C2.6 Introduction to Schemes Sheet 2

Hilary 2024

(1) (A) Let $(X, \mathcal{O}_X)$ be a locally ringed space, with $U \subseteq X$ an open subset, $s \in \mathcal{O}_X(U)$ a section. Show:

(a) $\{x \in U : s_x = 0 \in \mathcal{O}_{X,x}\} \subseteq U$ is open;

(b) $\{x \in U : s(x) = 0 \in \kappa(x)\} \subseteq U$ is closed.

Solution. (a): If $s_x = 0$ then (by definition of the stalk), there is $V \subseteq U$ open, with $x \in V$ and $s|_V = 0$. Then for all $y \in V$, $s_y = 0$.

(b): If $s(x) \neq 0$ then s_x is a unit in $\mathcal{O}_{X,x}$, so there is t_x with $s_x t_x = 1 \in \mathcal{O}_{X,x}$. There is then $V \subseteq U$ open, with $x \in V$ and an element $t \in \mathcal{O}_X(V)$, with $t|_x = t$. Possibly shrinking V further we have $st = 1$ in $\mathcal{O}_X(V)$. Then for all $y \in V, s_y t_y = 1$, so $s(y) \neq 0$.

(2) (B) Fill in the details of the proof of the following theorem (see lecture notes).

Theorem. *For all locally ringed spaces X and rings R, there is a natural bijection*

$$\mathrm{Maps}_{\mathsf{LocallyRingedSpaces}}(X, \operatorname{Spec} R) \cong \mathrm{Maps}_{\mathsf{Ring}}(R, \mathcal{O}_X(X)).$$

Solution. There is an obvious natural map

$$\begin{aligned} \mathrm{Maps}_{\mathsf{LocallyRingedSpaces}}(X, \operatorname{Spec} R) &\to \mathrm{Maps}_{\mathsf{Ring}}(R, \mathcal{O}_X(X)), \\ g &\mapsto g^{\#}_{\operatorname{Spec} R}. \end{aligned} \tag{1}$$

We will construct an inverse to this map. Before starting, we prove a useful Lemma. Given any scheme $(X, \mathcal{O}_X)$ and any $f \in \mathcal{O}_X(X)$ we define the subset

$$D_f := \{x \in X : f(x) \neq 0 \in \kappa(x)\};$$

by Exercise 1, this is an open subset of X.

Lemma. *$f|_{D_f} \in \mathcal{O}_X(D_f)$ is invertible.*

Proof of Lemma. The solution to Exercise 1(b) shows that f is invertible locally on D_f. If open subsets U, V and sections g, h are given with $f \cdot h = 1$ in $\mathcal{O}_X(U)$ and $f \cdot g = 1$ in $\mathcal{O}_X(V)$, then $h = g$ in $\mathcal{O}_X(U \cap V)$: this follows from the computation $h = h(fg) = (fh)g = g$ in $\mathcal{O}_X(U \cap V)$. Therefore, the sheaf property implies that the collection of local inverses glue to give an inverse to f in $\mathcal{O}_X(D_f)$. □

Set $Y = \operatorname{Spec} R$. Suppose that we are given a morphism $R = \mathcal{O}_Y(Y) \xrightarrow{\varphi} \mathcal{O}_X(X)$. By postcomposing with the canonical morphism $\mathcal{O}_X(X) \to \mathcal{O}_{X,x}$ we obtain a morphism $\mathcal{O}_Y(Y) \to \mathcal{O}_{X,x}$. Taking the preimage of the maximal ideal $\mathfrak{m}_x \subseteq \mathcal{O}_{X,x}$ gives a point $y \in \operatorname{Spec} R = Y$. Hence we obtain a morphism of sets $g : X \to Y$ by setting $g(x) = y$.

Now we check that g is continuous. By construction we have a commutative diagram of rings

$$\begin{array}{ccc} \kappa(g(x)) & \hookrightarrow & \kappa(x) \\ \uparrow & & \uparrow \\ \mathcal{O}_Y(Y) & \xrightarrow{\varphi} & \mathcal{O}_X(X) \end{array}$$

in which the vertical arrows are the canonical morphisms. In particular, given $f \in R = \mathcal{O}_Y(Y)$, we see, by chasing this diagram, that

$$\begin{aligned} g^{-1}(D_f) &= \{x \in X : f(g(x)) \neq 0 \in \kappa(g(x))\} \\ &= \{x \in X : (\varphi f)(x) \neq 0 \in \kappa(x)\} = D_{\varphi f}, \end{aligned} \tag{2}$$

is open by Exercise 1.

Now, we construct the morphism of structure sheaves. Consider the chain

$$\mathcal{O}_Y(D_f) = R_f \xrightarrow{\varphi_f} \mathcal{O}_X(X)_{\varphi f} \to \mathcal{O}_X(D_{\varphi f}) = \mathcal{O}_X(g^{-1}D_f) = g_*\mathcal{O}_X(D_f). \tag{3}$$

Here the first morphism φ_f is obtained by localising φ, the second morphism is obtained via the universal property of localization (using the Lemma), and the third is by the computation above. It is not hard to see that this composite is compatible with restrictions. Hence, we have constructed a morphism of schemes $g : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$. By taking inverse limits over the basic opens in (3), we see that $g_Y^{\#} = \varphi$.

On the other hand, suppose that we are given $h : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ and set $\varphi = h_Y^{\#}$. By the above construction we obtain a morphism of locally ringed spaces $g : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$. We claim that $g = h$. Since h is a morphism of locally ringed spaces then $h_x^{\#} : \mathcal{O}_{Y,h(x)} \to \mathcal{O}_{X,x}$ is local. On the other hand, this is compatible with the morphism on global sections and so we are forced to have $h(x) = \varphi^{-1}(\mathfrak{m}_x)$. Therefore the map $h = g$ on topological spaces. Given this, the same considerations show that $h_x^{\#} = g_x^{\#}$ for all x. Since morphisms of sheaves are determined by their values on stalks we conclude.

(3) (B) Let X be a topological space. Suppose that we are given an open cover $\{U_\alpha\}_\alpha$ together with sheaves $\mathcal{F}_\alpha$ on each U_α and isomorphisms

$$\varphi_{\alpha\beta} : \mathcal{F}_\alpha|_{U_\alpha \cap U_\beta} \xrightarrow{\sim} \mathcal{F}_\beta|_{U_\beta \cap U_\alpha},$$

satisfying the *cocycle condition* $\varphi_{\beta\gamma} \circ \varphi_{\alpha\beta} = \varphi_{\alpha\gamma}$ on $U_\alpha \cap U_\beta \cap U_\gamma$. Show that the $\{\mathcal{F}_\alpha\}_\alpha$ glue to a sheaf $\mathcal{F}$ on X.

Solution. Let $W \subseteq X$ be an open subset. We define the values of the glued sheaf to be

$$\mathcal{F}(W) := \left\{(s_\alpha)_\alpha : s_\alpha \in \mathcal{F}_\alpha(W \cap U_\alpha), \varphi_{\alpha\beta}(s_\alpha|_{W\cap U_\alpha \cap U_\beta}) = s_\beta|_{W \cap U_\beta \cap U_\alpha}\right\}, \tag{4}$$

here the restriction mappings for $\mathcal{F}$ are inherited from those of each $\mathcal{F}_\alpha$; the sheaf condition for $\mathcal{F}$ follows from the sheaf condition for each $\mathcal{F}_\alpha$.

It remains to show that the projection $\mathcal{F}|_{U_\alpha} \xrightarrow{\sim} \mathcal{F}_\alpha$ is an isomorphism. Let $W \subseteq U_\alpha$. Then the condition in the definition of $\mathcal{F}(W)$ implies that $s_\beta = \varphi_{\alpha\beta}(s_\alpha|_{W\cap U_\alpha})$. Therefore the projection $\mathcal{F}|_{U_\alpha} \to \mathcal{F}_\alpha$ is injective. Given any section $s \in \mathcal{F}_\alpha(W)$, the cocycle condition implies that, setting $s_\alpha := s$ and $s_\beta := \varphi_{\alpha\beta}(s_\alpha|_{W\cap U_\beta})$, we obtain a collection satisfying the condition in the definition of $\mathcal{F}(W)$. Therefore the projection $\mathcal{F}|_{U_\alpha} \to \mathcal{F}_\alpha$ is surjective.

(4) (B) Use exercise 3 to show that, given schemes $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ together with open subsets $U \subseteq X$, $V \subseteq Y$, and an isomorphism

$$(U, \mathcal{O}_X|_U) \xrightarrow{\sim} (V, \mathcal{O}_Y|_V),$$

one can perform gluing to obtain a scheme whose underlying topological space is $(X \sqcup Y)/(U \sim V)$, and whose structure sheaf is the glued structure sheaf.

Solution. We endow $Z := (X \sqcup Y)/(U \sim V)$ with the quotient topology. Note that the natural maps of topological spaces $X \to Z, Y \to Z$, are injective and open. Hence, in what follows we will identify X, Y with open subspaces of Z.

By Exercise 3, we obtain a glued sheaf $\mathcal{O}_Z$ on $Z := (X \sqcup Y)/(U \sim V)$ whose sections over an open $W \subseteq Z$ are given by the set-theoretic equalizer

$$\begin{aligned}\mathcal{O}_Z(W) := \operatorname{eq}(&\mathcal{O}_X(W\cap X) \times \mathcal{O}_Y(W \cap Y)\\ &\rightrightarrows \mathcal{O}_X(W\cap X) \times \mathcal{O}_X(W\cap X \cap Y) \times \mathcal{O}_Y(W \cap Y \cap X) \times \mathcal{O}_Y(W\cap Y)),\end{aligned}$$

Now the set-theoretic equalizer naturally carries the structure of a ring (this is the ring-theoretic equalizer). Hence $(Z, \mathcal{O}_Z)$ is a ringed space. Then, it is easy to see that the projection $\mathcal{O}_Z|_X \xrightarrow{\sim} \mathcal{O}_X$ defined as in Exercise 3, is a morphism of sheaves of rings. This consideration shows that $(X, \mathcal{O}_X)$ is isomorphic to the open subscheme of $(Z, \mathcal{O}_Z)$ on the open subset $X \subset Z$. In particular $(Z, \mathcal{O}_Z)$ is locally isomorphic to a scheme and therefore a scheme.

(5) (B) Prove that the following schemes are not affine:

(a) $\mathbb{A}^2_{\mathbb{Z}} \setminus \{(0,0)\}$, viewed as an open subscheme of $\mathbb{A}^2_{\mathbb{Z}}$;

(b) The projective line: glue $\mathbb{A}^1_{\mathbb{Z}}$ and $\mathbb{A}^1_{\mathbb{Z}}$ by identifiying the open subsets $\mathbb{A}^1_{\mathbb{Z}} \setminus \{0\} = \operatorname{Spec}\mathbb{Z}[t, t^{-1}]$ and $\mathbb{A}^1_{\mathbb{Z}} \setminus \{0\} = \operatorname{Spec}\mathbb{Z}[u, u^{-1}]$ via the isomorphism induced by $t \leftrightarrow u^{-1}$.

(c) The line with two origins: glue $\mathbb{A}^1_{\mathbb{Z}}$ and $\mathbb{A}^1_{\mathbb{Z}}$ by identifiying the open subsets $\mathbb{A}^1_{\mathbb{Z}} \setminus \{0\} = \operatorname{Spec}\mathbb{Z}[t, t^{-1}]$ and $\mathbb{A}^1_{\mathbb{Z}} \setminus \{0\} = \operatorname{Spec}\mathbb{Z}[u, u^{-1}]$ via the isomorphism induced by $t \leftrightarrow u$.

Solution. (a) Suppose that $U = \mathbb{A}^2_{\mathbb{Z}} \setminus \{(0,0)\}$ is affine. Say $\mathbb{A}^2_{\mathbb{Z}} = \operatorname{Spec}\mathbb{Z}[x, y]$. By the sheaf property applied to the covering $U = D_x \cup D_y$ one has

$$\mathcal{O}_U(U) = \operatorname{eq}(\mathbb{Z}[x, y, 1/x] \times \mathbb{Z}[x, y, 1/y] \rightrightarrows \mathbb{Z}[x, y, 1/x, 1/y]) \cong \mathbb{Z}[x, y]. \tag{5}$$

To be more precise, this calculation shows that the inclusion $\iota : U \to \mathbb{A}^2_{\mathbb{Z}}$ induces an isomorphism on global sections: $\mathcal{O}_{\mathbb{A}^2_{\mathbb{Z}}}(\mathbb{A}^2_{\mathbb{Z}}) \xrightarrow{\sim} (\iota_*\mathcal{O}_U)(\mathbb{A}^2_{\mathbb{Z}})$. If U were affine, by the anti-equivalence between affine schemes and rings this would imply that the inclusion $U \to \mathbb{A}^2_{\mathbb{Z}}$ is an isomorphism of schemes, in particular that $U \to \mathbb{A}^2_{\mathbb{Z}}$ is an isomorphism of topological spaces, which is evidently false since it is not surjective.

(b) By the gluing construction of Exercise 4 (and after minor simplification), we can calculate

$$\mathcal{O}_{\mathbb{P}^1_{\mathbb{Z}}}(\mathbb{P}^1_{\mathbb{Z}}) = \mathrm{eq}(\mathbb{Z}[t] \times \mathbb{Z}[u] \rightrightarrows \mathbb{Z}[t, t^{-1}]) \cong \mathbb{Z}, \tag{6}$$

where the two maps are $(f(t), g(u)) \mapsto f(t)$ and $(f(t), g(u)) \mapsto g(t^{-1})$. Hence, if $\mathbb{P}^1_{\mathbb{Z}}$ was affine, then we would have $\mathbb{P}^1_{\mathbb{Z}} \cong \operatorname{Spec} \mathbb{Z}$, by the anti-equivalence between affine schemes and rings. In particular one would have an isomorphism of the underlying topological spaces. However, this is clearly false by comparing their Krull dimensions: $\operatorname{Spec} \mathbb{Z}$ has Krull dimension 1 whereas $\mathbb{P}^1_{\mathbb{Z}}$ has Krull dimension 2.

(c) Let us denote by X the line with two origins. By the gluing construction of Exercise 4 (and after minor simplification), we can calculate

$$\mathcal{O}_X(X) = \mathrm{eq}(\mathbb{Z}[t] \times \mathbb{Z}[u] \rightrightarrows \mathbb{Z}[t, t^{-1}]) \cong \mathbb{Z}[t], \tag{7}$$

where the two maps are $(f(t), g(u)) \mapsto f(t)$ and $(f(t), g(u)) \mapsto g(t)$. More precisely this shows that the inclusion $\mathbb{A}^1_{\mathbb{Z}} \to X$ induces an isomorphism on global sections. If X were affine, by the anti-equivalence between affine schemes and rings this would imply that the inclusion $\mathbb{A}^1_{\mathbb{Z}} \to X$ into the first copy, is an isomorphism of schemes, in particular that $\mathbb{A}^1_{\mathbb{Z}} \to X$ is an isomorphism of topological spaces, which is evidently false since it is not surjective.

(6) (B) Let k be a field with $\operatorname{char} k \neq 2$. Show that $\operatorname{Spec} k[x, y]/(y^2 - x^2 - x^3)$ is an integral scheme. Show that its preimage in $\operatorname{Spec} k[\![x, y]\!]/(y^2 - x^2 - x^3)$ is reducible. Briefly discuss the geometric intuition.

Solution. Note that $y^2 - x^2 - x^3$ is irreducible in the UFD $k[x, y] = k[x][y]$ since $x^2 + x^3$ is not a square in $k[x]$. Therefore $k[x, y]/(y^2 - x^2 - x^3)$ is an integral domain and $\operatorname{Spec}(k[x, y]/(y^2 - x^2 - x^3))$ is an integral scheme.

The preimage is just $\operatorname{Spec}(k[\![x, y]\!]/(y^2 - x^2 - x^3))$. Note that $y^2 - x^2 - x^3$ factors as $(y - x\sqrt{1 + x})(y + x\sqrt{1 + x})$ where $\sqrt{1 + x} = \sum_{k=0}^{\infty} \frac{(-1/2)_k}{k!}(-x)^k \in k[\![x]\!]$, (here $(\alpha)_k$ is the rising Pochhammer symbol), because $\frac{(-1/2)_k}{k!} \in \mathbb{Z}[1/2]$ (students should check this).

Therefore scheme is reducible: $\operatorname{Spec}(k[\![x, y]\!]/(y^2 - x^2 - x^3)) = \operatorname{Spec}(k[\![x, y]\!]/(y - x\sqrt{1 + x})) \sqcup \operatorname{Spec}(k[\![x, y]\!]/(y + x\sqrt{1 + x}))$.

Geometric intuition: The ring $k[\![x, y]\!]/(y^2 - x^2 - x^3)$ is the completed stalk of $k[x, y]/(y^2 - x^2 - x^3)$ at the closed point corresponding to 0. If we zoom in on a neighbourhood of 0 in the nodal cubic it looks like two crossing axes (and so reducible). In particular we expect $\operatorname{Spec}(k[\![x, y]\!]/(y^2 - x^2 - x^3))$ to resemble two copies of "the formal neighbourhood of 0" in $\mathbb{A}^1_k$.

(7) (B) Let R be a ring. Construct the scheme $\mathbb{P}^n_R$ by glueing $n+1$ copies of $\mathbb{A}^n_R = \operatorname{Spec} R[x_0, \ldots, x_n]$, where for the i^{th} copy, we use coordinates $y_1 = x_0/x_1, \ldots, x_n/x_i$, (omitting x_i/x_i).

Solution. Let U_i be the i^{th} copy of $\mathbb{A}^n_R$ with coordinates on it given as in the question. For $0 \leq i, j \leq n$ we define the open subset $U_{ij} := D_{x_j/x_i} \subseteq U_i$. The transition isomorphisms are given by

$$\varphi_{ij} : R[x_0/x_j, \dots, \widehat{x_j/x_j}, \dots, x_n/x_j, (x_i/x_j)^{-1}] \xrightarrow{\sim} R[x_0/x_i, \dots, \widehat{x_i/x_i}, \dots, x_n/x_i, (x_j/x_i)^{-1}]$$

where φ_{ij} is determined by $\varphi_{ij}(x_k/x_j) = (x_k/x_i) \times (x_j/x_i)^{-1}$ for $k \neq j$. In order to check that the cocycle condition is satisfied, i.e., that $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$, it suffices to note that $(x_j/x_k)^{-1} \times (x_i/x_j)^{-1} = (x_i/x_k)^{-1}$ in $\mathcal{O}(D_{x_i/x_k} \cap D_{x_j/x_k})$. Hence, using the notion of gluing data for schemes as given in [Sta18, Tag 01JA], we may glue to obtain the n-dimensional projective space $\mathbb{P}^n_R$.

References

[Sta18] The Stacks Project Authors. *Stacks Project*. `https://stacks.math.columbia.edu`, 2018.