_i\}_{i \in I}$ fully ordered open cover of $X$.

$U_{i_0 \dots i_p} := U_{i_0} \cap \dots \cap U_{i_p}$.

def. The group of Čech p-cochains is

$$C^p_{\mathcal{U}}(X, \mathcal{F}) := \prod_{i_0 < \dots < i_p} \mathcal{F}(U_{i_0 \dots i_p}), \quad p \geq 0$$

cochain

The differential is

$$C^p \xrightarrow{d^p} C^{p+1} \qquad \text{(denoted as } d\text{)}$$

$\alpha \in C^p$

$$(d\alpha)_{i_0 \dots i_{p+1}} := \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0 \dots \hat{i}_k \dots i_{p+1}} \Big|_{U_{i_0 \dots i_{p+1}}}$$

Ex:
- $C_0 = \prod_i \mathcal{F}(U_i) \xrightarrow{d} \prod_{i<j} \mathcal{F}(U_{ij}) = C_1$

$$(s_i) \longmapsto \left(s_j\big|_{U_{ij}} - s_i\big|_{U_{ij}}\right)$$

- $C_1 = \prod_{i<j} \mathcal{F}(U_{ij}) \xrightarrow{d} \prod_{i<j<k} \mathcal{F}(U_{ijk}) = C_2$

$$(s_{ij}) \longmapsto \left(s_{jk}\big|_{U_{ijk}} - s_{ik}\big|_{U_{ijk}} + s_{ij}\big|_{U_{ijk}}\right)$$

Easy to check: $d^2 = 0$, so $C^*_{\mathcal{U}}(X, \mathcal{F})$ is a complex:

$$\underset{\text{(co) boundaries}}{\operatorname{Im} d^{n-1}} \subseteq \underset{\text{(co)cycles}}{\operatorname{Ker} d^n}$$

def. The Čech cohomology groups are

$$H^p_{\mathcal{U}}(X, \mathcal{F}) := \operatorname{Ker}(d : C^p \to C^{p+1}) \Big/ \operatorname{Im}(d : C^{p-1} \to C^p)$$

Observations

1) $H^0_{\mathcal{U}}(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$

$\mathcal{F}$ is a sheaf

$$H^0 = \operatorname{Ker}(d^0) = \{(s_i) \in \mathcal{F}(U_i) \mid s_i|_{U_{ij}} = s_j|_{U_{ij}}\} = \mathcal{F}(X)$$

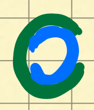
2) $H^m_{\mathcal{U}}(X, \mathcal{F}) = 0$ $m \geq |I|$ for finite $I$ by construction — there are no $\mathcal{U}_{i_0 \dots i_p}$, $p \geq |I|$.

3) $H^*_{\mathcal{U}}(X, \mathcal{F})$ does NOT depend on the choice of ordering of $\mathcal{U}$ — fact (Alex' notes)

Rem. pick bad $\mathcal{U}$ $\rightsquigarrow$ get bad $H^*$:

$\mathcal{U} = \{X\} \Rightarrow$ only detect $H^0_{\mathcal{U}}(X, \mathcal{F}) = \mathcal{F}(X)$ — no new invariants, boring!

**Ex:** $X = S^1$, $\mathcal{F} = \underline{\mathbb{Z}}$, $\mathcal{U} = \{U, V\}$



Then $C^0 = C^1 = \mathbb{Z}^2$ with

$$d: C^0 \to C^1$$
$$(a, b) \mapsto (b-a,\ b-a)$$ — very explicit.

$H^0 = H^1 = \mathbb{Z}$ — like singular coh :)

**Exercise:** $\mathcal{F} = \mathcal{O}_{\mathbb{P}^1_k}(-2)$, $\mathcal{U} = \mathbb{A}^1 \cup \mathbb{A}^1$ standard cover

$H^0 = 0$ but $H^1 = k$ - more info than just $K_0$!

help to compute:

$$C^0_{\mathcal{U}}(X, \mathcal{O}(-2)) = k\left[\frac{x_1}{x_0}\right] \times k\left[\frac{x_0}{x_1}\right]$$
$$C^1_{\mathcal{U}}(X, \mathcal{O}(-2)) = k\left[\frac{x_1}{x_0}\right]_{\frac{x_1}{x_0}} = k\left[\frac{x_1}{x_0}, \frac{x_0}{x_1}\right]$$
$$d(f, g) = g - f \cdot \frac{x_1^2}{x_0^2}$$

calculate diff, kernel and cokernel!

(we'll compute this later more generally)

# §Cohomology of affine schemes

Thm. $X = \operatorname{Spec} R$, $\mathcal{F} \in \operatorname{QCoh}(X)$,
$\mathfrak{U} = \cup U_i$ finite affine open cover of $X$ $\Rightarrow$

$$H^n_{\mathfrak{U}}(X, \mathcal{F}) = 0 \qquad n \geq 1.$$

Intuition: $H^*(\mathbb{C}^n) = 0$ $* \geq 1$ in alg topology

How to show $H^* = 0$ (general idea)

def. $C^*$ a complex: $\{C^i\}_{i \in \mathbb{Z}}$, $d: C^i \to C^{i+1}$, $d^2 = 0$.
$f = \{f^n: C^n \to C^n\}_n$ is a chain map if $f \circ d = d \circ f$.
Such $f$ induces $f: H^n \to H^n$ $\forall n$ via $f[c] = [fc]$.

$h = \{h^n: C^n \to C^{n-1}\}_n$ is a chain homotopy between chain maps $f, g$ if

$$f - g = d \circ h + h \circ d.$$

If $h$ exists, $f = g: H^n \to H^n$,
Because $dc = 0 \Rightarrow [fc - gc] = [dhc] = 0$ $\forall c$.

Trick: to show $H^* = 0$, find a chain htpy between id and 0 maps on $C^*$.
Such $C^*$ is then called exact or acyclic.

**Proof** that $H^m_{\mathcal{U}}(X,\mathcal{F}) = 0$ for $X$ affine, $\mathcal{F} \in QCoh(X)$, $m > 0$.

Let $X = \operatorname{Spec} A$. Assume $\mathcal{U} = \bigcup_{i=1}^{n} D(f_i)$, $f_i \in A$.

$\mathcal{F}$ qcoherent on $\operatorname{Spec} A \Rightarrow \mathcal{F} = \widetilde{M}$, $M$ A-module.

Need to show:

$0 \to M \to \prod_{i_0} M_{f_{i_0}} \to \prod_{i_0 i_1} M_{f_{i_0} f_{i_1}} \to \dots$ is exact.

Suffices to show: exact after $(-)_p$ $\forall p$ (stalks!)

Fix $p$. Choose $i_{fix}$: $f_{i_{fix}} \notin p$. $M_{f_{i_{fix}}, p} = M_p$.

Define homotopy

$$h : \prod M_{f_{i_0} \dots f_{i_{p+1}}, p} \to \prod M_{f_{i_0} \dots f_{i_p}, p}$$

via $h(s)_{i_0 \dots i_p} = s_{i_{fix} i_0 \dots i_p}$ (projection map!)

Then $(dh + hd)(s) = s = (id - 0)(s)$ $\Rightarrow$ by Trick we win!

by construction

General $\mathcal{U}$: refine to distinguished opens (we skip proof)

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**Cor.** of the proof:

by similar method, can show that $X$ irreducible scheme, $\underline{A}$ constant sheaf on $X \Rightarrow$

$$H^m_{\mathcal{U}}(X, \underline{A}) = 0 \quad m > 0.$$

(homol alg) uses the computation for affines!

**Thm** $X$ separated, qcompact; $\mathcal{F} \in \mathrm{QCoh}(X) \Rightarrow$
$\check{H}^*_{\mathcal{U}}(X;\mathcal{F})$ is independent of the choice of finite affine open cover $\mathcal{U}$: $\check{H}^*(X;\mathcal{F})$.

(there's a proof in Alex Ritter's notes)

**Rem.** for general $X$ have to take colimit along different $\mathcal{U}$, but sep. & qcomp schemes is good enough for us :)

$$\check{H}^*(X,\mathcal{F}) := \operatorname*{colim}_{\mathcal{U}} \check{H}^*_{\mathcal{U}}(X;\mathcal{F})$$

actual definition of Čech cohomology

maps $\mathcal{U} \to \mathcal{V}$ are refinements: $\forall j\ \exists i\ V_j \subseteq U_i$
gives $\check{C}^*_{\mathcal{U}}(X;\mathcal{F}) \to \check{C}^*_{\mathcal{V}}(X;\mathcal{F})$ via restriction maps

**NB:** usually it's a notation for a different concept – "sheaf cohomology", but they agree when $X$ separated & Noetherian, $\mathcal{F} \in \mathrm{QCoh}(X)$.

non-examinable

**Cool fact:** $X$ top space, hom equiv. to a CW-cpx (e.g. $X$ manifold) $\Rightarrow$

$$\check{H}^*(X;\underline{A}) \simeq H^*(X;A)$$

constant sheaf — singular coh

## § Long exact sequence on $H^*$

**Lemma.** $U \subseteq X$ open affine,

$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ exact sequence in $QCoh(X)$ $\Rightarrow$

$0 \to \mathcal{F}_1(U) \to \mathcal{F}_2(U) \to \mathcal{F}_3(U) \to 0$ is exact.

**Proof:** enough to check locally (stalks!) $\Rightarrow$ can assume $\mathcal{F}_i|_U = \widetilde{M}_i$, and $0 \to \widetilde{M}_1 \to \widetilde{M}_2 \to \widetilde{M}_3 \to 0$ is exact iff $0 \to M_1 \to M_2 \to M_3 \to 0$ is.

(because exactness can be checked on stalks / localizations at primes).

**Rem.** $X$ non-affine $\Rightarrow$ $\Gamma(X, -)$ is in general only left exact. What happens on the right?

**Thm:** $X$ separated qcomp.,

$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ s.e.s. in $QCoh(X)$ $\Rightarrow$

$\exists$ l.e.s. $0 \to \underset{\mathcal{F}_1(X)}{\underset{\shortparallel}{H^0(X, \mathcal{F}_1)}} \to \underset{\mathcal{F}_2(X)}{\underset{\shortparallel}{H^0(X, \mathcal{F}_2)}} \to \underset{\mathcal{F}_3(X)}{\underset{\shortparallel}{H^0(X, \mathcal{F}_3)}} \to H^1(X, \mathcal{F}_1) \to \ldots$

**Proof:** take $\mathcal{U} = \{U_i\}$ affine open cover

Fact: $X$ separated $\Rightarrow$ any $U_{i_0 \ldots i_p}$ is also affine

By Lemma, $\forall I$ $0 \to \mathcal{F}_1(U_I) \to \mathcal{F}_2(U_I) \to \mathcal{F}_3(U_I) \to 0$ exact

$\Rightarrow$ $0 \to C^*(\mathcal{F}_1) \to C^*(\mathcal{F}_2) \to C^*(\mathcal{F}_3) \to 0$ exact

$\Rightarrow$ claim follows by Homological Algebra. :)

(see C 3.1 notes)

# Cohomology of projective spaces

## Product on Čech cohomology

$(X, \mathcal{O}_X)$ ringed space $\rightsquigarrow$ well-defined map

$$H^p_{\{U_i\}}(X, \mathcal{F}) \times H^q_{\{U_i\}}(X, \mathcal{G}) \longrightarrow H^{p+q}_{\{U_i\}}(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})$$

$$(s_I) \quad , \quad (t_I) \longmapsto (s_I \otimes t_J)$$

Rem. When $\mathcal{F} = \mathcal{G} = \underline{\mathbb{Z}}$ we have $\underline{\mathbb{Z}} \otimes_{\mathcal{O}_X} \underline{\mathbb{Z}} \simeq \underline{\mathbb{Z}}$ and for $X \sim$ CW cpx this recovers product on singular cohomology.

Thm. Consider $\mathcal{O}(d)$ on $\mathbb{P}^r_k$, $d \in \mathbb{Z}$, $r \geq 1$. Then

a) $H^0(\mathbb{P}^r_k, \mathcal{O}(d)) \simeq \mathbb{Z}[x_0, \dots, x_r]_d \leftarrow$ degree $d$ homog poly (0 if $d<0$)

b) $H^i(\mathbb{P}^r_k, \mathcal{O}(d)) = 0$ for $0 < i < r$

c) $H^r(\mathbb{P}^r_k, \mathcal{O}(-r-1)) \simeq k$

d) The canonical map „Serre's duality"

$$H^0(\mathbb{P}^r_k, \mathcal{O}(d)) \times H^r(\mathbb{P}^r_k, \mathcal{O}(-d-r-1)) \to H^r(\mathbb{P}^r_k, \mathcal{O}(-r-1)) \simeq k$$

is a perfect pairing (non-degenerate bilinear) of f.g. free $k$-modules, (i.e. these $k$-vector spaces are dual to each other).

Rem. The same is true for $\mathbb{P}^n_R$ $\forall R$ instead of $k$.

Rem. $H^i(\mathbb{P}^r, \mathcal{O}(d)) = 0$ for $i > r$ because $\mathbb{P}^r$ is covered by $r+1$ open affines.

**Proof:** consider $\mathcal{F} = \bigoplus_{n\in\mathbb{Z}} \mathcal{O}(n)$ qcoh sheaf on $\mathbb{P}^r_k$.

Fact: $H^*$ commutes with $\oplus$ on a noetherian scheme, so enough to compute $H^*(\mathcal{F})$. Let $S = k[x_0, \dots, x_r]$, and let $U_i = \{x_i \neq 0\}$ – standard cover.

Claim: $\mathcal{F}(U_{i_0 \dots i_p}) = S_{x_{i_0} \cdot \ldots \cdot x_{i_p}}$ – localization at this element

and this is an isom. of graded rings, where $S$ has a natural grading: $\deg(x_{d_1}^{l_1} \cdots x_{d_m}^{l_m}) = l_1 + \dots + l_m$.

We get: $C^\cdot(\mathcal{U}, \mathcal{F})$: $\prod S_{x_{i_0}} \to \prod S_{x_{i_0} \cdot x_{i_1}} \to \dots \to S_{x_{i_0} \cdot \ldots \cdot x_{i_r}}$.

a) $H^0 = \ker d^0 \simeq S$ and this isom respects grading

c) $H^r = \operatorname{coker} d^{r-1} = \operatorname{coker}\left(\prod_i S_{x_0 \dots \hat{x}_i \dots x_r} \to S_{x_0 \dots x_r}\right)$.

$S_{x_0 \dots x_r}$ is a free $k$-module with basis $x_0^{l_0} \cdot \ldots \cdot x_r^{l_r}$, $l_i \in \mathbb{Z}$.

$\operatorname{Im}(d^{r-1})$ is the submodule generated by $\}$ s.t. $\exists i : l_i \geq 0$.

Hence $H^r(\mathbb{P}^r, \mathcal{F}) = \bigoplus k \cdot \{x_0^{l_0} \cdots x_r^{l_r} \mid l_i < 0 \ \forall i\}$

and in degree $-r-1$ the only such monomial is $(x_0 \cdot \ldots \cdot x_r)^{-1}$, hence $H^r(\mathbb{P}^r, \mathcal{O}(-r-1)) \simeq k \cdot \{\frac{1}{x_0 \cdot \ldots \cdot x_r}\}$.

d) If $d<0$, then $H^0(\mathbb{P}^r, \mathcal{O}(d)) = 0$ and also $H^r(\mathbb{P}^r, \mathcal{O}(-d-r-1)) = 0$ since $-d-r-1 > -r-1$ and so there are no „negative" monomials of that degree.

If $d \geq 0$, $H^0(\mathbb{P}^r, \mathcal{O}(d)) = \bigoplus k \cdot \{x_0^{m_0} \cdot \ldots \cdot x_r^{m_r} \mid m_i \geq 0, \sum m_i = d\}$

and that pairing is given by

$(x_0^{m_0} \cdot \ldots \cdot x_r^{m_r}) \cdot (x_0^{l_0} \cdot \ldots \cdot x_0^{l_r}) = x_0^{m_0 + l_0} \cdot \ldots \cdot x_r^{m_r + l_r}$

where $\sum l_i = -d-r-1$, and $(\ ,\ ) \mapsto 0$ if $\exists i : m_i + l_i \geq 0$.

Hence $\{x_0^{m_0} \cdot \ldots \cdot x_r^{m_r}\}$ has a dual basis $\{x_0^{-m_0-1} \cdot \ldots \cdot x_r^{-m_r-1}\}$.

b) induction on $r$ (sketch)

For $r=1$ ok. For $r>1$, use exact sequence:

$$0 \to \mathcal{O}_{\mathbb{P}^r}(-1) \xrightarrow{\cdot x_r} \mathcal{O}_{\mathbb{P}^r} \to i_* \mathcal{O}_H \to 0$$

for $H = Z(x_r)$ and $i: H \hookrightarrow \mathbb{P}^r$.

line bundle

The sequence is exact after $- \otimes \mathcal{O}(n)$,
then one takes LES on cohomology
and applies induction.
(details can be found in Hartshorne)