C2.6 Introduction to Schemes Sheet 1

Hilary 2024

(1) (A) Describe all the points of the spaces $\mathbb{A}^1_{\mathbb{R}}$ and $\mathbb{A}^1_{\mathbb{C}}$, and compute their residue fields.
What are the closed points?
What are the generic points?

(2) (A) Prove that for rings R_1, R_2,

$$\operatorname{Spec}(R_1 \times R_2) \cong \operatorname{Spec} R_1 \sqcup \operatorname{Spec} R_2.$$

(3) (B) (a) Prove the following Proposition from the lectures:

Proposition. *1)* $\mathfrak{p} \subset R$ *prime* $\implies$ $\overline{\{\mathfrak{p}\}} = Z(\mathfrak{p})$, *and* $\{\mathfrak{p}\}$ *is the only generic point of* $Z(\mathfrak{p})$.
2) A closed subset $Z \subset \operatorname{Spec} R$ *is irreducible*[1] *if and only if* $Z = Z(\mathfrak{p})$ *for some* $\mathfrak{p}$.
3) $\operatorname{Spec} R$ *is irreducible if and only if the nilradical* $\operatorname{Nil} R := \sqrt{(0)}$ *is prime.*

(b) Hence, if R is an integral domain, then $\operatorname{Spec}(R)$ is irreducible. Is the converse true?

(4) (B) (a) Prove the following Proposition from the lectures:

Proposition. *There is a contravariant functor*

$$\begin{aligned} \operatorname{Spec} : \mathsf{Ring}^{op} &\to \mathsf{Top} \\ R &\mapsto \operatorname{Spec} R \\ (\varphi : R \to S) &\mapsto \begin{pmatrix} \varphi^* : \operatorname{Spec} S & \to \operatorname{Spec} R \\ \mathfrak{p} & \mapsto \varphi^{-1}\mathfrak{p} \end{pmatrix}. \end{aligned}$$

In particular, show that φ^* is a continuous map of topological spaces. (Hint: show that the preimage of a distinguished open set is a distinguished open set).

(b) Is the image of a closed set under φ^* always closed? If not, what can we say about its closure?

(5) (B) Prove the following Proposition from the lectures:

Proposition. *Let* $\varphi : R \to S$ *be a ring homomorphism, with* $\Phi := \varphi^* : \operatorname{Spec} S \to \operatorname{Spec} R$.

1) If φ *is surjective, then*

$$\Phi : \operatorname{Spec} S \xrightarrow{\sim} Z(\operatorname{Ker} \varphi) \subseteq \operatorname{Spec} R.$$

where the first arrow is a homeomorphism.

[1] Not a union of two closed proper subsets.

2) If φ is injective, then $\Phi(\operatorname{Spec} S) \subseteq \operatorname{Spec} R$ is dense.
Moreover, $\operatorname{Im}\Phi$ is dense if and only if $\operatorname{Ker}\varphi \subseteq \operatorname{Nil} R$.

(6) (B) Let X be a topological space and let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism in $\mathsf{Ab}(X)$, the category of sheaves of abelian groups on X.

1) Prove that
$$(\operatorname{Ker}\varphi)_x \cong \operatorname{Ker}(\varphi_x) \quad \text{and} \quad (\operatorname{Im}\varphi)_x \cong \operatorname{Im}(\varphi_x),$$
for all $x \in X$.

2) Prove that φ is injective (resp. surjective), if and only if φ_x is injective (resp. surjective) for all $x \in X$.

3) Deduce the following Corollary:

Corollary. *A sequence $\mathcal{F} \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\psi} \mathcal{H}$ in $\mathsf{Ab}(X)$ is exact*[2] *if and only if $\mathcal{F}_x \to \mathcal{G}_x \to \mathcal{H}_x$ is exact for all $x \in X$.*

(7) (C) Let X be a topological space and let $\mathcal{F}$ be a presheaf of sets on X. For each open subset $U \subseteq X$, we define

$$\mathcal{F}^+(U) := \left\{ s = (s_x)_x \in \prod_{x \in U} \mathcal{F}_x : \text{“locally } s \text{ is a section of } \mathcal{F}\text{”} \right\},$$

where “locally s is a section of $\mathcal{F}$” means that, for all $x \in U$, there exists an open neighbourhood $x \in V \subseteq U$, and a section $t \in \mathcal{F}(V)$, such that for all $y \in V$ we have $s_y = t_y$ in $\mathcal{F}_y$.

1) Briefly explain why $\mathcal{F}^+$ is equipped with natural restriction morphisms making it into a presheaf, and why there is a canonical morphism of presheaves $\mathcal{F} \to \mathcal{F}^+$.

2) Prove that $\mathcal{F}^+$ is a sheaf on X and that $\mathcal{F}_x = \mathcal{F}^+_x$ for all $x \in X$.

(This in fact defines a functor $\mathcal{F} \mapsto \mathcal{F}^+$, called *sheafification*, which is left adjoint to the inclusion of the full subcategory $\mathsf{Sh}(X) \subseteq \mathsf{PSh}(X)$).

[2] i.e., $\operatorname{Im}\varphi = \operatorname{Ker}\psi$.