C2.6 Introduction to Schemes Sheet 3

Hilary 2024

(1) (A) Describe the schematic fibers of $\operatorname{Spec}\mathbb{Z}[x] \to \operatorname{Spec}\mathbb{Z}$ (Try to draw a picture of it).

Solution. The fiber over (p) is $\operatorname{Spec}\mathbb{F}_p[x]$ and the fiber over (0) is $\operatorname{Spec}\mathbb{Q}[x]$. The points of these fibers correspond to monic irreducible polynomials mod p, resp. irreducible monic polynomials with integer coefficients in $\mathbb{Q}[x]$. A famous picture of this fibration can be found in Mumford's red book.

(2) (B) Prove the following statements:

1) $\mathbb{A}^n$ and $\mathbb{P}^n$ are separated (over $\operatorname{Spec}\mathbb{Z}$). Deduce that $\mathbb{A}^n_S$ and $\mathbb{P}^n_S$ are separated S-schemes for any S affine.
2) Open and closed embeddings of schemes are separated maps.
3) Compositions of separated maps are separated.

Solution For 1), $\mathbb{A}^n$ is separated because it is affine. For $\mathbb{P}^n$, take the standard covering of $\mathbb{P}^n$ by affine opens

$$U_i := \operatorname{Spec}\mathbb{Z}[x_0/x_i, \ldots, \widehat{x_i/x_i}, \ldots, x_n/x_i], \tag{1}$$

it suffices to check (c.f. [Stacks, Tag 01KP]), that the map $\mathcal{O}_{\mathbb{P}^n}(U_i) \otimes_{\mathbb{Z}} \mathcal{O}_{\mathbb{P}^n}(U_j) \to \mathcal{O}_{\mathbb{P}^n}(U_{ij})$ obtained by multiplying the restrictions, is surjective. Indeed, one has

$$\begin{aligned}\mathbb{Z}[x_0/x_i, \ldots, \widehat{x_i/x_i}, \ldots, x_n/x_i] \otimes \mathbb{Z}[x_0/x_j, \ldots, \widehat{x_j/x_j}, \ldots, x_n/x_j]& \\ \to \mathbb{Z}[x_0/x_i, \ldots, \widehat{x_i/x_i}, \ldots, x_n/x_i, (x_j/x_i)^{-1}]& \end{aligned} \tag{2}$$

sending $x_k/x_i \otimes 1 \mapsto x_k/x_i$ and $1 \otimes x_\ell/x_j \mapsto (x_\ell/x_i) \times (x_j/x_i)^{-1}$, and this is clearly surjective.

I would interpret "seperated S-scheme" to mean a an S-scheme X such that the structure morphism $X \to S$ is separated. In that case, the separatedness of $\mathbb{P}^n_S$ and $\mathbb{A}^n_S$ as S-schemes, follows from the above, and the fact that separated morphisms are stable under base change.

For 2) we claim that any morphism $j : X \to Y$ of schemes which is injective on the underlying topological spaces, is separated. For, if $z \in X \times_Y X$, then $p_1(z) = p_2(z) =: x$ (by the definition of the fiber product). Set $y := j(x)$. Then we can choose affine open neighbourhoods $x \in U \subseteq X$, $y \in V \subseteq Y$ such that $j(U) \subseteq V$. Thus $z \in U \times_V U$ so that $X \times_Y X$ is the union of such affines. Since $\Delta^{-1}_{X/Y}(U \times_V U) = U$ and $U \to U \times_V U$ is a closed immersion (morphisms of affines are always separated, as can be seen directly), this shows that $\Delta_{X/Y}$ is a closed immersion and therefore j is separated.

For 3) suppose we are given separated maps $f : X \to Y$ and $g : Y \to Z$. Consider the diagram

$$\begin{array}{ccccc} X & \xrightarrow{\Delta_{X/Y}} & X \times_Y X & \longrightarrow & X \times_Z X \\ & & \downarrow & \lrcorner & \downarrow \\ & & Y & \xrightarrow{\Delta_{Y/Z}} & Y \times_Z Y \end{array} \tag{3}$$

The top composite is $\Delta_{X/Z}$ and we would like to show this is a closed immersion. By assumption $\Delta_{X/Y}$ is a closed immersion. The second top horizontal arrow is a closed immersion because $\Delta_{Y/Z}$ is a closed immersion, and closed immersions are stable under base change. Therefore, since closed immersions are stable under composition, $\Delta_{X/Z}$ is a closed immersion.

(3) (B) Prove that the "bug-eyed line" obtained by gluing two copies of $\mathbb{A}^1$ along $\mathbb{A}^1 \setminus \{0\}$, is not separated.

Solution. One way to see this, is that the valuative criterion fails. Let X be the space in the question, and let $g_1 : \mathbb{A}^1 \to X$, $g_2 : \mathbb{A}^1 \to X$ be the two inclusions of $\mathbb{A}^1$ into X. Let $R := \mathbb{Q}[\![x]\!]$ with fraction field $K := \mathbb{Q}(\!(x)\!)$. Consider the composite $f : \operatorname{Spec} K \to \mathbb{A}^1 \setminus \{0\} \to X$ where the first map is induced by the inclusion of rings $\mathbb{Z}[x, x^{-1}] \to \mathbb{Q}(\!(x)\!)$. Then the morphism f admits two distinct extensions to $\operatorname{Spec} R$, namely the two composites

$$f_i : \operatorname{Spec} R \to \mathbb{A}^1 \xrightarrow{g_i} X \quad \text{for } i = 1, 2, \tag{4}$$

where the first morphism corresponds to the inclusion of rings $\mathbb{Z}[x] \to \mathbb{Q}[\![x]\!]$. The fact that these are distinct morphisms can be seen by looking at the image of the special point $s \in \operatorname{Spec} R$. Therefore X is not separated.

(4) (B) Prove the following criterion: A ring homomorphism $\varphi^\# : A \to B$ is flat if and only if the corresponding morphism of affine schemes $\varphi : \operatorname{Spec} B \to \operatorname{Spec} A$ is flat.

Solution. We first prove a Lemma:

Lemma. *Let R be a ring and let M be an R-module. Then $M = 0$ if and only if $M_\mathfrak{p} = 0$ for all $\mathfrak{p} \in \operatorname{Spec} R$.*

Proof. The "only if" direction being obvious, we prove the "if" direction. Let $x \in M$ and let $I := \operatorname{Ann}_R(x) \subset R$. By the assumption, and the definition of localization, for all $\mathfrak{p} \in \operatorname{Spec} R$ there exists $f \in R \setminus \mathfrak{p}$ such that $fx = 0$. This implies that I is not contained in any prime ideal and therefore, is the unit ideal. In particular $x = 1.x = 0$, so $M = 0$. □

As a corollary of this, we obtain:

Lemma. *Let R be a ring. A sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ of R-modules is exact if and only if $0 \to M_{1,\mathfrak{p}} \to M_{2,\mathfrak{p}} \to M_{3,\mathfrak{p}} \to 0$ is exact for all $\mathfrak{p} \in \operatorname{Spec} R$.*

Proof. This is a straightforward exercise using the previous Lemma and the fact that localization is exact, and hence commutes with kernels and cokernels. □

Now suppose that $\varphi : \operatorname{Spec} B \to \operatorname{Spec} A$ is flat and let $0 \to M_1 \to M_2 \to M_3 \to 0$ be an exact sequence of A-modules. We would like to show that the sequence

$$0 \to M_1 \otimes_A B \to M_2 \otimes_A B \to M_3 \otimes_A B \to 0 \tag{5}$$

is an exact sequence of B-modules. By the previous Lemma this is exact if and only if

$$0 \to M_1 \otimes_{A_{\varphi(\mathfrak{p})}} B_{\mathfrak{p}} \to M_2 \otimes_{A_{\varphi(\mathfrak{p})}} B_{\mathfrak{p}} \to M_3 \otimes_{A_{\varphi(\mathfrak{p})}} B_{\mathfrak{p}} \to 0 \tag{6}$$

is exact for all $\mathfrak{p} \in \operatorname{Spec} R$. However, this holds since the morphisms $A_{\varphi(\mathfrak{p})} \to B_{\mathfrak{p}}$ are flat, by assumption.

Conversely suppose that $\varphi^\# : A \to B$ is a flat ring morphism, let $\mathfrak{p} \in \operatorname{Spec} B$ and let $0 \to N_1 \to N_2 \to N_3 \to 0$ be an exact sequence of $A_{\varphi(\mathfrak{p})}$-modules. By restriction along $A \to A_{\varphi(\mathfrak{p})}$ we view this as an exact sequence of A-modules, and by flatness then

$$0 \to N_1 \otimes_A B \to N_2 \otimes_A B \to N_3 \otimes_A B \to 0 \tag{7}$$

is exact. Since localization is exact we conclude that

$$0 \to N_1 \otimes_{A_{\varphi(\mathfrak{p})}} B_{\mathfrak{p}} \to N_2 \otimes_{A_{\varphi(\mathfrak{p})}} B_{\mathfrak{p}} \to N_3 \otimes_{A_{\varphi(\mathfrak{p})}} B_{\mathfrak{p}} \to 0 \tag{8}$$

is an exact sequence of $B_{\mathfrak{p}}$-modules. Therefore $A_{\varphi(\mathfrak{p})} \to B_{\mathfrak{p}}$ is flat.

(5) (B) Show that $\operatorname{Spec} \mathbb{Z}[x,y]/(x^2 - y^2 - 5) \to \operatorname{Spec} \mathbb{Z}$ is flat.

Is $\operatorname{Spec} \mathbb{Z}[x,y]/(2x^2 - 2y^2 - 10) \to \operatorname{Spec} \mathbb{Z}$ flat?

Explain the geometric intuition behind these examples by looking at the dimensions of fibers.

Solution. Since $\mathbb{Z}$ is a PID, a $\mathbb{Z}$-module is flat if and only if it is torsionfree.

For the first part, we are thus reduced to prove the following: for all $f \in \mathbb{Z}[x,y]$ and $N \in \mathbb{Z}$, $Nf \in (x^2 - y^2 - 5)$ if and only if $f \in (x^2 - y^2 - 5)$. But this follows since $(x^2 - y^2 - 5)$ is irreducible (and hence prime) in the UFD $\mathbb{Z}[x,y] = \mathbb{Z}[y][x]$, as $y^2 - 5$ is not a square in $\mathbb{Z}[y]$.

For the second part, the morphism is not flat since $x^2 - y^2 - 5$ is 2-torsion in $\mathbb{Z}[x,y]/(2x^2 - 2y^2 - 10)$.

For geometric intuition: flatness is supposed to correspond to dimensions of fibers not jumping unexpectedly. One notes that the fibers of $\operatorname{Spec}(\mathbb{Z}[x,y]/(x^2 - y^2 - 5)) \to \operatorname{Spec}(\mathbb{Z})$ over a prime (p) (or (0)) are $\operatorname{Spec}(\mathbb{F}_p[x,y]/(x^2 - y^2 - 5))$, (or $\operatorname{Spec}(\mathbb{Q}[x,y]/(x^2 - y^2 - 5))$), which is always has Krull dimension 1.

(6) (B) A morphism $f : X \to S$ is called *finite* if S has an affine cover $S = \bigcup_{i \in \mathcal{I}} \operatorname{Spec} B_i$ such that, for all i, $f^{-1}(\operatorname{Spec} B_i) \simeq \operatorname{Spec} A_i$ is an affine scheme and A_i is finitely generated as a module over B_i.

a) Give some examples of finite morphisms.

b) Show that a finite morphism has finite fibers. Is the converse true?

c) Assume that X and S are Noetherian. Using the valuative criterion for properness, show that finite morphisms are proper.

Moreover, the following is true (don't prove):

Theorem. *Let $f : X \to S$ be a morphism of schemes with S locally Noetherian. Then f is finite if and only if f is proper with finite fibers.*

Solution. a) Any closed immersion is finite. Another example would be the map $\mathbb{A}^1_{\mathbb{C}} \setminus \{0\} \to \mathbb{A}^1_{\mathbb{C}} \setminus \{0\}$ induced by the ring morphism $\mathbb{C}[t, t^{-1}] \to \mathbb{C}[t, t^{-1}]$ sending $t \mapsto t^2$, it is the quintessential example of a finite étale covering map.

b) Working locally, we may assume that $X = \operatorname{Spec} A$ and $S = \operatorname{Spec} B$ are both affine and A is finitely generated as a module over B. Let $p : \operatorname{Spec} K \to \operatorname{Spec} B$ be a point and let $f^{-1}(p) = \operatorname{Spec}(C)$, where $C := A \otimes_B K$; this is then a finite dimensional algebra over K. If $\mathfrak{p} \in \operatorname{Spec} C$ then $C/\mathfrak{p}$ is then a finite dimensional (over K) integral domain and therefore a field. Hence all prime ideals of C are maximal. Note that C is Artinian, by finite dimensionality. Now consider a minimal element $\mathfrak{m}_1 \ldots \mathfrak{m}_k$ in the family of finite products of finitely many maximal ideals. If $\mathfrak{m}$ is a further maximal ideal then

$$\mathfrak{m}\mathfrak{m}_1 \ldots \mathfrak{m}_k \subseteq \mathfrak{m}_1 \ldots \mathfrak{m}_k \tag{9}$$

and therefore by minimality $\mathfrak{m}\mathfrak{m}_1 \ldots \mathfrak{m}_k = \mathfrak{m}_1 \ldots \mathfrak{m}_k$ so $\mathfrak{m}_1 \ldots \mathfrak{m}_k \subseteq \mathfrak{m}$. By primality and maximality it then follows that $\mathfrak{m} = \mathfrak{m}_i$ for some $1 \leq i \leq k$, so $\operatorname{Spec} C$ is finite.

The converse is not true, for example, any open immersion has finite fibers (they are either a singleton or empty), but open immersions are almost never finite morphisms, eg. consider the inclusion $\mathbb{A}^1 \setminus \{0\} \to \mathbb{A}^1$ - we can see that $\mathbb{Z}[t, t^{-1}]$ is not finite as a $\mathbb{Z}[t]$-module.

c) Working locally once again, it suffices to treat the case when $X = \operatorname{Spec} A$ and $S = \operatorname{Spec} B$ are both affine. Since $B \to A$ is a finite ring extension, it is integral, (by the characteristic polynomial trick). Let R be a DVR with fraction field K, and suppose we are given a commutative diagram

$$\begin{array}{ccc} K & \longleftarrow & A \\ \uparrow & & \uparrow \\ R & \longleftarrow & B \end{array} \tag{10}$$

in which the right vertical arrow is induced by f and left vertical arrow is the inclusion. The commutativity, plus the fact that $B \to A$ is integral, implies that the image of $A \to K$ factors uniquely through the integral closure of R in K. However, R is integrally closed in its field of fractions since it is a DVR, so $B \to A$ factors uniquely through R, establishing the valuative criterion.

(7) (B) a) Let X be a complete variety over a field k (recall that this means X is an integral proper separated scheme, of finite type over k). Show that all global sections of X are constant.

b) Deduce that if an affine variety is complete, then it is a point (or $\emptyset$).

Solution. a) Global sections correspond to morphisms $f : X \to \mathbb{A}^1_k$, and therefore, we will show that any such morphism is constant.

First, let us extend f to a morphism $g : X \to \mathbb{P}^1_k$. Since $X \to \operatorname{Spec} k$ is separated, g is separated. Therefore the graph $\Gamma_g := (g \times \mathrm{id})^{-1}\Delta_g \subseteq X \times \mathbb{P}^1_k$ is a closed subset. By universal-closedness then $\operatorname{im} g := p_2\Gamma_g$ is a closed subset of $\mathbb{P}^1_k$. We endow it with the induced-reduced subscheme structure. Since $\mathbb{P}^1_k$ is a complete variety, and $\operatorname{im} g$ is a closed subset, then $\operatorname{im} g$ is also complete.

On the other hand, we have that $\operatorname{im} g$ is contained in $\mathbb{A}^1_k \subseteq \mathbb{P}^1_k$. If $\operatorname{im} g = \mathbb{A}^1_k$ then this would imply that $\mathbb{A}^1_k$ is complete, which is false (one can consider the image of $V(xy) \subseteq \mathbb{A}^1_k \times \mathbb{A}^1_k$ under the second projection). Hence $\operatorname{im} g$ is a proper closed subset of $\mathbb{A}^1_k$ and hence must be a finite collection of points. Since X is topologically irreducible this implies that $\operatorname{im} g$ is single point, so g (hence also f) is constant.

b) If an affine variety X is complete, then $X \cong \operatorname{Spec}(\Gamma(X, \mathcal{O}_X))$ and by the preceding, $\Gamma(X, \mathcal{O}_X) = 0$ or k. So $X = \emptyset$ or $X = \mathrm{pt}$.

References

[Stacks] The Stacks Project Authors. *Stacks Project*. `https://stacks.math.columbia.edu`. 2018.