C2.6 Introduction to Schemes Sheet 3

Hilary 2024

(1) (A) Describe the schematic fibers of $\operatorname{Spec}\mathbb{Z}[x] \to \operatorname{Spec}\mathbb{Z}$ (Try to draw a picture of it).

(2) (B) Prove the following statements:

1) $\mathbb{A}^n$ and $\mathbb{P}^n$ are separated (over $\operatorname{Spec}\mathbb{Z}$). Deduce that $\mathbb{A}^n_S$ and $\mathbb{P}^n_S$ are separated S-schemes for any S affine.
2) Open and closed embeddings of schemes are separated maps.
3) Compositions of separated maps are separated.

(3) (B) Prove that the "bug-eyed line" obtained by gluing two copies of $\mathbb{A}^1$ along $\mathbb{A}^1 \setminus \{0\}$, is not separated.

(4) (B) Prove the following criterion: A ring homomorphism $\varphi^\# : A \to B$ is flat if and only if the corresponding morphism of affine schemes $\varphi : \operatorname{Spec} B \to \operatorname{Spec} A$ is flat.

(5) (B) Show that $\operatorname{Spec}\mathbb{Z}[x, y]/(x^2 - y^2 - 5) \to \operatorname{Spec}\mathbb{Z}$ is flat.

Is $\operatorname{Spec}\mathbb{Z}[x, y]/(2x^2 - 2y^2 - 10) \to \operatorname{Spec}\mathbb{Z}$ flat?

Explain the geometric intuition behind these examples by looking at the dimensions of fibers.

(6) (B) A morphism $f : X \to S$ is called *finite* if S has an affine cover $S = \bigcup_{i\in\mathcal{I}} \operatorname{Spec} B_i$ such that, for all i, $f^{-1}(\operatorname{Spec} B_i) \simeq \operatorname{Spec} A_i$ is an affine scheme and A_i is finitely generated as a module over B_i.

a) Give some examples of finite morphisms.
b) Show that a finite morphism has finite fibers. Is the converse true?
c) Assume that X and S are Noetherian. Using the valuative criterion for properness, show that finite morphisms are proper.

   Moreover, the following is true (don't prove):

   Theorem. *Let $f : X \to S$ be a morphism of schemes with S locally Noetherian. Then f is finite if and only if f is proper with finite fibers.*

(7) (B) a) Let X be a complete variety over a field k (recall that this means X is an integral proper separated scheme, of finite type over k). Show that all global sections of X are constant.

b) Deduce that if an affine variety is complete, then it is a point (or $\emptyset$).