C2.6 Introduction to Schemes Sheet 1

Hilary 2024

(1) (A) Describe all the points of the spaces $\mathbb{A}^1_{\mathbb{R}}$ and $\mathbb{A}^1_{\mathbb{C}}$, and compute their residue fields.
What are the closed points?
What are the generic points?

Solution. Since $\mathbb{R}[x], \mathbb{C}[x]$ are both UFDs of Krull dimension 1, the unique generic point in both cases is $\eta = (0)$ and the closed points are in bijection with nonzero monic irreducibles.

Hence for $\mathbb{A}^1_{\mathbb{C}}$, the closed points are $\{(x-a) : a \in \mathbb{C}\}$, and for $\mathbb{A}^1_{\mathbb{R}}$ they are

$$\{(x-a) : a \in \mathbb{R}\} \cup \{(x-b)(x-\bar{b}) : b \in \mathbb{C} \setminus \mathbb{R}\}.$$

For $\mathbb{A}^1_{\mathbb{C}}$ one has residue fields $\kappa((x-a)) = \mathbb{C}[x]_{(x-a)}/(x-a) \cong \mathbb{C}$ and $\kappa(\eta) = \mathbb{C}(x)$.

For $\mathbb{A}^1_{\mathbb{R}}$ one has residue fields $\kappa((x-a)) = \mathbb{R}[x]_{(x-a)}/(x-a) \cong \mathbb{R}$, $\kappa(\eta) \cong \mathbb{R}(x)$, and for $\mathfrak{p} = ((x-b)(x-\bar{b}))$ one has

$$\kappa(\mathfrak{p}) \cong \mathbb{R}[x]_{\mathfrak{p}}/((x-b)(x-\bar{b})) \cong \mathbb{C}.$$

(2) (A) Prove that for rings R_1, R_2,

$$\operatorname{Spec}(R_1 \times R_2) \cong \operatorname{Spec} R_1 \sqcup \operatorname{Spec} R_2.$$

Solution. Consider the elements $e_1 = (1_{R_1}, 0)$ and $e_2 = (0, 1_{R_2}) \in R_1 \times R_2$. Since $e_1 + e_2 = 1_{R_1 \times R_2}$ and $e_1 e_2 = 0$ one has $\operatorname{Spec}(R_1 \times R_2) \cong Z(e_2) \sqcup Z(e_1)$. Now one has an isomorphism

$$R_1 \cong (R_1 \times R_2)/e_2,$$

and there is a canonical continuous bijection

$$\operatorname{Spec}((R_1 \times R_2)/e_1) \cong \{\mathfrak{p} \in \operatorname{Spec}(R_1 \times R_2) : \mathfrak{p} \ni e_2\} = Z(e_2).$$

Hence $\operatorname{Spec}(R_1 \times R_2) \cong \operatorname{Spec} R_1 \sqcup \operatorname{Spec} R_2$.

(3) (B) (a) Prove the following Proposition from the lectures:

Proposition. *1)* $\mathfrak{p} \subset R$ *prime* $\implies \overline{\{\mathfrak{p}\}} = Z(\mathfrak{p})$, *and* $\{\mathfrak{p}\}$ *is the only generic point of* $Z(\mathfrak{p})$.

2) A closed subset $Z \subset \operatorname{Spec} R$ *is irreducible*[1] *if and only if* $Z = Z(\mathfrak{p})$ *for some* $\mathfrak{p}$.

3) $\operatorname{Spec} R$ *is irreducible if and only if the nilradical* $\operatorname{Nil} R := \sqrt{(0)}$ *is prime.*

(b) Hence, if R is an integral domain, then $\operatorname{Spec}(R)$ is irreducible. Is the converse true?

[1] Not a union of two closed proper subsets.

Solution (a). 1) It is clear that $\overline{\{\mathfrak{p}\}} \subseteq Z(\mathfrak{p})$. Conversely if $\mathfrak{p} \in\subseteq Z(I)$ then $Z(\mathfrak{p}) \subseteq Z(I)$. So $Z(\mathfrak{p}) \subseteq \overline{\{\mathfrak{p}\}}$. For the last part, if $Z(\mathfrak{p}) = Z(\mathfrak{q})$ then $\mathfrak{q} \subseteq \mathfrak{p}$ and $\mathfrak{p} \subseteq \mathfrak{q}$, since $Z(I) \subseteq Z(J)$ if and only if $\sqrt{J} \subseteq \sqrt{I}$.

2) For the "if" part , note that $Z(\mathfrak{p}) = \overline{\{\mathfrak{p}\}}$ is irreducible because $\{\mathfrak{p}\}$ is. For the "only if" part, let $Z(I) \subset \operatorname{Spec} R$ be a proper closed subset. One has $f, g \notin \sqrt{I}$ if and only if $D_f \cap Z(I), D_g \cap Z(I) \neq \emptyset$. Hence by the irreducibility $D_{fg} \cap Z(I) \neq \emptyset$ and so $fg \notin \sqrt{I}$. So $\sqrt{I}$ is prime and hence, using the formula $\sqrt{I} = \bigcap_{\mathfrak{p} \supset I} \mathfrak{p}$, we deduce that I is prime and $Z(I)$ is irreducible.

3) This follows from 2) since $\operatorname{Spec} R = Z(\operatorname{Nil} R)$.

(b) This is not true, e.g., consider $R = k[x]/(x^2)$ - one has $\operatorname{Spec} R = \mathrm{pt}$ is irreducible, but R is not a domain.

(4) (B) (a) Prove the following Proposition from the lectures:

Proposition. *There is a contravariant functor*

$$\begin{aligned} \operatorname{Spec} : \mathsf{Ring}^{op} &\to \mathsf{Top} \\ R &\mapsto \operatorname{Spec} R \\ (\varphi : R \to S) &\mapsto \begin{pmatrix} \varphi^* : \operatorname{Spec} S & \to \operatorname{Spec} R \\ \mathfrak{p} & \mapsto \varphi^{-1}\mathfrak{p} \end{pmatrix}. \end{aligned}$$

In particular, show that φ^* is a continuous map of topological spaces. (Hint: show that the preimage of a distinguished open set is a distinguished open set).

(b) Is the image of a closed set under φ^* always closed? If not, what can we say about its closure?

Solution. (a) If $\mathfrak{p} \in \operatorname{Spec} S$ is prime, then $\varphi^{-1}\mathfrak{p}$ is also prime: one has $ab \in \varphi^{-1}\mathfrak{p} \iff \varphi(ab) = \varphi(a)\varphi(b) \in \mathfrak{p} \iff \varphi(a) \in \mathfrak{p}$ or $\varphi(b) \in \mathfrak{p} \iff a \in \varphi^{-1}\mathfrak{p}$ or $b \in \varphi^{-1}\mathfrak{p}$. Moreover one has $(\varphi\psi)^* = \psi^*\varphi^*$ since $\psi^{-1}\varphi^{-1}\mathfrak{p} = (\varphi\psi)^{-1}\mathfrak{p}$. Clearly $\mathrm{id}^* = \mathrm{id}$. Hence, the only thing to show is continuity. Setting $\Phi := \varphi^*$, one has, for $f \in \operatorname{Spec} R$:

$$\begin{aligned} \Phi^{-1}(Z(f)) &= \{\mathfrak{p} \in \operatorname{Spec} S : \varphi^{-1}\mathfrak{p} \in Z(f)\} \\ &= \{\mathfrak{p} \in \operatorname{Spec} S : f \in \varphi^{-1}\mathfrak{p}\} \\ &= \{\mathfrak{p} \in \operatorname{Spec} S : \varphi(f) \in \mathfrak{p}\} = Z(\varphi(f)). \end{aligned}$$

Hence $\Phi^{-1}(D_f) = D_{\varphi(f)}$. Since the basic opens form a basis for the topology, the result follows.

(b) The image of a closed subset need not be closed. For instance one can consider the inclusion of the generic point $\eta : \operatorname{Spec} k(x) \to \operatorname{Spec} k[x]$.

We claim that for any closed subset $Z(I) \subseteq \operatorname{Spec} S$, one has

$$\overline{\Phi(Z(I))} = Z(\varphi^{-1}(I)).$$

Indeed, for any $f \in S$ one has $\Phi(V(I)) \subseteq Z(f) \iff V(I) \subseteq \Phi^{-1}(Z(f)) = Z(\varphi(f))$, by the previous part; which holds if and only if $\sqrt{(\varphi(f))} \subseteq \sqrt{I}$. In turn, this holds if and only if there exists n such that $\varphi(f)^n = \varphi(f^n) \in I \iff f^n \in \varphi^{-1}(I) \iff Z(\varphi^{-1}(I)) \subseteq Z(f^n) = Z(f)$. Since subsets of the form $Z(f)$ are a basis for the closed subsets, one has $\overline{\Phi(Z(I))} = Z(\varphi^{-1}(I))$.

(5) (B) Prove the following Proposition from the lectures:

Proposition. *Let* $\varphi : R \to S$ *be a ring homomorphism, with* $\Phi := \varphi^* : \operatorname{Spec} S \to \operatorname{Spec} R$.

1) If φ *is surjective, then*

$$\Phi : \operatorname{Spec} S \xrightarrow{\sim} Z(\operatorname{Ker}\varphi) \subseteq \operatorname{Spec} R.$$

where the first arrow is a homeomorphism.

2) If φ *is injective, then* $\Phi(\operatorname{Spec} S) \subseteq \operatorname{Spec} R$ *is dense.*
Moreover, $\operatorname{Im}\Phi$ *is dense if and only if* $\operatorname{Ker}\varphi \subseteq \operatorname{Nil} R$.

Solution. 1) If φ is surjective then $S \cong R/\operatorname{Ker}\varphi$. The canonical bijection between sets of ideals

$$\{\overline{J} \subset R/\operatorname{Ker}\varphi\} \leftrightarrow \{J \subset R : J \supset \operatorname{Ker}\varphi\},$$

respects inclusions and sends prime ideals to prime ideals. In particular, it induces a homeomorphism $\operatorname{Spec}(R/\operatorname{Ker}\varphi) \cong Z(\operatorname{Ker}\varphi)$.

2) Note that $\operatorname{Spec} S = Z(\{0\})$. Hence, using the previous question, one has

$$\overline{\Phi(\operatorname{Spec} S)} = Z(\varphi^{-1}(0)) = Z(\{0\}) = \operatorname{Spec} R,$$

since φ is injective.

For the second part we have

$$\begin{aligned} \overline{\Phi(\operatorname{Spec} S)} = \operatorname{Spec} R &\iff \overline{\Phi(Z(\{0\}))} = \operatorname{Spec} R \\ &\iff Z(\operatorname{Ker}\varphi) = Z(\{0\}) \\ &\iff \operatorname{Nil}(R) = \sqrt{\operatorname{Ker}\varphi} \\ &\iff \operatorname{Nil}(R) \supseteq \operatorname{Ker}\varphi, \end{aligned}$$

since the inclusion $\operatorname{Nil}(R) \subseteq \sqrt{\operatorname{Ker}\varphi}$ always holds, and $\operatorname{Nil} R$ is radical.

(6) (B) Let X be a topological space and let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism in $\mathsf{Ab}(X)$, the category of sheaves of abelian groups on X.

1) Prove that

$$(\operatorname{Ker}\varphi)_x \cong \operatorname{Ker}(\varphi_x) \quad \text{and} \quad (\operatorname{Im}\varphi)_x \cong \operatorname{Im}(\varphi_x),$$

for all $x \in X$.

2) Prove that φ is injective (resp. surjective), if and only if φ_x is injective (resp. surjective) for all $x \in X$.

3) Deduce the following Corollary:

Corollary. *A sequence* $\mathcal{F} \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\psi} \mathcal{H}$ *in* $\mathsf{Ab}(X)$ *is exact*[2] *if and only if* $\mathcal{F}_x \to \mathcal{G}_x \to \mathcal{H}_x$ *is exact for all* $x \in X$.

[2] i.e., $\operatorname{Im}\varphi = \operatorname{Ker}\psi$.

Solution. Before starting, we prove a useful Lemma:

Lemma. *If $\mathcal{F}, \mathcal{G} \subseteq \mathcal{H}$ are subsheaves, then $\mathcal{F} = \mathcal{G}$ iff $\mathcal{F}_x = \mathcal{G}_x$, for all $x \in X$.*

Proof. By considering the sheaf $\mathcal{F} + \mathcal{G} \subseteq \mathcal{H}$ we reduce to the case when $\mathcal{F}$ is a subsheaf of $\mathcal{G}$. Let $U \subseteq X$ be an open subset. All we need to show is that the inclusion $\mathcal{F}(U) \subseteq \mathcal{G}(U)$, is surjective.

Let $t \in \mathcal{G}(U)$ and $x \in U$. Then (by assumption) there exists $s_x \in \mathcal{F}_x$ with $s_x = t_x$. Say that s_x is represented by s^x on a neigbourhood V_x of x. Then s^x and $t|_{V_x}$ are two elements of V_x whose germs at x are the same. Thus replacing V_x by a smaller neighbourhood of x. (if necessary), we may assume that $s^x = t|_{V_x}$ in $\mathcal{G}(V_x)$. Now U is covered by the open sets V_x, and on each V_x we have a section $s^x \in \mathcal{F}(V_x)$. If x, y are two points, then $s^x|_{V_x \cap V_y} = t|_{V_x \cap V_y} = s^y|_{V_y \cap V_x}$, so by the sheaf property, there exists $s \in \mathcal{F}(U)$ with $s|_{V_x} = s^x$, for each x. Finally, $s = t$, since their restrictions to the V_x agree, by the sheaf property again. □

1) It is obvious that $(\ker \varphi)_x = \varinjlim_{U \ni x} \ker \varphi_U \subseteq \ker(\varphi_x)$. We now show that $(\ker \varphi)_x \supseteq \ker(\varphi_x)$. Let $s_x \in \ker(\varphi_x)$. Then there exists an open $U \ni x$ and a section $s \in \mathcal{F}(U)$ with $s|_x = s_x$ and $\varphi_U(s)|_x = 0$. Then there exists an open $V \ni x$, $U \subseteq V$, with $\varphi_U(s)|_V = 0$. So $\varphi_V(s|_V) = 0$, (as φ is a morphism of sheaves), so $s|_V \in \ker \varphi_V = (\ker \varphi)(V)$.

It is obvious that $(\operatorname{im} \varphi)_x \subseteq \operatorname{im}(\varphi_x)$. The reason for this, is that the sheafification functor preserves stalks, and so $(\operatorname{im} \varphi)_x = \varinjlim_{U \ni x} \varphi(U)$, which is clearly contained in $\operatorname{im}(\varphi_x)$. For the converse, let $\varphi_x(s_x) \in \operatorname{im}(\varphi_x)$. Then there exists an open $U \ni x$ and $s \in \mathcal{F}(U)$ with $s|_x = s_x$. Then $\varphi_U(s)|_x = \varphi_x(s_x)$, so $\varphi_x(s_x) \in (\operatorname{im} \varphi)_x$.

2) Using the Lemma, φ is injective iff $\ker \varphi = 0$ iff $(\ker \varphi)_x = \ker(\varphi_x) = 0$ (for all x).

φ is surjective iff $\operatorname{im} \varphi = \mathcal{G}$, which is true iff $(\operatorname{im} \varphi)_x = \mathcal{G}_x$ for all x, (by the Lemma), i.e., $\operatorname{im}(\varphi_x) = \mathcal{G}_x$ for all x, by 1).

3) The sequence is exact iff $\operatorname{im} \varphi = \ker \psi$. Since these are both subsheaves of $\mathcal{G}$, by the Lemma, this holds if and only if $(\operatorname{im} \varphi)_x = (\ker \psi)_x$, for all $x \in X$, i.e., if and only if $\operatorname{im}(\varphi_x) = \ker(\psi_x)$, for all $x \in X$, i.e. if and only if the sequence is exact on stalks.

(7) (C) Let X be a topological space and let $\mathcal{F}$ be a presheaf of sets on X. For each open subset $U \subseteq X$, we define

$$\mathcal{F}^+(U) := \left\{ s = (s_x)_x \in \prod_{x \in U} \mathcal{F}_x : \text{“locally } s \text{ is a section of } \mathcal{F}\text{”} \right\},$$

where “locally s is a section of $\mathcal{F}$” means that, for all $x \in U$, there exists an open neighbourhood $x \in V \subseteq U$, and a section $t \in \mathcal{F}(V)$, such that for all $y \in V$ we have $s_y = t_y$ in $\mathcal{F}_y$.

1) Briefly explain why $\mathcal{F}^+$ is equipped with natural restriction morphisms making it into a presheaf, and why there is a canonical morphism of presheaves $\mathcal{F} \to \mathcal{F}^+$.
2) Prove that $\mathcal{F}^+$ is a sheaf on X and that $\mathcal{F}_x = \mathcal{F}^+_x$ for all $x \in X$.

(This in fact defines a functor $\mathcal{F} \mapsto \mathcal{F}^+$, called *sheafification*, which is left adjoint to the inclusion of the full subcategory $\mathsf{Sh}(X) \subseteq \mathsf{PSh}(X)$).

Solution. 1) I have just copied this from [Sta18, Tag 007X]. Note that the condition "locally s is a section of $\mathcal{F}$" is a condition for each $x \in U$, and that given $x \in U$ the truth value of this condition only depends on the values s_y for y in any open neighbourhood of x. Thus, it is clear that, if $V \subseteq U \subseteq X$ are open, the projection maps $\prod_{x \in U} \mathcal{F}_x \to \prod_{y \in V} \mathcal{F}_y$, map elements of $\mathcal{F}^+(U)$ into elements of $\mathcal{F}^+(V)$. Hence, $\mathcal{F}^+$ is a presheaf. The morphism $\mathcal{F}(U) \to \prod_{x \in U} \mathcal{F}_x$, sending a section to the collection of its stalks, clearly has image in $\mathcal{F}^+(U)$, and if $V \subseteq U \subseteq X$ are opens then the diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \longrightarrow & \mathcal{F}^+(U) \\ \downarrow & & \downarrow \\ \mathcal{F}(V) & \longrightarrow & \mathcal{F}^+(V) \end{array}$$

commutes, and so we obtain a morphism of presheaves $\mathcal{F} \to \mathcal{F}^+$.

2) This is obvious from the definitions! If you need convincing read [Sta18, Tag 007Z], or read something about the espace étalé of a presheaf.

References

[Sta18] The Stacks Project Authors. *Stacks Project*. `https://stacks.math.columbia.edu`, 2018.