Axiomatic Set Theory

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Chapter 1

Introduction

[1]

An introductory course on set theory, including statements of all the standard ZF axioms, the development of the transfinite ordinal and cardinal numbers, transfinite induction and recursion, and equivalents of the Axiom of Choice. So is an introductory course in logic going at least as far as the Completeness Theorem for first-order predicate calculus.

I plan to edit these lecture notes from time to time throughout the term.

Videos from the time of the lockdowns are still (I believe) up on the website. They were done by Dr Suabedissen, following a different set of lecture notes but on the same syllabus.

One of our main aims in this course is to prove the following:

Theorem 1.1 *(Gödel 1938) If set theory without the Axiom of Choice (ZF) is consistent (i.e. does not lead to a contradiction), then set theory with the axiom of choice (ZFC) is consistent.*

Importance of this result: Set theory is the axiomatization of mathematics, and without AC no-one seriously doubts its truth, or at least consistency. However, much of mathematics requires AC (eg. every vector space has a basis, every ideal can be extended to a maximal ideal). Probably most mathematicians don't doubt the truth, or at least consistency, of set theory with AC, but it does lead to some bizarre, seemingly paradoxical results—eg. the Banach-Tarski paradox **(explain)**. Hence it is comforting to have Gödel's theorem.

I formulate the axioms of set theory below. For the moment we have:

(AC.) *Axiom of Choice* (Zermelo) If X is a set of non-empty pairwise disjoint sets, then there is a set Y which has exactly one element in common with each element of X.

To complement Gödel's theorem, there is also the following result which is beyond this course:

[1] See Andreas Blass, "On the inadequacy of inner models", JSL 37 no. 3 (Sept 72) 569–571.

Proposition 1.2 *(Cohen 1963) If ZF is consistent, so is ZF with $\neg AC$.*

We shall also discuss Cantor's continuum problem which is the following.

Cantor defined the cardinality, or size, of an arbitrary set. The cardinality of A is denoted $|A|$. He showed that $|\mathbb{R}| > |\mathbb{N}|$, but could not find any set S such that $|\mathbb{R}| > |s| > |\mathbb{N}|$, so conjectured:

(CH.) *Cantor's Continuum Hypothesis* For any set S, either $|S| \leq |\mathbb{N}|$, or $|S| \geq |\mathbb{R}|$.

Again Gödel (1938) showed:

Theorem 1.3 *If ZF is consistent, so is ZF+AC+CH,*

and Cohen (1963) showed:

Proposition 1.4 *If ZF is consistent, so is ZF+AC+$\neg$CH.*

We shall prove Gödel's theorem but not Cohen's.

Of course Gödel's theorem on CH was perhaps not so mathematically pressing as his theorem on AC since mathematicians rarely want to assume CH, and if they do, then they say so.

We first make Gödel's theorem precise, by defining set theory and its language.

These notes were originally created by Peter Koepke, and adapted by Alex Wilkie and the current lecturer.

Chapter 2

Basics

See D. Goldrei *Classic Set Theory*, Chapman and Hall 1996, or H.B. Enderton *Elements of Set Theory*, Academic Press, 1977.

2.1 The language of set theory

Definition 2.1.1 *The* language of set theory*, LST, is first-order predicate calculus with equality having the membership relation* $\in$ *(which is binary) as its only non-logical symbol.*

Thus the basic symbols of LST are: $=$, $\in$, $\vee$, $\neg$, $\forall$, (and), and an infinite list $v_0, v_1, \ldots, v_n, \ldots$ of variables (although for clarity we shall often use $x, y, z, t, \ldots, u, v, \ldots$ etc. as variables).

Definition 2.1.2 *The* well-formed formulas*, or just* formulas*, of LST are those expressions that can be built up from the* atomic formulas*:* $v_i = v_j$*,* $v_i \in v_j$*, using the rules:*

1. *if* ϕ *is a formula, so is* $\neg\phi$*,*
2. *if* ϕ *and* ψ *are formulas, so is* $(\phi \vee \psi)$*, and*
3. *if* ϕ *is a formula, so is* $\forall v_i\, \phi$*.*

2.2 Some standard abbreviations

We write

1. $(\phi \wedge \psi)$ for $\neg(\neg\phi \vee \neg\psi)$;
2. $(\phi \rightarrow \psi)$ for $(\neg\phi \vee \psi)$;
3. $(\phi \leftrightarrow \psi)$ for $((\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi))$;

4. $\exists x\, \phi$ for $\neg\forall x\, \neg\phi$;

5. $\exists! x\, \phi$ for $\forall y(\phi \leftrightarrow x = y)$;

6. $\exists x \in y\, \phi$ for $\exists x\, (x \in y \wedge \phi$;

7. $\forall x \in y\, \phi$ for $\forall x\, (x \in y \to \phi)$;

8. $\forall x, y\, \phi$ (etc.) for $\forall x \forall y\, \phi$;

9. $x \notin y$ for $\neg x \in y$.

We shall also often write ϕ as $\phi(x)$ to indicate free occurrences of a variable x in ϕ. The formula $\phi(z)$ (say) then denotes the result of substituting every free occurrence of x in ϕ by z. Similarly for $\phi(x, y)$, $\phi(x, y, z)$,. . . , etc.

2.3 The Axioms

(A1.) *Extensionality*

$$\forall x, y\, (x = y \leftrightarrow \forall t(t \in x \leftrightarrow t \in y))$$

Two sets are equal iff they have the same members.

(A2.) *Empty set*

$$\exists x\, \forall y\, y \notin x$$

There is a set with no members, *the empty set*, denoted $\varnothing$.

(A3.) *Pairing*

$$\forall x, y\, \exists z\, \forall t\, (t \in z \leftrightarrow (t = x \vee t = y))$$

For any sets x, y there is a set, denoted $\{x, y\}$, whose only elements are x and y.

(A4.) *Union*

$$\forall x\, \exists y\, \forall t\, (t \in y \leftrightarrow \exists w(w \in x \wedge t \in w))$$

For any set x, there is a set, denoted $\bigcup x$, whose members are the members of the members of x.

(A5.) *Separation Scheme* If $\phi(\mathbf{x}, \mathbf{y})$ is a formula of LST, the following is an axiom:

$$\forall \mathbf{x}\, \forall u\, \exists z\, \forall y\, (y \in z \leftrightarrow (y \notin u \wedge \phi(\mathbf{x}, y))$$

For given sets $\mathbf{x}, u$ there is a set, denoted $\{y \in u : \phi(\mathbf{x}, y)\}$, whose elements are those elements y of u which satisfy the formula $\phi(\mathbf{x}, y)$.

(A6.) *Replacement Scheme* If $\phi(x, y)$ is a formula of LST (possibly with other free variables $\mathbf{u}$, say) then the following is an axiom:

$$\forall \mathbf{u}\, [\forall x, y, y' \, ((\phi(x, y) \wedge \phi(x, y')) \to y = y') \to \forall s\, \exists z\, \forall y\, (y \in z \leftrightarrow \exists x \in s \phi(x, y))]$$

The set z is denoted $\{y : \exists x\, \phi(x, y) \wedge x \in s\}$.

(A7.) *Power Set*

$$\forall x\, \exists y\, \forall t\, (t \in y \leftrightarrow \forall z\, (z \in t \to z \in x))$$

For any set x there is a set, denoted $\mathbb{P}(x)$, whose members are exactly the subsets of x.

(A8.) *Infinity*

$$\exists x\, [\exists y\, (y \in x \wedge \forall z\, (z \notin y) \wedge \forall y\, (y \in x \to \exists z\, (z \in x \wedge \forall t\, (t \in z \leftrightarrow (t \in y \vee t = y))))]$$

There is a set x such that $\varnothing \in x$ and whenever $y \in x$, they $y \cup \{y\} \in x$. (Such a set is called a *successor set.*

(A9.) *Foundation*

$$\forall x\, (\exists z\, z \in x \to \exists z\, (z \in x \wedge \forall y \in z\, y \notin x))$$

If the set x is non-empty, then for some $z \in x$, z has no members in common with x.

(A10.) *Axiom of Choice*

$$\forall u\, [[\forall x \in u\, \exists y\, y \in x \wedge \forall x, y\, ((x \in u \wedge y \in u \wedge x \neq y) \to \forall z\, (z \notin x \vee \notin y))] \to \exists v\, \forall x \in u\, \exists! y\, (y \in x \wedge y \in v)]$$

We write ZF* for the collection of axioms A1–A8; ZF for A1–A9; ZFC for A1–A10.

2.4 Proofs in principle and proofs in practice

Definition 2.4.1 *Suppose that T is one of the above collections of axioms. If σ is a sentence of LST (ie. a formula without free variables), we say that σ is* a theorem of T*, or that σ* can be proved from T*, and write $T \vdash \sigma$, if there is a finite sequence $\sigma_1, \ldots, \sigma_n$ of LST formulas such that σ_n is σ, and each σ_i is either in T or else follows from earlier formulas in the sequence by a rule of logic.*

Proposition 2.4.2 *Every theorem of ZF is a theorem of ZFC and every theorem of ZF* is a theorem of ZF.*

Definition 2.4.3 *To say that T* is consistent *means that for no sentence ϕ of LST is $(\phi \wedge \neg\phi)$ a theorem of T.*

Proposition 2.4.4 *T is consistent if and only if there is some sentence which is not provable from T.*

This now makes theorem 1.1 precise: we must show that if ZF is consistent, then so is ZFC.

2.5 Interpretations

The Completeness Theorem for first-order predicate calculus (also due to Gödel) states:

Theorem 2.5.1 *A sentence σ of any first-order language is provable from a collection of sentences S in the same language if and only if every model of S is a model of σ.*

Equivalently, S is consistent if and only if S has a model.

Definition 2.5.2 *A* structure for LST *is specified by a domain of discourse M over which the quantifiers $\forall x \ldots$ and $\exists x \ldots$ range, and a binary relation E on M to interpret the membership relation $\in$.*

If σ is a sentence of LST which is true under this interpretation we say that σ is true in $\langle M, E\rangle$ *or* $\langle M, E\rangle$ is a model of σ, *and write $\langle M, E\rangle \vDash \sigma$.*

If T is a collection of sentences of LST we also write $\langle M, E\rangle \vDash T$ iff $\langle M, E\rangle \vDash \sigma$ for each sentence σ in T.

If $\phi(x_1, \ldots, x_n)$ is a formula of LST with free variables among $x_1, \ldots, x_n$ and $a_1, \ldots, a_n$ are in the domain M, we also write $\langle M, E\rangle \vDash \phi(a_1, \ldots, a_n)$ to mean $\phi(x_1, \ldots, x_n)$ is true of $a_1, \ldots, a_n$ in the interpretation $\langle M, E\rangle$.

Example 2.5.3 *Suppose M contains just the two distinct elements a and b, and E is specified by $a \to b$, ie. $E(a, b)$, not $E(b, a)$, not $E(a, a)$, not $E(b, b)$. Then $\langle M, E\rangle \vDash$ Empty Set, ie. $M \vDash \exists x \forall y y \notin x$, since it is true that there is an x in M (namely a) such that for all $y \in M$, not $E(y, x)$. It is also easy to see that $\langle M, E\rangle \vDash$ Extensionality and $\langle M, E\rangle \vDash \neg$Pairing. Notice that, by the completeness theorem, this implies that Paring is not provable from Extensionality and Empty Set since we have found a model of the latter two axioms which is not a model of the former.*

Exercise 2.5.4 *Let $\mathbb{Q}$ be the set of rational numbers and $<$ the usual ordering of $\mathbb{Q}$. Which axioms of ZF are true in $\langle \mathbb{Q}, <\rangle$?*

The completeness theorem provides a method for establishing theorem 1.1. For we can rephrase that theorem as: If ZF has a model then so does ZFC. Indeed we shall construct a subcollection L of V^* such that if we assume $\langle V^*, \in\rangle \vDash$ ZF, then $\langle L, \in\rangle \vDash$ ZFC. (Actually our proof will yield somewhat more which ought to be enough to satisfy any purist. Namely, it will produce an effective procedure for converting any proof of a contradiction (ie. a sentence of the form $(\phi \wedge \neg\phi)$) from ZFC to a proof of a contradiction from ZF.)

We now turn to the development of some basic set theory from the axioms ZF*.

2.6 New sets from old

The axioms of ZF are of three types: (a) those that assert that all sets have a certain property (Extensionality, Foundation), (b) those that sets with certain properties exist (Empty Set, Infinity), and (c) those that tell us how we may construct new sets out of given sets (Pairing, Union, Separation, Replacement, Power Set). Our aim here is to combine the operations implicit in the axioms of type (c) to obtain more ways of constructing sets and to introduce notations for these constructions (just as, for example, we introduced the notation $\bigcup x$ for the set y given by Union).

Notation 2.6.1 *We write $\{x : \phi(x)\}$ for the collection (or* class*) of sets x satisfying the LST formula $\phi(x)$.*[1]

As we have seen, such a class need not be a set. However, in the following definitions it can be shown (from the axioms ZF*) that we always do get a set. This amounts to showing that for some set a, if b is any set such that $\phi(b)$ holds (ie. $V^* \vDash \phi(b)$) then $b \in a$, so that $\{x : \phi(x)\} = \{x \in a : \phi(x)\}$ which is a set by A5. I leave all the required proofs as exercises—they can also be found in the books.

In the following, $A, B, \ldots, a, b, c, \ldots, f, g, a_1, a_2, \ldots, a_n, \ldots$ etc. all denote sets.

1. $\{a_1, \ldots, a_n\} \ := \{x : x = a_1 \vee \ldots \vee x = a_n\}$.
2. $a \cup b \ := \bigcup\{a, b\} = \{x : x \in a \vee x \in b\}$.
3. $a \cap b \ := \{x : x \in a \wedge x \in b\}$.
4. $a \setminus b \ := \{x : x \in a \wedge x \notin b\}$.
5. $\bigcap a \ := \begin{cases} \{x : \forall y \in a x \in y\} \text{ if } a \neq \varnothing \\ \quad \text{undefined if } a = \varnothing \end{cases}$.
6. $\langle a, b\rangle \ := \{\{a\}, \{a, b\}\}$. (**Lemma.** $\langle a, b\rangle = \langle c, d\rangle \leftrightarrow (a = c \wedge b = d)$.)
7. $a \times b \ := \{x : \exists c \in a \exists d \in b x = \langle c, d\rangle\}$. (**Remark:** Of course the proof via Comprehension that $a \times b$ is a set requires not only "bounding the x's", but also showing that the expression "$\exists c \in a \exists d \in b x = \langle c, d\rangle$" can be written as a formula of LST (with parameters a, b).)

[1] Actually, $\phi(x)$ will be allowed to have parameters (ie. names for given sets), so is not strictly a formula of LST. Notice, however, that parameters are allowed in Separation and Replacement (the "**x**" and "**u**").

8. $a \times b \times c \; := a \times (b \times c), \ldots$, etc.

9. $a^2 \; := a \times a$, $a^3 \; := a \times a \times a, \ldots$, etc.

10. We write $a \subseteq b$ for $\forall x \in a(x \in b)$.

11. c is a *binary relation* on a we take to mean $c \subseteq a^2$. (Similarly for ternary,..., n-ary, ... relations.)

12. If A is a binary relation on a we usually write xAy for $\langle x, y\rangle \in A$.

    A is called a *(strict) partial order* on a iff

    (a) $\forall x, y \in a(xAy \to \neg yAx)$,

    (b) $\forall x, y, z \in a((xAy \wedge yAx) \to xAz)$.

    If in addition we have (3) $\forall x, y \in a(x = y \vee xAy \vee yAx)$, then A is called a *(strict) total* (or *linear) order* of a.

13. Write $f : a \to b$ (f is a *function* with *domain* a and *codomain* b, or simply f is a *function from* a *to* b) if $f \subseteq a \times b$ and $\forall c \in a \exists! d \in b \langle c, d\rangle \in f$. Write $f(c)$ for this unique d.

14. If $f : a \to b$, f is called *injective* (or *one-to-one*) if $\forall c, d \in a(c \neq d \to f(c) \neq f(d))$, *surjective* (or *onto*) if $\forall d \in b \exists c \in a f(c) = d$, and *bijective* if it is both injective and surjective.

15. We write $a \sim b$ if $\exists f(f : a \to b \wedge f$ bijective$)$.

16. ${}^a b \; := \{f : f : a \to b\}$.

17. A set a is called *a successor set* if

    (a) $\varnothing \in a$ and

    (b) $\forall b(b \in a \to b \cup \{b\} \in a)$.

    Axiom A8 states that a successor set exists and it can be further shown that a unique such set, denoted ω, exists with the property that $\omega \subseteq a$ for every successor set a. The set ω is called the set of natural numbers. If $n, m \in \omega$ we often write $n + 1$ for $n \cup \{n\}$ and $n < m$ for $n \in m$ and 0 for $\varnothing$ (in this context). The relation $\in$ (ie. $<$) is a total order of ω (more precisely $\{\langle x, y\rangle : x \in \omega, y \in \omega \wedge x \in y\}$ is a total order of ω).

18. The set ω satisfies the *principle of mathematical induction*, ie. if $\psi(x)$ is any formula of LST such that $\psi(0) \wedge \forall n \in \omega(\psi(n) \to \psi(n+1))$ holds, then $\forall n \in \omega \psi(n)$ holds.

19. The set ω also satisfies the *well-ordering principle*, ie. for any set a, if $a \subseteq \omega$ and $a \neq \varnothing$ then $\exists b \in a \forall c \in a(c > b \vee c = b)$.

20. *Definition by recursion*

    Suppose that $f : A \to A$ is a function and $a \in A$. Then there is a unique function $g : \omega \to A$ such that:

    (a) $g(0) = a$, and

    (b) $\forall n \in \omega g(n+1) = f(g(n))$.

    (Thus, $g(n) = \underbrace{f(f \cdots (f}_{n \text{ times}} a)) \cdots)).$)

    More generally, if $f : B \times \omega \times A \to A$ and $h : B \to A$ are functions, then there is a unique function $g : B \times \omega \to A$ such that

    (a) $\forall b \in B g(b, 0) = h(b)$, and

    (b) $\forall b \in B \forall n \in \omega g(b, n+1) = f(b, n, g(b, n))$.

    Using this result one can define the addition, multiplication and exponentiation functions.

    (**Remark** I have adopted here the usual convention of writing $g(b, n+1)$ for $g(\langle b, n+1 \rangle)$. Similarly for f.)

21. A set a is called *finite* iff $\exists n \in \omega a \sim n$.

22. A set a is called *countably infinite* iff $a \sim \omega$.

23. A set a is called *countable* iff a is finite or countably infinite. (Equivalently: iff $\exists f (f : a \to \omega \wedge f$ injective).)

    (**Theorem** $\mathbb{P}\omega$ is not countable. In fact, for no set A do we have $A \sim \mathbb{P}A$. (Cantor))

Chapter 3

Classes, class terms and recursion

3.1 Notation and basic concepts

Definition 3.1.1 *We call collections of the form $\{x : \phi(x)\}$, where ϕ is a formula of LST,* classes*.*

Definition 3.1.2 *V^*=the collection of all sets (assuming only ZF*).*

Proposition 3.1.3 *Every set is a class.*

Proof. $a = \{x : x \in a\}$. (so $\phi(x)$ is $x \in a$ here). $\square$

We must be careful in their use—we cannot quantify over them but some operations will still apply.

Notation 3.1.4 *If $U_1 = \{x : \phi(x)\}$ and $U_2 = \{x : \psi(x)\}$, then*

$$\begin{aligned} U_1 \cap U_2 &= \{x : \phi(x) \wedge \psi(x)\} \\ U_1 \cup U_2 &= \{x : \phi(x) \vee \psi(x)\} \\ U_1 \times U_2 &= \{x : \exists y(y = \langle s, t\rangle \wedge \phi(s) \wedge \psi(t))\} \end{aligned} \tag{3.1}$$

and so on. ($x \in U_1$ means $\phi(x)$ and $U_1 \subseteq U_2$ means $\forall x(\phi(x) \to \psi(x))$).

Classes are only a notation—we can always eliminate their use.

Proposition 3.1.5 *V^* is a class.*

Proof. $V^* = \{x : x = x\}$. $\square$

Definition 3.1.6 *If F, U_1, U_2 are classes with the properties that $F \subseteq U_1 \times U_2$ and $\forall x \in U_1 \exists! y \in U_2 \langle x, y\rangle \in F$, then F is called a* class term*, or just a term, and we write $F(x) = y$ instead of $\langle x, y\rangle \in F$.*

We also write $F : U_1 \to U_2$, although F may not be a function, as U_1 may not be a set.

So if $F = \{x : \exists y_1, y_2(x = \langle y_1, y_2\rangle \wedge y_2 = \bigcup y_1)\}$, so for all sets $F(x) = \bigcup x$, then F is a class term. We need class terms for *higher* recursion.

3.2 Recursion

(Use only ZF* throughout.)

Theorem 3.2.1 *Suppose $G : U \to U$ is a class term and $a \in U$. Then there is a term $F : \omega \to U$ (which is therefore a* function*) such that*

1. *$F(0) = a$ and*
2. *$\forall n \in \omega F(n + 1) = G(F(n))$.*

Some applications:

Definition 3.2.2 *A set a is called* transitive *if $\forall x \in a \forall y \in x y \in a$. (ie. $x \in a \to x \subseteq a$, or $a = \bigcup a$.)*

Lemma 3.2.3 *ω is transitive; and if $n \in \omega$, then n is transitive.*

Theorem 3.2.4 *For any set a, there is a unique set b, denoted $TC(a)$, and called the* transitive closure *of a, such that*

1. *$a \subseteq b$,*
2. *b is transitive,*
3. *whenever $a \subseteq c$ and c is transitive, then $b \subseteq c$.*

Proof. Uniqueness is clear since if $a \subseteq b_1$ and $a \subseteq b_2$, b_1 and b_2 transitive and both satisfying (3), then $b_1 \subseteq b_2$ and $b_2 \subseteq b_1$, so $b_1 = b_2$.

For existence **(give idea: $b = a \cup \bigcup a \cup \bigcup\bigcup a \cup \ldots$)** let G be the class term given by $G(x) = \bigcup x$ (for $x \in V^*$). Apply 3.2.1, to get a term F such that

1. $F(0) = a$, and
2. $\forall n \in \omega F(n + 1) = G(F(n)) = \bigcup F(n)$.

By replacement, there is a set B such that $B = \{y : \exists x \in \omega F(x) = y\}$.

Let $b = \bigcup B = \bigcup\{F(n) : n \in \omega\}$. Then

1. Since $a = F(0)$ and $F(0) \in B$, we have $a \in B$, so $a \subseteq \bigcup B = b$.

2. Suppose $x \in b$ and $y \in x$. We must show $y \in b$. But $x \in b$ implies $x \in \bigcup B$ implies $x \in F(n)$ for some $n \in \omega$ implies $x \subseteq \bigcup F(n)$, so $y \in \bigcup F(n)$, so $y \in F(n+1)$, so $y \in \bigcup B$, so $y \in b$.

3. Suppose $a \subseteq c$, c transitive.

   We prove by induction on n that $F(n) \subseteq c$.

   $F(0) = a \subseteq c$.

   Suppose $F(n) \subseteq c$.

   We want to show that $F(n+1) \subseteq c$, so suppose $x \in F(n+1)$, ie $x \in \bigcup F(n)$. Then for some $y \in F(n)$, $x \in y$. Thus $x \in y \in F(n) \subseteq c$, so $x \in y \in c$, so $x \in c$, since c is transitive, as required.

   Thus, by induction, $\forall n \in \omega F(n) \subseteq c$, so $\bigcup\{F(n) : n \in \omega\} \subseteq c$, ie. $b \subseteq c$, as required.

□

Recursion on $\in$.

Theorem 3.2.5 *(Requires Foundation—ie. assume ZF) For $\psi(x)$ any formula of LST (with parameters) if $\forall x(\forall y \in x\psi(y) \to \psi(x))$, then $\forall x\psi(x)$. (The hypothesis trivially implies $\psi(\varnothing)$.)*

Proof. Suppose $\forall x(\forall y \in x\psi(y) \to \psi(x))$, but that there is some set a such that $\neg\psi(a)$. Then $a \neq \varnothing$. Let $b = TC(a)$, so $a \subseteq b$, and hence $b \neq \varnothing$. Let $C = \{x \in b : \neg\psi(x)\}$. Then $C \neq \varnothing$, since otherwise we would have $\forall x \in b\psi(x)$, hence $\forall x \in a\psi(x)$ (since $a \subseteq b$), and hence $\psi(a)$, contradiction.

By foundation there is some $d \in C$ such that $d \cap C = \varnothing$, ie. $d \in b$, $\neg\psi(d)$, but $\forall x \in dx \in b$ (since b is transitive) and $x \notin C$. But this means $\forall x \in d\psi(x)$, so $\psi(d)$—contradiction. □

Our present aim is to prove that if ZF* is consistent then so is ZF—so we won't use 3.2.5. Instead we find another generalization of induction.

Definition 3.2.6 *Suppose that a is a set and R is a binary relation on a. Then R is called a* well-ordering *of a if*

1. *R is a total ordering of a.*

2. *If b is a non-empty subset of a, then b contains an R-least element. ie. $\exists x \in b\forall y \in b(y = x \vee xRy)$.*

Remark: AC iff every set is well-orderable.

Definition 3.2.7 *Suppose that R_1 is a total order of a, and R_2 is a total order of b. Then we say that $\langle a, R_1\rangle$ is* order-isomorphic *to $\langle b, R_2\rangle$, written $\langle a, R_1\rangle \sim \langle b, R_2\rangle$, if there is a bijective function $f : a \to b$ such that $\forall x, y \in a(x < y \leftrightarrow f(x) < f(y))$.*

Definition 3.2.8 *We say x* is an ordinal*, $On(x)$, or $x \in On$, if*

1. *x is transitive, and*

2. *$\in$ is a well-ordering of x.*

We usually use α, β, etc., for ordinals.
On is a class.

Theorem 3.2.9 *(Enderton)*

1. *If R is a well-order of the set a, then there is a unique ordinal α such that $\langle a, R\rangle \sim \langle \alpha, \in\rangle$.*

2. *$\varnothing \in On$. (Write $\varnothing = 0$.)*

3. *$\alpha \in On \to \alpha+1 \in On$ (so all natural numbers are ordinals, by induction).*

4. *If a is a set and $a \subseteq On$, then $\bigcup a \in On$. (Hence $\omega \in On$.)*

5. *If $\alpha, \beta \in On$, either $\alpha = \beta$, $\alpha \in \beta$, or $\beta \in \alpha$, and exactly one occurs.*

6. *If $\alpha, \beta, \gamma \in On$, and $\alpha \in \beta$ and $\beta \in \gamma$, then $\alpha \in \gamma$.*

7. *If $\alpha, \beta \in On$, $\alpha \subseteq \beta$ iff $\alpha \in \beta$ or $\alpha = \beta$.*

8. *If $\alpha \in On$ and $a \in \alpha$, then $a \in On$.*

(Note that (3) implies that On is not a set.)

Theorem 3.2.10 *(Which is required to prove the above.) Suppose that $\phi(x)$ is a formula of LST, such that $\forall \alpha \in On(\forall \beta \in \alpha \phi(\beta) \to \phi(\alpha))$. Then $\forall \alpha \in On \phi(\alpha)$.*

Proof. Suppose $\forall \alpha \in On(\forall \beta \in \alpha \phi(\beta) \to \phi(\alpha))$, and suppose that there is some $\gamma \in On$ such that $\neg\phi(\gamma)$. Let $X = \{\alpha \in \gamma : \neg\phi(\alpha)\}$, then X is a set and $X \subseteq \gamma$. Also $X \neq \varnothing$, since if $\forall \alpha \in \gamma \phi(\gamma)$, then $\phi(\gamma)$.

Let α_0 be the least element of X. Then $\alpha_0 \in X$, so $\neg\phi(\alpha_0)$, and for all $\alpha \in X$ $\alpha = \alpha_0$ or $\alpha_0 \in \alpha$.

Now let α be any member of α_0. Then $\alpha \in \gamma$, since γ is transitive. Now we cannot have $\alpha \in X$, for then $\alpha_0 \in \alpha$ or $\alpha_0 \in \alpha$, and $\in$ would not be a strict total ordering of γ.

So we have $\alpha \in \gamma$, $\alpha \notin X$, so $\phi(\alpha)$ holds.

In other words $\forall \alpha \in \alpha_0 \phi(\alpha)$. But then $\phi(\alpha_0)$, giving us a contradiction. $\square$

Definition 3.2.11 *(1) An ordinal α is called a* successor ordinal *if $\alpha = \beta \cup \{\beta\}$ for some (necessarily unique) ordinal β. (Write $\alpha = \beta + 1$.)*

(2) An ordinal α is called a limit ordinal *if $\alpha \neq \varnothing$ and α is not a successor ordinal.*

Theorem 3.2.10 is often applied in the following way:
To prove $\forall \alpha \in \mathbf{On}\, \phi(\alpha)$:

1. Show $\phi(0)$
2. Show $\forall \alpha(\phi(\alpha) \to \phi(\alpha + 1))$
3. Show $\forall \alpha < \delta \phi(\alpha) \to \phi(\delta)$

Theorem 3.2.12 *(Definition by recursion on* **On***) Suppose $F : V^* \to V^*$ is a class term, and $a \in V^*$. Then there is a unique class term $G : \mathbf{On} \to V^*$ such that*

1. $G(0) = a$
2. $G(\alpha + 1) = F(G(\alpha))$
3. $G(\delta) = \bigcup_{\alpha\in\delta}$ *for δ a limit.*

*Proof.*Proof Let $\phi(g, \alpha)$ be the formula of LST expressing:
"g is a function with domain $\alpha + 1$ such that $\forall \beta < \alpha g(\beta + 1) = F(g(\beta))$ and if β is a limit $g(\beta) = \bigcup\{g(\alpha) : \alpha < \beta\}$ and $g(0) = a$".
((*) Note that if $\phi(g, \alpha)$ and $\beta \leq \alpha$, then $\phi(g\restriction\beta + 1, \beta)$.)

Lemma 3.2.13 $\forall \alpha \in \mathbf{On}\, \exists! g\, \phi(g, \alpha)$.

Proof. Induction on α.
$\alpha = 0$: Clearly $g = \{\langle 0, a\rangle\}$ is the only set satisfying $\phi(g, 0)$.
Suppose true for α. Let g be the unique set satisfying $\phi(g, \alpha)$. (Note $g : \alpha+1 \to V^*$.) Certainly $g^* = g\cup\{\langle\alpha+1, F(g(\alpha))\rangle\}$ satisfies $\phi(g^*, \alpha+1)$. If g' also satisfied $\phi(g', \alpha + 1)$, then $\phi(g'\restriction\alpha + 1, \alpha)$ holds, so by the inductive hypothesis $g = g'\restriction\alpha + 1$. But $\phi(g', \alpha + 1)$ implies $g'(\alpha + 1) = F(g'(\alpha)) = F(g(\alpha))$. So $g' = g \cup \{\langle\alpha + 1, F(g(\alpha))\rangle\} = g^*$, as required.
Suppose δ is a limit and $\forall \alpha < \delta\exists! g\phi(g, \alpha)$. For given $\alpha < \delta$ let the unique g be g_α. Notice that $S = \{g_\alpha : \alpha < \delta\}$ is a set by Replacement. But $\alpha_1 < \alpha_2$ implies $g_{\alpha_1} = g_{\alpha_2}\restriction\alpha_1 + 1$. Let $g^* = \bigcup S$. Then g^* is a function with domain $\{\alpha : \alpha < \delta\} = \delta$, and $\forall \alpha < \delta g^*(\alpha + 1) = F(g^*(\alpha))$ and if β is a limit $< \delta$, then $g^*(\beta) = \bigcup\{g^*(\alpha) : \alpha < \beta\}$ and $g^*(0) = a$. (Since for any $\alpha < \delta$, g^* coincides with g_α on $\alpha+1$, and the g_α's satisfy the condition by the inductive hypothesis.) Further g^* is the only such function by (*).
Now define $g = g^* \cup \{\langle\delta, \bigcup\{g^*(\alpha) : \alpha < \delta\}\rangle\}$. Then g is unique such that $\phi(g, \delta)$.
Now set $G = \{\langle x, \alpha\rangle : \exists g(\phi(g, \alpha) \wedge g(\alpha) = x))$.
Then G satisfies the required conditions since by the lemma for each $\alpha \in On$, $G\restriction\alpha + 1$ is the unique g such that $\phi(g, \alpha)$.
We get uniqueness of G by induction. □

Theorem 3.2.14 *Suppose $F : V^* \to V^*$ and $H : V^* \to V^*$ are class terms. Then there is a unique class term $G : V^* \times \mathbf{On} \to V^*$ such that*

1. $G(x, 0) = H(x)$
2. $G(x, \alpha + 1) = F(x, G(x, \alpha))$
3. $G(x, \delta) = \bigcup_{\alpha<\delta} G(x, \alpha)$ *for δ a limit.*

Some applications:

Definition 3.2.15 *Ordinal addition: Set $F(x, y) = y \cup \{y\}$, $H(x) = x$. We get G such that*

1. $G(x, 0) = x$
2. $G(x, \alpha + 1) = G(x, \alpha) \cup \{G(x, \alpha)\}$
3. $G(x, \delta) = \bigcup_{\alpha<\delta} G(x, \alpha)$.

Suppose $\alpha, \beta \in \mathbf{On}$. Write $\alpha + \beta$ for $G(\alpha, \beta)$. Then:

1. $\alpha + 0 = \alpha$
2. $\alpha + (\beta + 1) = (\alpha + \beta) + 1$
3. $\alpha + \delta = \bigcup_{\beta<\delta} \alpha + \beta$.

Definition 3.2.16 *Ordinal multiplication:*

1. $\alpha.0 = 0$ *(So $H(x) = 0$)*
2. $\alpha.(\beta + 1) = \alpha.\beta + \alpha$ *(So $F(x, y) = y + x$)*
3. $\alpha.\delta = \bigcup_{\beta<\delta} \alpha.\beta$.

Chapter 4

The Cumulative Hierarchy and the consistency of the Axiom of Foundation

4.1

We apply Theorem 3.2.12 with $a = \varnothing$ and $F(x) = \mathbb{P}x$, to obtain the following:

Definition 4.1.1 *We define a class term $V : \mathbf{On} \to V^*$ so that*

1. $V(0) = \varnothing$
2. $V(\alpha + 1) = \mathbb{P}V(\alpha)$, *and*
3. $V(\delta) = \bigcup_{\alpha<\delta} V(\alpha)$ *for δ a limit.*

We write V_α for $V(\alpha)$. Each V_α is a set and we also write V for the class $\{x : \exists \alpha \in \mathbf{On}\, x \in V_\alpha\}$ "$=$" $\bigcup_{\alpha\in\mathbf{On}} V_\alpha$.

Theorem 4.1.2 *For each $\alpha \in \mathbf{On}$,*

1. V_α *is transitive,*
2. $V_\alpha \subseteq V_{\alpha+1}$,
3. $\alpha \in V_{\alpha+1}$.

Proof. Simultaneous induction on α.

$\alpha = 0$ $V_0 = \varnothing$, which is transitive. $V_0 \subseteq V_1$, and $0 = \varnothing \in \{\varnothing\} = V_1$.

Suppose true for α.

(1) Suppose $x \in y \in V_{\alpha+1}$. $V_{\alpha+1} = \mathbb{P}V_\alpha$, so $x \in y \subseteq V_\alpha$, so $x \in V_\alpha$. Since $V_\alpha \subseteq V_{\alpha+1}$ by the inductive hypothesis, we get $x \in V_{\alpha+1}$ as required.

(2) Suppose $x \in V_{\alpha+1}$. Then $x \subseteq V_\alpha$. But $V_\alpha \subseteq V_{\alpha+1}$ by the inductive hypothesis, so $x \subseteq V_{\alpha+1}$. Hence $x \in V_{(\alpha+1)+1}$, as required.

(3) $\alpha \in V_{\alpha+1}$ by hypothesis. So $\alpha \subseteq V_{\alpha+1}$, since $V_{\alpha+1}$ is transitive. Thus $\alpha \cup \{\alpha\} \subseteq V_{\alpha+1}$. Hence $\alpha + 1 = \alpha \cup \{\alpha\} \in V_{(\alpha+1)+1}$, as required.

—Hence the result is true for $\alpha + 1$.

Suppose δ a limit and (1), (2) and (3) are true for all $\alpha < \delta$.

(1) Suppose $x \in y \in V_\delta = \bigcup_{\alpha<\delta} V_\alpha$. Then $x \in y \in V_\alpha$ for some $\alpha < \delta$. So $x \in V_\alpha$ by ind hyp. But $V_\alpha \subseteq V_\delta$, so $x \in V_\delta$.

(2) Suppose $x \in V_\delta$. Since $y \in x \in V_\delta \to y \in V_\delta$, we have $x \subseteq V_\delta$, so $x \in V_{\delta+1}$. Thus $V_\delta \subseteq V_{\delta+1}$.

(3) Now for all $\alpha < \delta$, $\alpha \in V_{\alpha+1}$, by the inductive hypothesis. So $\forall \alpha < \delta \alpha \in V_\delta$ (since $V_{\alpha+1} \subseteq V_\delta$). Thus $\delta \subseteq V_\delta$ (note $\delta = \{\alpha : \alpha < \delta\}$) and so $\delta in \mathbb{P} V_\delta = V_{\delta+1}$, as required. □

Corollary 4.1.3 *(1) V is a transitive class (ie. $x \in y \in V \to x \in V$) containing all the ordinals.*

(2) $\forall \alpha < \beta\, V_\alpha \subseteq V_\beta$.

Theorem 4.1.4 $(V, \in) \models$ *ZF.*

Proof. (Note that $(V, \in)$ is a substructure of $(V^*, \in)$, so for $a, b \in V$, $(V, \in) \models a \in b$ iff $a \in b$, and $(V, \in) \models a = b$ iff $a = b$.)

Extensionality. Suppose $x, y \in V$, and $\langle V, \in\rangle \models \forall t(t \in x \leftrightarrow t \in y)$ (*). We must show $\langle V, \in\rangle \models x = y$, ie $x = y$. Suppose $x \neq y$. Say $a \in x$, $a \notin y$. Since $a \in x \in V$ we have $a \in V$ (by Corollary 4.1.3). But by (*), $\forall t \in V$, $t \in x \leftrightarrow t \in y$. In particular $a \in x \leftrightarrow a \in y$—contradiction.

So $x = y$.

Empty Set. We must show $\langle V, \in\rangle \models \exists x \forall y y \notin x$. Since $\varnothing \in V$, we have $\varnothing \in V$, and clearly $\forall y \in V, \notin \varnothing$.

Pairing. Suppose $a, b \in V$. We must show $\langle V, \in\rangle \models \exists z \forall t(t \in z \leftrightarrow (t = a \vee t = b))$. Let $c = \{a, b\}$. Now by 4.1.3 (ii), there is some α such that $a, b \in V_\alpha$. So $c \subseteq V_\alpha$, so $c \in V_{\alpha+1}$, so $c \in V$. It remains to show $\forall t \in V(t \in c \leftrightarrow (t = a \vee t = b))$, which is clear since this is true $\forall t \in V^*$.

Union. $\langle V, \in\rangle \models$ Unions—exercise.

Power Set. Suppose $a \in V$. We must show $\langle V, \in\rangle \models \exists y \forall t(t \in y \leftrightarrow \forall z(z \in t \to z \in a))$.

Now suppose $a \in V_\alpha$.

Exercise: $\forall \alpha \in On$, if $b \in a \in V_\alpha$, then $b \in V_\alpha$.

It follows that $\forall b \in \mathbb{P}(a)$, $b \in V_\alpha$. Thus $\mathbb{P}(a) \subseteq V_\alpha$, so $\mathbb{P}(a) \in V_{\alpha+1}$. So $\mathbb{P}(a) \in V$. Let $c = \mathbb{P}(a)$.

We show $\langle V, \in\rangle \models \forall t(t \in c \leftrightarrow \forall z(z \in t \to z \in a))$.

So suppose $t \in V$.

$\Rightarrow$): If $\langle V, \in\rangle t \in c$, then $t \in c$, so $t \subseteq a$, ie. $\forall z \in V^*(z \in t \to z \in a)$, thus $\forall z \in V(z \in t \to z \in a)$.

$\Leftarrow$): Suppose $\langle V, \in\rangle \models \forall z(z \in t \to z \in a)$ (*) (ie. $\langle V, \in\rangle \models t \subseteq a$). We show that really, $t \subseteq a$. Suppose $d \in t$. Since $t \in V$, we have $d \in V$ (by 4.1.3 (i)). Hence, by (*), $d \in a$. Thus $t \subseteq a$, so $t \in c$, so $\langle V, \in\rangle \models t \in c$ as required.

[Remark: Won't always be the case that $\mathbb{P}(a)$ in substructure is real $\mathbb{P}(a)$—fudge this for now?]

Infinity. Exercise (Note: $\omega \in V_{\omega+1}$, so $\omega \in V$).

Foundation. Suppose $a \in V$, $a \neq \varnothing$. We must find $b \in a$ such that $b \cap a = \varnothing$.

[Since then $b \in V$, by transitivity, and $\langle V, \in\rangle \vDash \forall y \in by \notin a$.]

Let $x \in a$. Then $x \in V$, so $x \in V_\alpha$ for some α. This shows $\exists \alpha \in On, a \cap V_\alpha \neq \varnothing$. Choose β minimal such that $a \cap V_\beta \neq \varnothing$. Then β is a successor ordinal since, for δ a limit, $a \cap V_\delta = a \cap \bigcup_{\alpha<\delta} V_\alpha = \bigcup_{\alpha<\delta}(a \cap V_\alpha)$, so if $a \cap V_\delta \neq \varnothing$, then $a \cap V_\alpha \neq \varnothing$ for some $\alpha < \delta$.

Say $\beta = \gamma + 1$. Now choose $\beta \in a \cap V_\beta$.

We claim that $b \cap a = \varnothing$. Suppose $x \in a \cap b$. Now $b \in V_\beta$, so $b \subseteq V_\gamma$, so $x \in V_\gamma$. But $x \in a$, so $a \cap V_\gamma \neq \varnothing$—a contradiction to the minimality of β.

Separation. Suppose $\phi(x_1, \ldots, x_n, y)$ is a formula of LST and $a_1, \ldots, a_n \in V$, and $u \in V$. We want $b \in V$ such that

$$\langle V, \in\rangle \vDash \forall y(y \in b \leftrightarrow (y \in u \wedge \phi(a_1, \ldots, a_n, y))).$$

(Give wrong proof.)

Definition 4.1.5 Relativization of formulas *Suppose U is a class, say $U = \{x : \Phi(x)\}$, and $\phi(v_1, \ldots, v_k)$ is a formula of LST. We define the formula $\phi^U(v_1, \ldots, v_k)$ (or $\phi^\Phi(v_1, \ldots, v_k)$), which has the same free variables as ϕ, as follows (by recursion on ϕ):*

1. *If ϕ is $v_i = v_j$ or $v_i \in v_j$, then ϕ^U is just ϕ.*
2. *If ϕ is $\neg\psi$, then ϕ^U is $\neg\psi^U$.*
3. *If ϕ is $(\psi \vee \psi')$, then ϕ^U is $(\psi^U \vee (\psi')^U)$.*
4. *If ϕ is $\forall v_i \psi$, then ϕ^U is $\forall v_i(\Phi(v_i) \to \psi^U)$.*

(We tacitly assume ϕ and Φ have no bound variables in common.)

Lemma 4.1.6 *For any $\phi(v_1, \ldots, v_k)$ and $a_1, \ldots, a_k \in U$, $\langle U, \in\rangle \vDash \phi(a_1, \ldots, a_k)$ iff $\phi^U(a_1, \ldots, a_k)$.*

Proof. Obvious. □

To return to the proof of A5 in $\langle V, \in\rangle$: Suppose $u \in V_\alpha$. Let $b = \{y \in u : \phi^V(a_1, \ldots, a_k, y)\}$. Then $b \subseteq u \in V_\alpha$, so $b \in V_\alpha$ (by an exercise), so $b \in V$.

Suppose $y \in V$.

We want to show $\langle V, \in\rangle \vDash y \in b \leftrightarrow (y \in u \wedge \phi(a_1, \ldots, a_n, y))$.

$\Rightarrow$): Suppose $y \in b$. Then $y \in u$, and $\phi^V(a_1, \ldots, a_n, y)$. Hence, by lemma 4.1.6, $\langle V, \in\rangle \vDash y \in u \wedge \phi(a_1, \ldots, a_n, y)$.

$\Leftarrow$): Suppose $\langle V, \in\rangle \vDash y \in u \wedge \phi(a_1, \ldots, a_n, y)$. Then $y \in u$ and $\phi^V(a_1, \ldots, a_n, y)$ (by 4.1.6), so $y \in b$, as required.

Replacement. Suppose $\phi(x,y)$ is a formula of LST (possibly involving parameters from V).

Suppose $\langle V, \in\rangle \models \forall x, y, y'((\phi(x,y) \wedge \phi(x,y')) \rightarrow y = y')$.

Let $\psi(x,y)$ be $\overbrace{x \in V}^{V(x)} \wedge \overbrace{y \in V}^{V(y)} \wedge \phi^V(x,y)$. [Note $V(x)$ has no parameters.]

Then we have (in V^*) $\forall x, y, y'((\psi(x,y) \wedge \psi(x,y')) \rightarrow y = y')$, by lemma 4.1.6.

Let $s \in V$.

Hence there is a set z such that

$$\forall y(y \in z \leftrightarrow \exists x \in s\psi(x,y)) \qquad (*)$$

(by replacement in V^*). We want to show $z \in V$.

Now by (*), if $y \in z$, then $\exists x \in s\psi(x,y)$, so $\exists x \in s(x \in V \wedge y \in V \wedge \phi^V(x,y)$, so $y \in V$. We want to show $z \in V$.

Thus for each $y \in z$, $\exists \alpha \in On, y \in V_\alpha$.

Let $\chi(u,v)$ be "$u \in z \wedge v$ is the least ordinal such that $u \in V_v$".

Then by replacement in V^*, there is a set S such that

$$\forall v(\exists u \in z(\chi(u,v)) \leftrightarrow v \in S).$$

Clearly S is a set of ordinals, so $\bigcup S$ is an ordinal, β say.

Clearly $\forall y \in z, y \in V_\beta$. Hence $z \subseteq V_\beta$, so $z \in V_{\beta+1}$, so $z \in V$.

We must show $\langle V, \in\rangle \models \forall y(y \in z \leftrightarrow \exists x \in s\phi(x,y))$.

$\Rightarrow$): So suppose $y \in V$ and $y \in z$.

By (*), $\exists x \in s\psi(x,y)$, ie. $\exists x \in s(x \in V \wedge y \in V \wedge \phi^V(x,y))$, so $\langle V, \in\rangle \models \exists x \in s\phi(x,y)$.

$\Leftarrow$): Conversely, if $y \in V$, and $\langle V, \in\rangle \models \exists x \in s\phi(x,y)$, then $\exists x \in S(x \in V \wedge \phi^V(x,y))$, so $\exists x \in s(x \in V \wedge y \in V \wedge \phi^V(x,y))$, ie $\exists x \in s\psi(x,y)$, so by (*), $y \in z$. $\square$

Corollary 4.1.7 *If ZF* is consistent, then so is ZF.*

Proof. If σ is an axiom of ZF, we have shown that ZF$^* \vdash \sigma^V$. Hence if $\sigma_1, \sigma_2, \ldots, \sigma_k$ were a proof of a contradiction from ZF, then (roughly) $\sigma_1^V, \ldots, \sigma_k^V$ could be converted into one from ZF*. $\square$

From now on we assume Foundation, and hence (exercise) that ZF=ZF*.

Chapter 5

Lévy's Reflection Principle

5.1

Theorem 5.1.1 *(Lévy's Reflection Principle, or (LRP)) (ZF—for each individual χ)*

Suppose $\tilde{W} : On \to V$ is a class term, and write W_α for $\tilde{W}(\alpha)$. Suppose $\tilde{W}$ satisfies:

1. *$\alpha < \beta \to W_\alpha \subseteq W_\beta$ $(\forall \alpha, \beta \in On)$*
2. *$W_\delta = \bigcup_{\alpha \in \delta} W_\alpha$ for all limit ordinals δ.*

Let $W = \bigcup_{\alpha \in On} W_\alpha$ $(= \{x : \exists \alpha \in On, x \in W_\alpha\}$, so W is a class; each W_α is a set.)

Suppose $\chi(v_1, \ldots, v_n)$ is a formula of LST (without parameters). Then, for any $\alpha \in On$, there is $\beta \in On$ such that $\beta \geq \alpha$, and such that $\forall a_1, \ldots a_n \in W_\beta$, $\langle W, \in \rangle \vDash \chi(a_1, \ldots, a_n)$ iff $\langle W_\beta, \in \rangle \vDash \chi(a_1, \ldots, a_n)$; ie. for lall $a_1, \ldots, a_n \in W_\beta$, $\chi^W(a_1, \ldots, a_n) \leftrightarrow \chi^{W_\beta}(a_1, \ldots, a_n)$.

Proof. For any formula ϕ of LST, by the *collection of subformulas of ϕ, $SF(\phi)$*, we mean all those formulas that go into the building up of ϕ. Formally

1. $SF(\phi) = \{\phi\}$ if ϕ is of the form $x = y$ or $x \in y$;
2. $SF(\neg\phi) = \{\neg\phi\} \cup SF(\phi)$;
3. $SF(\phi \vee \psi) = \{\phi \vee \psi\} \cup SF(\phi) \cup SF(\psi)$;
4. $SF(\forall x \phi) = \{\forall x \phi\} \cup SF(\phi)$.

Clearly $SF(\phi)$ is a finite colleciton for any formula ϕ, and $\phi \in SF(\phi)$.

Suppose now that S is any finite collection of formulas, which is closed under taking subformulas—ie. if $\phi \in S$, then $SF(\phi) \subseteq S$.

Define $T_S = \{\beta \in On : \forall \chi \in S \forall \mathbf{a} \in W_\beta(\chi^{W_\beta}(\mathbf{a}) \leftrightarrow \chi^W(\mathbf{a})\}$. (Abuse of notation here.) (T_S is a class since S is finite.)

We must show that T_S is unbounded in the ordinals. (LRP follows by taking $S = SF(\chi)$.)

We first show, however, that for any S as above, T_S is a *closed* class of ordinals, ie. it contains all its limits, ie. when X is a subset of T_S, then $\sup X \in T_S$.

We prove this by induction on the total number n of occurrences of connectives in formulas of S. We write this n as $\#S$.

If $n = 0$, then all formulas of S are of the form $x = y$ or $x \in y$ (for variables x and y), so $T_S = On$, so T_S is definitely closed.

Now suppose that $\#S = n + 1$. Let χ be a formula in S with maximal number of connectives.

Let $S' = S \setminus \{\chi\}$. Clearly S' is also closed under taking subformulas and $\#S' \leq n$. Also since $S' \subseteq S$, we have $T_{S'} \subseteq T_S$.

Let $X \subseteq T_S$, and suppose X has no greatest element. Note that $X \subseteq T_{S'}$, so $\sup X \in T_{S'}$ by the inductive hypothesis.

We want to show that $\sup X \in T_S$.

Case 1. χ is $\neg\psi$. Note $\psi \in S'$, so $T_S = T_{S'}$. So $\sup X \in T_S$.

Case 2. χ is $\psi_1 \vee \psi_2$. Then again $\psi_1, \psi_2 \in S'$, so we can easily check $T_S = T_{S'}$, and the result follows by the inductive hypothesis.

Case 3. χ is $\forall v_{n+1}\psi(v_1, \ldots, v_n, v_{n+1})$.

Then $\psi(v_1, \ldots, v_n, v_{n+1}) \in S'$. Let $\eta = \sup X$. Now since X has no greatest element η is a limit ordinal, so $W_\eta = \bigcup_{\alpha<\eta} W_\alpha = \bigcup_{\alpha\in X} W_\alpha$.

But by the inductive hypothesis we have for all $\phi \in S'$, for all $\mathbf{a} \in W_\eta$

$$\phi^{W_\eta}(\mathbf{a}) \leftrightarrow \phi^W(\mathbf{a}) \qquad (*)$$

We clearly only have to show:

$$\forall \mathbf{a} \in W_\eta \chi^{W_\eta}(\mathbf{a}) \leftrightarrow \chi^W(\mathbf{a}). \qquad (\dagger)$$

Now since $X \subseteq T_S$ we have

$$\forall \beta \in X \forall \mathbf{a} \in W_\beta \chi^{W_\beta}(\mathbf{a}) \leftrightarrow \chi^W(\mathbf{a}). \qquad (**)$$

Proof of $\leftarrow$ in ($\dagger$)

Suppose $\mathbf{a} \in W_\eta$ and $\chi^W(\mathbf{a})$. Thus

$$(\forall v_{n+1}\psi(\mathbf{a}, v_{n+1}))^W\text{, ie. } \forall v_{n+1} \in W\psi^W(\mathbf{a}, v_{n+1}).$$

But $W_\eta \subseteq W$, so $\forall v_{n+1} \in W_\eta\psi^W(\mathbf{a}, v_{n+1})$. Let $a_{n+1} \in W_\eta$. Then $\psi^W(\mathbf{a}, a_{n+1})$. But $\psi \in S'$ (since ψ is a subformula of χ different from χ), so by (*) $\psi^{W_\eta}(\mathbf{a}, a_{n+1})$. Since this holds for any $a_{n+1} \in W_\eta$ we have $\forall v_{n+1} \in W_\eta\psi^{W_\eta}(\mathbf{a}, v_{n+1})$, ie. χ^{W_η} as required.

Proof of $\rightarrow$ in ($\dagger$)

Suppose $\mathbf{a} \in W_\eta$ and $\chi^{W_\eta}(\mathbf{a})$. Since $W_\eta = \bigcup_{\alpha\in X} W_\alpha$ we have $\mathbf{a} \in W_\beta$ for some βinX. Now $\forall v_{n+1} \in W_\eta\psi^{W_\eta}(\mathbf{a}(v_{n+1})$. Since $W_\beta \subseteq W_\eta$, we have $\forall v_{n+1} \in W_\beta\psi^{W_\beta}\psi^{W_\eta}(\mathbf{a}, v_{n+1})$. Now let $a_{n+1} \in W_\beta$. Then $\psi^{W_\beta}(\mathbf{a}, a_{n+1})$. Hence by (*), $\psi^W(\mathbf{a}, a_{n+1})$. But $\beta \in X \subseteq T_{S'}$ (and $\psi \in S'$), so $\psi^{W_\beta}(\mathbf{a}, a_{n+1})$. Since $a_{n+1} \in W_\beta$ was arbitrary, we have $\forall v_{n+1} \in W_\beta \psi^{W_\beta}(\mathbf{a}, v_{n+1})$, ie. $\chi^{W_\beta}(\mathbf{a})$. Hence by (**), $\chi^W(\mathbf{a})$ as required.

This completes the proof that T_S is a closed subclass of On, for any finite subcollection S of formulas closed under taking subformulas.**(Isolate this out)**

We now show $\forall \alpha \in On \exists \beta \in On(\beta > \alpha \wedge \beta \in T_S)$.

The proof is again by induction on $\#S$, and the only difficult case is when χ is $\forall v_{n+1}\psi(\mathbf{v}, v_{n+1})$ and $S = S' \setminus \{\chi\}$, S' closed under taking subformulas.

By our inductive hypothesis we have

$$\forall \alpha \exists \beta > \alpha \beta \in T_{S'}. \qquad (***)$$

It remains to show that given any $\alpha \in On, \exists \beta > \alpha \beta \in T_{S'}$, such that $\forall \mathbf{a} \in W_\beta(\chi^{W_\beta}(\mathbf{a}) \leftrightarrow \chi^W(\mathbf{a})$. (For then such a β will be in T_S.)

Let $\alpha \in On$ be given.

Now $\chi(v)$ is $\forall v_{n+1}\psi(v_1, \ldots, v_n, v_{n+1})$.

Define the term $f : On \times V^n \to On$ so that $\forall \gamma \in On \forall a_1, \ldots, a_n \in V$ $f(\gamma, a_1, \ldots, a_n)$ is the least $\theta \in On$ such that $\theta > \gamma$ and $\exists a_{n+1} \in W_\theta$ such that $\neg\psi^W(a_1, \ldots, a_n, a_{n+1})$, if such a θ exists.

Now define the term $F : On \to On$ so that $\forall \gamma \in On$ $F(\gamma)$ is the least $\theta \in T_{S'}$ such that $\theta > \sup\{f(\gamma, a_1, \ldots, a_n) : \langle a_1, \ldots, a_n \rangle \in W_gamma^n\}$. (This last thing is a set by replacement since W_γ^n is. θ exists using (***).)

Notice that for all γ, $F(\gamma) > \gamma$, $F(\gamma) \in T_{S'}$, and if $a_1, \ldots, a_n \in W_\gamma$, and $\forall v_{n+1} \in W_{F(\gamma)}\psi^W(a_1, \ldots, a_n, v_{n+1})$, then $\forall v_{n+1} \in W\psi^W(a_1, \ldots, a_n, v_{n+1})$. (††)

(For otherwise, $\exists a_{n+1} \in W \neg\psi^W(a_1, \ldots, a_n, a_{n+1})$, so for some minimal η, $\exists a_{n+1} \in W^\eta \neg\psi^W(a_1, \ldots, a_n, a_{n+1})$ (since $W = \bigcup_{\eta \in On} W_\eta$), so $F(\gamma) \geq f(\gamma, a_1, \ldots, a_n) \geq \eta$, so $\exists a_{n+1} \in W_{F(\gamma)} \neg\psi^W(a_1, \ldots, a_n, a_{n+1})$ since $W_{F(\gamma)} \supseteq W_\eta$—contradiction.) **(Isolate out as a lemma)**

Now by the recursion theorem on ω define the function $g : \omega \to On$ by

1. $g(0) = F(\alpha)$,
2. $g(n+1) = F(g(n))$;

let $X = \operatorname{ran} g$. Clearly X has no greatest element and $X \subseteq T_{S'}$. Let $\beta = \sup X$. Since $T_{S'}$ is closed (proved above), we have $\beta \in T_{S'}$. We also have $\beta > \alpha$, and:

For all $a_1, \ldots, a_n \in W_\beta$, if $\forall v_{n+1} \in W_\beta\psi^W(a_1, \ldots, a_n, v_{n+1})$, then $\forall v_{n+1} \in W\psi^W(a_1, \ldots, a_n, v_{n+1})$. (****)

Proof. Suppose $a_1, \ldots, a_n \in W_\beta$. Since $W_\beta = \bigcup_{\gamma \in X} W_\gamma$, we have $a_1, \ldots, a_n \in W_\gamma$, for some $\gamma \in X$. Suppose $\forall v_{n+1} \in W_\beta\psi^W(a_1, \ldots, a_n, v_{n+1})$.

Since $F(\gamma) \in X$, and hence $W_{F(\gamma)} \subseteq W_\beta$, we have $\forall v_{n+1} \in W_{F(\gamma)}\psi^W(a_1, \ldots, a_n, v_{n+1})$. Hence by (††) we have $\forall v_{n+1} \in W\psi^W(a_1, \ldots, a_n, v_{n+1})$, as required. □

Now show that (****) implies $\beta \in T_S$ as required (exercise). □

Chapter 6

Gödel's Constructible Universe

6.1

Definition 6.1.1 *For any set a and $n \in \omega$ we define ${}^n a$ to be $\{f : f : n \to a\}$, and ${}^{<\omega}a = \bigcup_{n\in\omega} {}^n a$.*

(Exercise: this is a set.)

We shall construct a class term $G : \omega \times V \times V \to V$ such that

$$\forall n \in \omega \, \forall a, s \in V \, G(m, a, s) \subseteq a.$$

Further to each formula $\psi(v_0, \ldots, v_{n-1}, v_n)$ of LST with free variables amongst $v_0, \ldots, v_n$ (with $n \geq 1$), there will be assigned a number $m \in \omega$ ($m = \ulcorner\psi(v_0, \ldots, v_n)\urcorner$) with the property that for all $a, s \in V$, $G(m, a, s) = \{b \in a : \langle a, \in\rangle \models \psi(s(0), \ldots, s(n-1), b)\}$ if $s \in {}^{<\omega}a$ and $\mathrm{dom} s \geq n$ and $\varnothing$ otherwise.

Definition 6.1.2 *We define the class term* $\mathrm{Def} : V \to V$ *by*

$$\mathrm{Def}(a) = \{G(m, a, s) : m \in \omega, \ s \in {}^{<\omega}a\}.$$

Thus $\mathrm{Def}(a)$ *consists of all the definable (with parameters) subsets of the structure* $\langle a, \in\rangle$.

Definition 6.1.3 *(The constructible hierarchy)*

We define the class term $L : On \to V$ (writing L_α for $L(\alpha)$) by recursion on On as follows:

1. $L_0 = \varnothing$;
2. $L_{\alpha+1} = \mathrm{Def}(L_\alpha)$;
3. $L_\delta = \bigcup_{\alpha<\delta} L_\alpha$ *for limit* δ.

L is called the Constructible Universe.
(explain why AC and CH hold in L.)
Throughout we assume ZF holds in V.

Lemma 6.1.4 *For all α, $\beta \in On$:*

1. $\alpha < \beta \to L_\alpha \subseteq L_\beta$;
2. $\alpha < \beta \to L_\alpha \in L_\beta$;
3. L_β *is transitive;*
4. $L_\beta \subseteq V_\beta$;
5. $On \cap L_\beta = \beta$.

Proof. Fix α. We prove (1)–(5) (simultaneously) by induction on β.

$\beta = 0$: trivial.

The successor case: Suppose (1)–(5) true for β.

(1) Suffices to show $L_\beta \subseteq L_{\beta+1}$. Suppose $x \in L_\beta$. Then $x \subseteq L_\beta$ (by IH(3)). Let $s = \{\langle 0, x\rangle$; then $s \in {}^{<\omega}L_\beta$ and dom$s = 1$. Then $A = G(\ulcorner v_1 \in v_0 \urcorner, L_\beta, s) \in \mathrm{Def}(L_\beta) = L_{\beta+1}$.

Also $A = \{b \in L_\beta : \langle L_\beta, \in\rangle \vDash b \in s(0)\} = \{b \in L_\beta : b \in x\} = x$ (since $x \subseteq L_\beta$).

Thus $x \in L_{\beta+1}$ as required.

(2) Suffices to show (by (1)) that $L_\beta \in L_{\beta+1}$. (Since if $\alpha < \beta$ then $L_\alpha \in L_\beta$ (by IH) and $L_\beta \subseteq L_{\beta+1}$ (by (1)).

Must show that $L_\beta \in \mathrm{Def}(L_\beta)$.

Let $s = \varnothing$. Then $G(\ulcorner v_1 = v_0 \urcorner, L_\beta, s) = \{b \in L_\beta : \langle L_\beta, \in\rangle = b = b\} = L_\beta$, so $L_\beta \in \mathrm{Def}(L_\beta)$, as required.

(3) If $x \in L_{\beta+1}$, then $x \subseteq L_\beta$. But $L_\beta \subseteq L_{\beta+1}$, by (1), so $x \subseteq L_{\beta+1}$. Thus $L_{\beta+1}$ is transitive.

(4) By IH $L_\beta \subseteq V_\beta$.

Also $x \in L_{\beta+1} \to x \subseteq L_\beta \to x \subseteq V_\beta \to x \in \mathbb{P}V_\beta = V_{\beta+1}$.

Thus $L_{\beta+1} \subseteq V_{\beta+1}$.

(5) By IH $On \cap L_\beta = \beta$.

Suppose $x \in On \cap L_{\beta+1}$. Then $x \in On$ and $x \subseteq L_\beta$.

But every member of x is an ordinal, so $x \subseteq L_\beta \cap On$, so $x \subseteq \beta$. Thus either $x \in \beta$ or $x = \beta$. In either case $x \in \beta \cup \{\beta\} = \beta + 1$. Thus $On \cap L_{\beta+1} \subseteq \beta + 1$.

Suppose $x \in \beta+1$. Then either $x \in \beta$, in which case $x \in On \cap L_\beta \subseteq On \cap L_{\beta+1}$ (by (1)), or $x = \beta$. So it remains to show $\beta \in L_{\beta+1}$.

Let $s = \varnothing$.

Then $A = G(\ulcorner On(v_0)\urcorner, L_\beta, s) = \{b \in L_\beta : \langle L_\beta, \in\rangle \vDash On(b)\}$, and $A \in \mathrm{Def}(L_\beta) = L_{\beta+1}$. We show $A = \beta$.

But $On(v_0)$ is a Σ_0-formula **(DEFINE THIS CONCEPT BEFORE NOW IF WE REALLY NEED IT)** (exercise this week) and hence absolute between transitive classes.

Thus $\forall b \in L_\beta$, $\langle L_\beta, \in\rangle \vDash On(\beta)$ iff $b \in On$.

Thus $A = L_\beta \cap On = \beta$ by IH, as required.

The Limit Step Suppose $\delta > 0$ is a limit ordinal and (1)–(5) hold for all $\beta < \delta$. Since $L_\delta = \bigcup_{\beta<\delta} L_\beta$, (1)–(5) for δ are all easy. □

Lemma 6.1.5 *For all $n \in \omega$, $L_n = V_n$.*

Proof. By induction on n.

For $n = 0$, this is clear.

Suppose now that $L_n = V_n$.

Now $L_{n+1} \subseteq V_{n+1}$ by 6.1.4.

Suppose $x \in V_{n+1}$. Then $x \subseteq V_n$, so x is finite. Also $x \subseteq L_n$ by IH. Say $x = \{a_0, \ldots, a_{k-1}\}$ ($k \in \omega$), so that $a_0, \ldots, a_{k-1} \in L_n$.

Let $s = \{\langle 0, a_0\rangle, \ldots, \langle k-1, a_{k-1}\rangle\}$, so $s \in {}^k L_n$.

Let $A = G(\ulcorner(v_k = v_0 \vee \cdots \vee v_k = v_{k-1}\urcorner, L_n, s) = \{b \in L_n : \langle L_n, \in\rangle \models (b = a_0 \vee \cdots \vee b = a_{k-1})\} = \{a_0, \ldots, a_{k-1}\} = x$.

Thus $x \in \mathrm{Def}(L_n) = L_{n+1}$.

Thus $V_{n+1} \subseteq L_{n+1}$.

So $V_{n+1} = L_{n+1}$. □

Lemma 6.1.6 *Suppose $a, c \in L$. Then*

1. *$\{a, b\} \in L$.*
2. *$\bigcup a \in L$.*
3. *$(\wp Pa \cap L) \in L$.*

Proof. (1) Suppose $a, c \in L_\alpha$. Define $s = \{\langle 0, a\rangle, \langle 1, c\rangle\}$, so $s \in {}^{<\omega} L_\alpha$.

Then $L_{\alpha+1} \ni G(\ulcorner v_2 = v_0 \vee v_2 = v_1\urcorner, L_alpha, s) = \{b \in L_\alpha : \langle L_\alpha, \ni\rangle \models b = a \vee b = c\} = L_\alpha \cap \{a, c\} = \{a, c\}$.

So $\{a, c\} \in L_{\alpha+1} \subseteq L$.

(2) Suppose $a \in L_\alpha$. Let $s = \{\langle 0, a\rangle\}$. Then $L_{\alpha+1} \ni G(\ulcorner \exists v_2 \in v_0(v_1 \in v_2)\urcorner, L_\alpha, s) = \{b \in L_\alpha : \langle L_\alpha, \in\rangle \models \exists v_2 \in a(b \in v_2)\} = A$, say.

We claim that $A = \bigcup a$.

Suppose that $b \in A$.

Then $\langle L_\alpha, \in\rangle \models \exists v_2 \in a(b \in v_2)$.

Say $d \in L_\alpha$ is such that $\langle L_\alpha, \in\rangle \models d \in a \wedge b \in d$.

Then $d \in a \wedge b \in d$, so $b \in \bigcup a$.

Conversely, suppose $b \in \bigcup a$. Then for some $d \in a$, $b \in d$. But L_α is transitive, and $a \in L_\alpha$, so $d \in L_\alpha$, and hence $b \in L_\alpha$.

So $\langle L_\alpha, \in\rangle \models d \in a \wedge b \in d$. Hence $\langle L_\alpha, \in\} \models \exists v_2 \in a(b \in v_2)$ (and $b \in L_\alpha$) so $b \in A$ as required.

Thus $\bigcup a \in L_{\alpha+1} \in L$.

(3) Let $f : \mathbb{P}a \to On$ be defined so that $f(x)$ is the least α such that $x \in L_\alpha$ if there is one, $f(x) = 0$ otherwise.

Then by replacement $\mathrm{ran}\, f$ is a set, and hence $\exists \beta \in On$ such that $\beta > \alpha$ for all $\alpha \in \mathrm{ran}\, f$.

Clearly $\mathbb{P}a \cap L \subseteq L_\beta$ (using 6.1.4 (1)).

We may also suppose that $a \in L_\beta$.

Let $s = \{\langle 0, a\rangle\}$.

Then $L_{\beta+1} \ni G(\ulcorner \forall v_2 \in v_1(v_2 \in v_0)\urcorner, L_\beta, s) = \{b \in L_\beta : \langle L_\beta, \in\rangle \models \forall v_2 \in b(v_2 \in a)\} = A$, say.

Suffices to show $A = \mathbb{P}a \cap L$.

Suppose $b \in A$. Then $b \in L_\beta$ (so $b \in L$) and $\langle L_\beta, \in\rangle \models \forall v_2 \in b(v_2 \in a)$.

Now suppose $d \in b$. Then $d \in L_\beta$ since L_β is transitive. Hence $\langle L_\beta, \in\rangle \models d \in b \wedge d \in a$, so $d \in a$.

Hence $b \subseteq a$, so $b \in \mathbb{P}a \cap L$. Thus $A \subseteq \mathbb{P}a \cap L$.

Conversely suppose $b \in \mathbb{P}a \cap L$. Then $b \in L_\beta$.

Also $\forall v_2 \in b(v_2 \in a)$. Hence $\forall v_2 \in L_\beta(v_2 \in b \to v_2 \in a)$, so $\langle L_\beta, \in\rangle \models \forall v_2 \in b(v_2 \in a)$.

So $b \in A$.

Hence $\mathbb{P}a \cap L = A$. $\square$

It is now easy to check that

Corollary 6.1.7 *Extensionality, empty-set, pairs, unions, power-set, infinity are all true in L (tho' PS is less easy).*

Lemma 6.1.8 $\langle L, \in\rangle \models$ *separation.*

Proof. Suppose $u \in L$, and $a_0, \dots, a_n \in L$. Say $u, a_0, \dots, a_n \in L_\alpha$. Let $\phi(v_0, \dots, v_{n+1})$ be a formula of LST. By Lévy's Reflection Principle, there is some $\beta \geq \alpha$ such that $\forall c, c_1, \dots, c_{n+1} \in L_\beta$

$$\langle L_\beta, \in\rangle \models (c \in c_{n+1} \wedge \phi(c_0, \dots, c_n, c)) \Leftrightarrow \langle L, \in\rangle \models (c \in c_{n+1} \wedge \phi(c_0, \dots, c_n, c)). \ (*)$$

Let $\psi(v_0, \dots, v_{n+2}) = (v_{n+2} \in v_{n+1} \wedge \phi(v_0, \dots, v_n, v_{n+2})$.

Let $s = \{\langle 0, a_0\rangle, \dots, \langle n, a_n\rangle, \langle n+1, u\rangle\}$.

Then $L_{\beta+1} \ni G(\ulcorner \psi(v_0, \dots, v_{n+2})\urcorner, L_\beta, s) = \{b \in L_\beta : \langle L_\beta, \in\rangle \models \psi(a_0, \dots, a_n, u, b)\} = \{b \in L_\beta : \langle L_\beta, \in\rangle \models (b \in u \wedge \phi(a_0, \dots, a_n, b)\} = A$, say. (So $A \in L$.)

Sufficient to show $\langle L, \in\rangle \models \forall x(x \in A \leftrightarrow (x \in u \wedge \phi(a_0, \dots, a_n, x)))$.

$\Rightarrow$): Suppose $x \in L$ and $x \in A$. Then $x \in L_\beta$, and $\langle L_\beta, in\rangle \models x \in u \wedge \phi(a_0, \dots, a_n, x)$.

By (*), $\langle L, \in\rangle \models x \in u \wedge \phi(a_0, \dots, a_n, x)$, as required.

$\Leftarrow$): Suppose $x \in L$, and $x \in u \wedge phi(a_0, \dots, a_n, x)$. Then $x \in L_\beta$, since $x \in L_\beta$ and L_β is transitive. Hence, using (*), $(L_\beta, \in\rangle \models x \in u \wedge \phi(a_0, \dots, a_n, x)$, so $x \in A$, as required. $\square$

Lemma 6.1.9 $\langle L, \in\rangle \models$ *replacement.*

Proof. Suppose $a_0, \dots, a_n \in L$, $\mathbf{a} = \langle a_0, \dots, a_n\rangle$, $u \in L$, $\phi(\mathbf{x}, y, z)$ a formula of LST, and $\langle L, \in\rangle \models \underbrace{\forall z, y, y'((\phi(\mathbf{a}, z, y) \wedge \phi(\mathbf{a}, z, y')) \to y = y')}_{\sigma}$.

Now choose β so large that $a_0, a_1, \ldots, a_n, u \in L_\beta$, and such that (by LRP) for all $z \in L_\beta$ $\langle L, \in\rangle \vDash \sigma \wedge \exists y(\phi(\mathbf{a}, z, y) \wedge z \in u) \Leftrightarrow \langle L_\beta, \in\rangle \vDash \sigma \wedge \exists y(\phi(\mathbf{a}, z, y) \wedge z \in u)$, and for all $c, d \in L_\beta$, $\langle L, \in\rangle \phi(\mathbf{a}, c, d)$ iff $\langle L_\beta, \in\rangle \vDash \phi(\mathbf{a}, c, d)$.

Now let $A = \{b \in L_\beta : \langle L_\beta, \in\rangle \vDash \exists z \in u(\phi(\mathbf{a}, z, b)\}$, so $A \in L_{\beta+1}$.

Then, as in the proof of separation, $\langle L, \in\rangle \vDash \forall z \in u(\exists y \phi(\mathbf{a}, z, y) \leftrightarrow \exists y \in A(\phi(\mathbf{a}, z, y))$, as required. $\square$

Lemma 6.1.10 $\langle L, \in\rangle \vDash$ *Foundation.*

Proof. Suppose $a \in L$. Choose $b \in V$ such that $b \in a \wedge b \cap a = \varnothing$. Since L is transitive, $b \in L$ and clearly $\langle L, \in\rangle \vDash b \in a \wedge b \cap a = \varnothing$. $\square$

Theorem 6.1.11 $\langle L, \in\rangle \vDash$ *ZF.*

Chapter 7

Absoluteness

7.1

Definition 7.1.1 *The* Σ_0-formulas *of LST are defined as follows:*

1. *$x \in y$, $x = y$, $\neg x \in y$, $\neg x = y$ are Σ_0-formulas for any variables x and y.*
2. *If ψ, ϕ are Σ_0-formulas, so are $\psi \wedge \phi$, $\psi \vee \phi$, $\forall x \in y \phi$ and $\exists x \in y \phi$ (where x and y are distinct variables).*
3. *Nothing else is a Σ_0 formula.*

Lemma 7.1.2 *If ϕ is a Σ_0 formula, then $\neg\phi$ is logically equivalent to a Σ_0 formula.*

Proof. Easy induction on ϕ. Note that $\neg\forall x \in y\phi$ is logically equivalent to $\exists x \in y \neg\phi$. □

Lemma 7.1.3 *If $\phi(x_1, \ldots, x_n)$ is a Σ_0-formula and U_1 and U_2 are* transitive *classes such that $U_1 \subseteq U_2$, then for all $a_1, \ldots, a_n \in U_1$,*

$$\langle U, \in\rangle \models \phi(a_1, \ldots, a_n) \Leftrightarrow \langle U_2, \in\rangle \models \phi(a_1, \ldots, a_n).$$

We say ϕ is absolute *between U_1 and U_2.*

Proof. Exercise—induction on ϕ. □

Definition 7.1.4 *The* Σ_1-formulas *of LST are defined as follows:*

1. *$x \in y$, $x = y$, $\neg x \in y$, $\neg x = y$ are Σ_1-formulas for any variables x and y.*
2. *If ψ, ϕ are Σ_1-formulas, so are $\psi \wedge \phi$, $\psi \vee \phi$, $\forall x \in y \phi$ and $\exists x \in y \phi$ (where x and y are distinct variables), and $\exists x \phi$.*
3. *Nothing else is a Σ_1 formula.*

Remark 7.1.5 *Note that every Σ_0 formula is Σ_1.*

Lemma 7.1.6 *If $\phi(x_1, \ldots, x_n)$ is a Σ_1-formula, and U_1 and U_2 are transitive classes with $U_1 \subseteq U_2$, then for all $a_1, \ldots, a_n \in U_1$*

$$\langle U_1, \in \rangle \models \phi(a_1, \ldots, a_n) \Rightarrow \langle U_2, \in \rangle \models \phi(a_1, \ldots, a_n).$$

(ie. ϕ is preserved up *or is* upward absolute *between U_1 and U_2.)*

Definition 7.1.7 *(1) A formula $\phi(\mathbf{x})$ is called Σ_0^{ZF} (respectively Σ_1^{ZF}) if there is a Σ_0 (or Σ_1) formula $\psi(\mathbf{x})$ such that ZF$\vdash \forall \mathbf{x}(\phi(\mathbf{x}) \leftrightarrow \psi(\mathbf{x}))$.*

(2) A formula ϕ is called Δ_1^{ZF} if ϕ and $\neg\phi$ are Σ_1^{ZF}.

(3) Suppose $n \in \omega$ and $F : V^n \to V$ is a class term. Then F is called Δ_1^{ZF} if the formula $\phi(x_1, \ldots, x_n, x_{n+1})$ defining $F(x_1, \ldots, x_n) = x_{n+1}$ is Δ_1^{ZF}, and if ZF proves that F is a class term.

Remark 7.1.8 *We need only verify that ϕ in part (3) is Σ_1^{ZF}, since $\neg\phi$ is Σ_1^{ZF} thus:*

$$ZF \vdash \forall x_1, \ldots, x_n, x_{n+1}(\neg\phi(x_1, \ldots, x_n, x_{n+1}) \leftrightarrow \exists y(\phi(x_1, \ldots, x_n, y) \wedge \neg y = x_{n+1}))$$

—and the bit on the right is Σ_1^{ZF} if ϕ is.

Remark 7.1.9 *Every Σ_0^{ZF} formula is Δ_1^{ZF} by 7.1.2 and 7.1.5.*

Theorem 7.1.10 *Suppose $\phi(x_1, \ldots, x_n)$ is Δ_1^{ZF} and U_1 and U_2 are* transitive *classes such that $U_1 \subseteq U_2$ and $\langle U_i, \in \rangle \models$ ZF $(i = 1, 2)$. Then for all $a_1, \ldots, a_n \in U_1$,*

$$\langle U, \in \rangle \models \phi(a_1, \ldots, a_n) \Leftrightarrow \langle U_2, \in \rangle \models \phi(a_1, \ldots, a_n).$$

(ie. ϕ is ZF-absolute.*)*

Proof. Let $\psi(x_1, \ldots, x_n)$ be Σ_1 such that ZF$\vdash \forall \mathbf{x}(\phi(\mathbf{x}) \leftrightarrow \psi(\mathbf{x})$ (*).

Then

$$\begin{aligned} \langle U_1, \in \rangle \models \phi(\mathbf{a}) &\Rightarrow \langle U_1, \in \rangle \models \psi(\mathbf{a}) && (*) \text{ and } \langle U_1, \in \rangle \models \text{ZF} \\ &\Rightarrow \langle U_2, \in \rangle \models \psi(\mathbf{a}) && \text{by 7.1.6} \\ &\Rightarrow \langle U_2, \in \rangle \models \phi(\mathbf{a}) && (*) \text{ and } \langle U_1, \in \rangle \models \text{ZF} \end{aligned} \tag{7.1}$$

Now let $\chi(x_1, \ldots, x_n)$ be Σ_1 such that ZF$\vdash \forall \mathbf{x}(\neg\phi(\mathbf{x}) \leftrightarrow \psi(\mathbf{x})$ (*).

Then as above,

$$\begin{aligned} \langle U_1, \in \rangle \models \neg\phi(\mathbf{a}) &\Rightarrow \langle U_1, \in \rangle \models \chi(\mathbf{a}) && (*) \text{ and } \langle U_1, \in \rangle \models \text{ZF} \\ &\Rightarrow \langle U_2, \in \rangle \models \chi(\mathbf{a}) && \text{by 7.1.6} \\ &\Rightarrow \langle U_2, \in \rangle \models \neg\phi(\mathbf{a}) && (*) \text{ and } \langle U_1, \in \rangle \models \text{ZF} \end{aligned} \tag{7.2}$$

□

Theorem 7.1.11 *The following formulas and class terms are all Σ_0^{ZF} (and hence Δ_0^{ZF}):*

1. $x = y$
2. $x \in y$
3. $x \subseteq y$
4. $F(x_1, \ldots, x_n) = \{x_1, \ldots, x_n\}$ *(for each n)*
5. $F(x_1, \ldots, x_n) = \langle x_1, \ldots, x_n \rangle$ *(for each n)*
6. *(where* $n \geq 1$ *and* $0 \leq i \leq n-1$*)* $F(x) = x_i$ *if* x *is an* n*-tuple* $\langle x_0, \ldots, x_{n-1} \rangle$, $\varnothing$ *otherwise.*
7. $F(x, y) = x \cup y$.
8. $F(x, y) = x \cap y$.
9. $F(x) = \bigcup x$.
10. $F(x) = \bigcap x$ *if* $x \neq \varnothing$, $F(x) = \varnothing$ *otherwise.*
11. $F(x, y) = x \setminus y$.
12. x *is an* n*-tuple.*
13. x *is an* n*-ary relation on* y.
14. x *is a function.*
15. $F(x) = \mathrm{dom} x$ *if* x *is a function,* $\varnothing$ *otherwise.*
16. $F(x) = \mathrm{ran}\, x$ *if* x *is a function,* $\varnothing$ *otherwise.*
17. $F(x, y) = x[y]$ $(= \{x(t) : t \in y\})$ *if* x *is a function,* $\varnothing$ *otherwise.*
18. $F(x, y) = x \restriction y$ *if* x *is a function,* $\varnothing$ *otherwise.*
19. $F(x) = x^{-1}$ *if* x *is a function,* $\varnothing$ *otherwise.*
20. $F(x) = x \cup \{x\}$.
21. x *is transitive.*
22. x *is an ordinal.*
23. x *is a successor ordinal.*
24. x *is a limit ordinal.*
25. $x : y \to z$.
26. $x : y \sim z$.

27. x is a natural number.

28. $x = \omega$.

29. x is a finite sequence of elements of y.

Proof. (Selections) (3) $x \subseteq y \Leftrightarrow \forall z \in x(z \in y)$ which is Σ_0.

Note that all the class terms F above are in ZF provably class terms, so we only have to show that the statement $F(\mathbf{x}) = y$ can be put in Σ_0 form.

(4) $F(x_1, \ldots, x_n) = y \Leftrightarrow x_1 \in y \wedge x_2 \in y \wedge \ldots \wedge x_n \in y \wedge \forall z \in y(z = x_1 \vee \ldots \vee z = x_n)$.

(5) $F(x_1, x_2) = y \Leftrightarrow \exists z_1 \in y \exists z_2 \in y(z_1 = \{x_1\} \wedge z_2 = \{x_1, x_2\} \wedge \forall t \in y(t = z_1 \vee t = z_2))$, which is Σ_0 by (4).

(12) x is a 2-tuple iff $\exists z_1 \in x \exists x_1 \in z_1 \exists x_2 \in z_1(x = \langle x_1, x_2 \rangle)$, which is Σ_0 by (5).

(13) x is a 2-ary relation on y iff $\forall z \in x \exists y_1 \in y \exists y_2 \in y(z = \langle y_1, y_2 \rangle)$, which is Σ_0 by (5).

(29) x is a natural number iff (x is an ordinal)$\wedge$(x is not a limit ordinal)$\wedge$($\forall y \in x$ y is not a limit ordinal), which is Σ_0 by (24), (26) and the fact that Σ_0^{ZF} formulas are closed under $\neg$. $\square$

Lemma 7.1.12 *Suppose F and G are Δ_1^{ZF} class terms. Then "$F(\mathbf{x}) = G(\mathbf{y})$" is Δ_1^{ZF}.*

Proof. Let $\psi(\mathbf{x}, z)$ and $\chi(\mathbf{y}, t)$ be Σ_1 formulas defining (in ZF) $F(\mathbf{x}) = y$ and $G(\mathbf{y}) = t$ respectively. Then

$$F(\mathbf{x}) = G(\mathbf{y}) \underbrace{\Leftrightarrow}_{ZF} \exists z(\psi(\mathbf{x}, z) \wedge \chi(\mathbf{y}, z)),$$

which is Σ_1, and

$$F(\mathbf{x}) \neq G(\mathbf{y}) \underbrace{\Leftrightarrow}_{ZF} \exists z \exists t(\psi(\mathbf{x}, z) \wedge \chi(\mathbf{y}, t) \wedge \neg z = t),$$

which is Σ_1.

Hence "$F(\mathbf{x}) = G(\mathbf{y})$" is Δ_1^{ZF}. $\square$

Theorem 7.1.13 *Suppose $F : V \times V \to V$ is a Δ_1^{ZF} class term. Then the class term G defined from F by recursion on On, ie:*

1. $G(0, x) = x$
2. $G(\alpha + 1, x) = F(G(\alpha, x), x)$ *for all* $\alpha \in On$
3. $G(\delta, x) = \bigcup_{\alpha < \delta} G(\alpha, x)$ *for all limit* $\delta \in On$
4. $G(y, x) = \varnothing$ *for all* $y \notin On$

is Δ_1^{ZF}.

Proof. As in the proof of 3.2.12 define $\phi(g, \alpha, x)$ by

$$\begin{array}{clc}
 & On(\alpha) & \chi_1 \\
\wedge & g \text{ is a function} & \chi_2 \\
\wedge & \mathrm{dom} g = \alpha \cup \{\alpha\} & \chi_3 \\
\wedge & g(0) = x & \chi_4 \\
\wedge & \forall \beta \in \alpha \exists y_1 \exists y_2 (y_1 = \beta \cup \{\beta\} \wedge y_2 = g(\beta) \wedge g(y_1) = F(y_2)) & \chi_5 \\
\wedge & \forall \beta \in \alpha (\beta \text{ is a limit ordinal} \rightarrow g(\beta) = \bigcup \{g(\alpha) : \alpha \in \beta\}). & \chi_6
\end{array} \tag{7.3}$$

χ_1 is Σ_0^{ZF} by 7.1.11 (24); χ_2 is Σ_0^{ZF} by (14); χ_3 is by (15), (22) and 7.1.12; χ_4 can be rewritten as $\exists y((\forall z \in y(\neg z \in z) \wedge g(y) = x)$ so is Σ_1^{ZF} by (17); χ_5 is Σ_1^{ZF} by (22), (17) and the fact that F is Σ_1^{ZF}, and using 7.1.12; χ_6 is Σ_1^{ZF} by (26) and the fact that "$g(\beta) = \bigcup \{g(\alpha) : \alpha \in \beta\}$" is equivalent to $\exists y \exists z (y = g[\beta] \wedge z = \bigcup y \wedge g(\beta) = z)$, which is Σ_1^{ZF} by (18), (9) and (17).

Hence $\phi(g, \alpha, x)$ is Σ_1^{ZF}.

Now recall from the proof of 3.2.12 that G can be defined by:

$$G(\alpha, x) = y \Leftrightarrow \exists g(\phi(g, \alpha, x) \wedge g(\alpha) = y) \vee (\neg On(\alpha) \wedge y = \varnothing).$$

This shows G is Σ_1^{ZF}, and hence Δ_1^{ZF} by 7.1.8. $\square$

Corollary 7.1.14 *Assuming the class term G (from the beginning of section 6) is Δ_1^{ZF}, then so is the class term $\bar{L} : On \to V$. (Strictly $\bar{L} : V \to V$, where $\bar{L}(x) = \varnothing$ if $x \notin On$.)*

Proof. By 7.1.13 it is sufficient to show Def is Δ_1^{ZF}. Recall that Def : $V \to V$ is defined by

$$\mathrm{Def}(a) = \{G(m, a, s) : m \in \omega, s \in {}^{<\omega}a\}.$$

Hence $\mathrm{Def}(a) = y$ iff $\exists w \exists x(w = \omega \wedge x = {}^{<\omega}a \wedge \forall m \in w \forall s \in x \exists t(t = G(m, a, s) \wedge t \in y)) \wedge \forall t \in y \exists m \in w \exists s \in x(t = G(m, a, s)))$.

Now $x = {}^{<\omega}a$ is Δ_1^{ZF} , so Def is Σ_1^{ZF} by 7.1.11 (29), (30), (31), and because G is.

Hence Def is Δ_1^{ZF} by 7.1.8. □

Definition 7.1.15 *V=L is the sentence of LST: $\forall x \exists \alpha(On(\alpha) \wedge x \in \bar{L}(\alpha))$ (writing L_α for $\bar{L}(\alpha)$).*

Theorem 7.1.16 $\langle L, \in\rangle \vDash$ *V=L.*

Proof. Suppose $a \in L$. We must show $\langle L, \in\rangle \vDash \exists \alpha(On(\alpha) \wedge a \in \bar{L}(\alpha))$. Now choose α such that $a \in L_\alpha$, ie. $\langle V, \in\rangle \vDash\ \in \bar{L}(\alpha)$.

Let X be the set $\bar{L}(\alpha)$ (ie. L_α). Then $X \in L_{\alpha+1}$ by 6.1.4 (2). Hence $X \in L$. Since $\langle V, \in\rangle \vDash a \in X$ we have $\langle L, \in\rangle \vDash a \in X$. Now $\langle V, \in\rangle \vDash \mathbf{On}(\alpha) \wedge X = \bar{L}(\alpha)$. But the formula "$x = \bar{L}(y)$" is Δ_1^{ZF}, and $On(\alpha)$ is Δ_1^{ZF}, so by 7.1.10 (since $\alpha, X \in L$),

$$\langle L, \in\rangle \vDash \mathbf{On}(\alpha) \wedge X = \bar{L}(\alpha) \wedge a \in X.$$

Hence $\langle L, in\rangle \vDash \exists \alpha \exists x(\mathbf{On}(\alpha) \wedge x = \bar{L}(\alpha) \wedge a \in x)$, so $\langle L, \in\rangle \vDash \exists \alpha(\mathbf{On}(\alpha) \wedge a \in \bar{L}(\alpha))$, as required. □

Corollary 7.1.17 *If ZF is consistent, so is ZF+V=L.*

(Same argument as for Foundation.)

Later we'll show ZF+V=L$\vdash$AC, GCH.

Chapter 8

Gödel numbering and the construction of Def

8.1

Notation 8.1.1 *If we say "$F : U_1 \times \cdots \times U_n \to V$ is a Δ_1^{ZF} term" we mean that the classes $U_1, \ldots, U_n$ are Δ_1^{ZF} (ie. defined by Δ_1^{ZF} formulas) and that "$F(x_1, \ldots, x_n) = y$" can be expressed by a Σ_1 formula.*

This clearly guarantees that the extension $F' : V^n \to V$ of F defined by $F'(x_1, \ldots, x_n) = F(x_1, \ldots, x_n)$ if $x_1 \in U_1$, …,$x_n \in U_n$ and $= \varnothing$ otherwise, is Δ_1^{ZF} in the sense given.)

Definition 8.1.2 *We first define $F : \omega^3 \to \omega$ by $F(n, m, l) = 2^n 3^n 5^l$. Then F is injective and easily seen to be Δ_1^{ZF}. Write $[n, m, l]$ for $F(n, m, l)$. We now define $\ulcorner\phi\urcorner$ by induction on ϕ:*

$$\begin{array}{rcl} \ulcorner v_i = v_j \urcorner & = & [0, i, j]; \\ \ulcorner v_i \in v_j \urcorner & = & [1, i, j]; \\ \ulcorner \phi \vee \psi \urcorner & = & [2, \ulcorner\phi\urcorner, \ulcorner\psi\urcorner]; \\ \ulcorner \neg\phi \urcorner & = & [3, \ulcorner\phi\urcorner, \ulcorner\phi\urcorner]; \\ \ulcorner \forall v_i \phi \urcorner & = & [4, i, \ulcorner\phi\urcorner]. \end{array} \tag{8.1}$$

Of course this definition does not take place in ZF and is not actually used in the following definition of Def. However it should be borne in mind in order to see what's going on.

Definition 8.1.3 *Define the class term* Sub $: V^4 \to V$ *by* $\mathrm{Sub}(a, f, i, c) = f(c/i)$ *if $f \in {}^{<\omega}a$, $c \in a$ and $i \in \omega$ and $= \varnothing$ otherwise; where if $f \in {}^{<\omega}a$, $c \in a$ and $i \in \omega$, $f(c/i) \in {}^{<\omega}a$ is defined by* $\mathrm{dom}(f(c/i)) = \mathrm{dom} f$, *and for $j \in \mathrm{dom} f$, $f(c/i)(j) = f(j)$ if $j \neq i$, and c if $j = i$.*

Lemma 8.1.4 Sub *is* Δ_1^{ZF}.

We now define a class term $Sat : \omega \times V \to V$. The idea is that if $m \in \omega$ and $m = \ulcorner \phi(v_0, \ldots, x_{n_1}) \urcorner$, for some formula ϕ of LST, and $a \in V$, then

$$Sat(m, a) = \{f \in {}^{<\omega}a : \mathrm{dom} f \geq n \wedge \langle a, \in \rangle \models \phi(f(0), \ldots, f(n-1))\}. \qquad (*)$$

We simply mimic the definition of satisfaction from predicate logic. (This definition uses a version of the recursion theorem which is slightly different from the usual one, and which I give later.)

Definition 8.1.5 *Firstly if $a \in V$, $m \in \omega$ but m is not of the form $[i, j, k]$, for any $i, j, k \in \omega$ with $i < 5$, then* $\mathrm{Sat}(m, a) = \varnothing$. *Otherwise, if $a \in V$ and $m = [i, j, k]$ with $i < 5$, then*

$$\begin{aligned}
\mathrm{Sat}([0, j, k], a) &= \{f \in {}^{<\omega}a : j, k \in \mathrm{dom} f \wedge f(j) = f(k)\}. \\
\mathrm{Sat}([1, j, k], a) &= \{f \in {}^{<\omega}a : j, k \in \mathrm{dom} f \wedge f(j) \in f(k)\}. \\
\mathrm{Sat}([2, j, k], a) &= \mathrm{Sat}(j, a) \cup \mathrm{Sat}(k, a). \\
\mathrm{Sat}([3, j, k], a) &= ({}^{<\omega}a \setminus \mathrm{Sat}(j, a)) \cap \{g \in {}^{<\omega}a : \exists f \in \mathrm{Sat}(j, a)\, \mathrm{dom} f \leq \mathrm{dom} g\}. \\
\mathrm{Sat}([4, j, k], a) &= \{f \in {}^{<\omega}a : j \in \mathrm{dom} f \wedge \forall x \in a, \mathrm{Sub}(a, f, j, x) \in \mathrm{Sat}(k, a)\}.
\end{aligned} \tag{8.2}$$

The generalized version of the recursion theorem (on ω) required here is:

Lemma 8.1.6 *Suppose that $\pi_1, \pi_2, \pi_3 : \omega \to \omega$ are Δ_1^{ZF} class terms and $H : V^4 \times \omega \to V$ is a Δ_1^{ZF} class term. Suppose further that $\forall n \in \omega \setminus \{0\}\ \pi_i(n) < n$ for $i = 1, 2, 3$. Then there is a Δ_1^{ZF} class term $F : \omega \times V \to V$ such that*

1. $F(0, a) = 0$
2. *and* $\forall n \in \omega \setminus \{0\}$

$$F(n, a) = H(F(\pi_1(n), (a)), F(\pi_2(n), (a)), F(\pi_3(n), (a)), a, n).$$

(Thus instead of defining $F(n, a)$ in terms of $F(n-1, a)$, we are defining $F(n, a)$ in terms of three specified previous values.)

Proof. Similar to the proof of the usual recursion theorem on ω. □

Thus the definition of Sat in 8.1.5 is an application of 8.1.6 with $\pi_1(n) = i$ if for some $j, k < n$, $[i, j, k] = n$, $= 0$ otherwise; and π_2 and π_3 are defined similarly, picking out j and k respectively from $[i, j, k]$, and with $H : V^4 \times \omega \to V$ defined so that

$$H(x, y, z, a, n) = \begin{cases}
\{f \in {}^{<\omega}a : \pi_2(n), \pi_3(n) \in \mathrm{dom} f \wedge f(\pi_2(n)) = f(\pi_3(n))\} & \text{if } \pi_1(n) = 0, \\
\{f \in {}^{<\omega}a : \pi_2(n), \pi_3(n) \in \mathrm{dom} f \wedge f(\pi_2(n)) \in f(\pi_3(n))\} & \text{if } \pi_1(n) = 1, \\
y \cup z & \text{if } \pi_1(n) = 2, \\
({}^{<\omega}a \setminus y) \cap \{g \in {}^{<\omega}a : \exists f \in y \mathrm{dom} f \leq \mathrm{dom} g\} & \text{if } \pi_2(n) = 3, \\
\{f \in {}^{<\omega}a : \pi_2(n) \in \mathrm{dom} f \wedge \forall x \in a Sub(a, f, \pi_2(n), x) \in z\} & \text{if } \pi_1(n) = 4, \\
0 & \text{otherwise.}
\end{cases}$$

(The F got from this H, π_1, π_2, π_3 (in 8.1.6) is Sat.)

It is completely routine to show that Sat so defined satisfies the required statement (*) (just before 8.1.5)—by induction on ϕ.

Before defining G we must introduce a term that picks out the largest $n \in \omega$ such that "v_n occurs free" in the "formula coded by m".

More formally:

Definition 8.1.7 *We define* $\mathrm{Fr}(m)$ *("the set of i such that v_i occurs free in the formula coded by m") as follows (again using 8.1.6):*

$$\begin{array}{rcl} \mathrm{Fr}([0,i,j]) & = & \{i,j\}; \\ \mathrm{Fr}([1,i,j]) & = & \{i,j\}; \\ \mathrm{Fr}([2,i,j]) & = & \mathrm{Fr}(i) \cup \mathrm{Fr}(j); \\ \mathrm{Fr}([3,i,j]) & = & \mathrm{Fr}(i); \\ \mathrm{Fr}([4,i,j]) & = & \mathrm{Fr}(j) \setminus i; \\ \mathrm{Fr}(x) & = & \varnothing, \textit{ if } x \textit{ not of the above form.} \end{array} \tag{8.3}$$

Lemma 8.1.8 $\mathrm{Fr}(x)$ *is a finite set of natural numbers for any set x.*

Definition 8.1.9 *Define*

$$\theta(x) = \max(\mathrm{Fr}(x)).$$

θ is Δ_1^{ZF}.

Lemma 8.1.10 *If ϕ is any formula of LST and $m = \ulcorner\phi\urcorner$, then $\theta(m)$ is the largest n such that v_n occurs as a free variable in ϕ, and that if $f \in \mathrm{Sat}(m,a)$, for any $a \in V$, then* $\mathrm{dom}f \geq 1 + \theta(m)$ *(ie.* $0, 1, \ldots, \theta(m) \in \mathrm{dom}f$*).*

Proof. This is proved by induction on ϕ and it is for this reason that we defined $\mathrm{Sat}([3,j,k],a)$ as we did (rather than just as ${}^{<\omega}a \setminus \mathrm{Sat}(j,a)$). $\square$

Definition 8.1.11 *We can now define G by*

$$G(m,a,s) = \begin{cases} \{b \in a : (s \cup \{\langle\theta(m), b\rangle\}) \in Sat(m,a)\} & \textit{if } s \in {}^{<\omega}a \textit{ and } \mathrm{dom}s = \theta(m) (= \{0, \ldots, \theta(m)-1\}), \\ \varnothing & \textit{otherwise.} \end{cases}$$

Lemma 8.1.12 *Then G is Δ_1^{ZF}.*

Proof. This follows because θ, Sat are Δ_1^{ZF}. $\square$

Lemma 8.1.13 *G has the required properties mentioned at the beginning of section 6.*

Proof. This is because of (*) (just before 8.1.5). $\square$

Another consequence of this is the following:

Lemma 8.1.14 *Suppose W is a transitive class such that* $\mathbf{On} \subseteq W$ *and* $W \models$ *ZF. Then* $L \subseteq W$*.*

Proof. Suppose $a \in L$, say $a \in L_\beta$.

We have ZF$\vdash \forall\alpha \in On \exists y(y = L_\alpha)$; hence $\langle W, \in\rangle \models \forall\alpha(On(\alpha) \to \exists y(y = L_\alpha))$.

But $On \subseteq W$, so $\beta \in W$, and "$On(\beta)$" is Δ_1^{ZF}, so $\langle W, \in\rangle \models \exists y = L_\beta$.

Let $b \in W$ be such that $\langle W, \in\rangle \models b = L_\alpha$.

But "$y = L_x$" is Δ_1^{ZF} (and W is transitive), so $\langle V, \in\rangle \models b = L_\alpha$, ie. $b = L_\alpha$.

So $a \in b \in W$. But W is transitive, so $a \in W$. $\square$

Chapter 9

ZF+V=L $\vdash$ AC

9.1

We first construct a class term $H : V \to V$ such that if $\langle a, R\rangle \in V$ and R is a well-ordering of the set a, then $H(\langle a, R\rangle) = \langle \omega \times {}^{<\omega}a, R'\rangle$, where R' is a well-ordering of $\omega \times {}^{<\omega}a$.

[We don't need absoluteness, though it holds]

Definition 9.1.1 *We define $H(x) = y$ iff x is not of the form $\langle a, R\rangle$, where R well-orders a, and $y = \varnothing$, or x is of this form, and y is an ordered pair the first coordinate of which is $\omega \times {}^{<\omega}a$ and the second coordinate is R', where $R' \subseteq (\omega \times {}^{<\omega}a)^2$, and satisfies: $\langle\langle n, s\rangle, \langle n', s'\rangle\rangle \in R'$ iff*

1. *$n < n'$, or*
2. *$n = n'$, and $\mathrm{dom} s < \mathrm{dom} s'$, or*
3. *$n = n'$, and $\mathrm{dom} s = \mathrm{dom} s' = k$, say, and $\exists j < k$ such that $\forall l < j(s(l) = s(l') \wedge \langle s(j), s'(j)\rangle \in R)$.*

(This is basically lexicographic order within chunks based on domain size.)

Theorem 9.1.2 *H has the required property.*

Now let $G : \omega \times V \times V \to V$ be as at the beginning of section 6.

Definition 9.1.3 *Define $J : \mathbf{On} \to V$ so that $J(0) = 0$, and $J(\alpha + 1)$ is the unique binary relation S on $L_{\alpha+1}$ such that for all $x, y \in L_{\alpha+1}$,*

1. *If $x \in L_\alpha$ and $y \notin L_\alpha$, then $\langle x, y\rangle \in S$;*
2. *If $x \in L_\alpha$ and $y \in L_\alpha$, then $\langle x, y\rangle \in S$ iff $\langle x, y\rangle \in J(\alpha)$;*

3. *If* $x, y \in L_{\alpha+1} \setminus L_\alpha$ *and* $H(\langle L_\alpha, J(\alpha)\rangle) = \langle \omega \times {}^{<\omega}L_\alpha, R\rangle$, *and* $\langle m, s\rangle \in \omega \times {}^{<\omega}a$ *is* R*-minimal such that* $G(m, s, L_\alpha) = x$, *and* $\langle m', s'\rangle \in \omega \times {}^{<\omega}a$ *is* R*-minimal such that* $G(m', s', L_\alpha) = y$, *then* $\langle x, y\rangle \in S$ *iff* $\langle\langle m, s\rangle, \langle m', s'\rangle\rangle \in R$.

And $J(\delta) = \bigcup_{\alpha<\delta} J(\alpha)$ *if* δ *is a limit.*

Then, from this definition, we immediately have by induction on α:

Lemma 9.1.4 *(ZF)* $\forall \alpha \in On$, $J(\alpha)$ *is a well-ordering of* L_α, *and* $J(\alpha) \subseteq J(\alpha+1)$, *and* $L_{\alpha+1}$ *is an initial segment of* $L_{\alpha+1}$ *under the ordering* $J(\alpha+1)$.

Corollary 9.1.5 *(ZF) The formula* $\Phi(x, y) \ := \exists\alpha(\alpha \in On \wedge \langle x, y\rangle \in J(\alpha))$ *is a well-ordering of* L. *(ie.* Φ *satisfies the axioms for a total ordering of* L, *and every* $a \in L$ *has a* Φ*-least element. In particular* $\forall a \in L$, $\{\langle x, y\rangle \in a^2 : \Phi(x, y)\}$ *is a well-ordering of* a.*)*

Theorem 9.1.6 *ZF+V=L* $\vdash$ *every set can be well-ordered, so ZF+V=L* $\vdash$ *AC.*

Proof. Immediate from 9.1.5. $\square$

Chapter 10

Cardinal Arithmetic

10.1

Recall $A \sim B$ means there is a bijection between A and B.

Definition 10.1.1 *An ordinal α is called a* cardinal *if for no $\beta < \alpha$ is $\beta \sim \alpha$.*

Cardinals are usually denoted κ, λ, μ. $Card$ denotes the class of all cardinals. Now every well-ordered set is bijective with an ordinal (using an order-preserving bijection). (Provable in ZF.) Hence if we assume ZFC, *as we do throughout this section*, then *every* set is bijective with an ordinal.

Definition 10.1.2 *(ZFC) The class term $|\ | : V \to On$ is defined so that $|x|$ is the least ordinal α such that $\alpha \sim x$.*

Lemma 10.1.3 *(ZFC) (1) The range of $|\ |$ is precisely the class of cardinals.*

(2) For all cardinals κ there is a cardinal μ such that $\mu > \kappa$. (κ^+ is the least such μ.)

(3) If X is a set of cardinals with no greatest element then $\sup X$ *is a cardinal.*

(4) $|\kappa| = \kappa$ for all cardinals κ.

Proof. (1) Exercise

(2) Consider $|\wp\kappa|$ (though this result is provable in ZFC)

(3) Let $\beta = \sup X$. Suppose $\exists\gamma < \beta(\gamma \sim \beta)$. Choose $\kappa \in X$, $\kappa > \gamma$. Then id_γ is an injection from γ to κ. However $\kappa \in X$, so $\kappa < \beta$, so by the Schröder-Bernstein Theorem $\kappa \sim \gamma$—contradicting the fact that κ is a cardinal.

(4) Exercise. □

(2) and (3) allow us to make the following

Definition 10.1.4 *(ZFC) The class term $\aleph : On \to Card$ is defined by (writing $\aleph_\alpha$ for $\aleph_\alpha$)*

1. $\aleph_0 = \omega$ *(ie.* $|\mathbb{N}|$*)*
2. $\aleph_{\alpha+1} = {\aleph_\alpha}^+$
3. $\aleph_\delta = \bigcup_{\alpha<\delta} \aleph_\delta$ *for* δ *a limit.*

Lemma 10.1.5 $\{\aleph_\alpha : \alpha \in On\}$ *is the class of all infinite cardinals (enumerated in increasing order). Thus* $\aleph_1$ *is the smallest uncountable cardinal.*

Proof. Exercise. □

Definition 10.1.6 *Suppose* κ, λ *are cardinals.*

1. $\kappa + \lambda = |(\kappa \times \{0\}) \cup (\lambda \times \{1\})|$.
2. $\kappa.\lambda = |\kappa \times \lambda|$.
3. $\kappa^\lambda = \left|{}^\lambda\kappa\right|$.

Theorem 10.1.7 *Suppose* κ, λ, μ *are non-zero cardinals. Then*

1. $\kappa^{\lambda+\mu} = \kappa^\lambda.\kappa^\mu$.
2. $\kappa^{\lambda.\mu} = (\kappa^\lambda)^\mu$.
3. $(\kappa.\lambda)^\mu = \kappa^\mu.\lambda^\mu$.
4. *(ZFC)* $2^\kappa > \kappa$.
5. *(ZFC) If* κ *or* λ *is infinite,* $\kappa + \lambda = \kappa.\lambda = \max\{\kappa, \lambda\}$.
6. $+$, $.$ *and exp are (weakly) order-preserving.*

Proof. See the books. □

Definition 10.1.8 *The* Generalized Continuum Hypothesis *(GCH) is the statement of LST: for all infinite cardinals* κ, $2^\kappa = \kappa^+$ *(ie.* $\forall\alpha \in On(2^{\aleph_\alpha} = \aleph_{\alpha+1})$*).*

Definition 10.1.9 *Suppose* $\beta > 0$ *is an ordinal and* $\sigma = \langle\kappa_\alpha : \alpha < \beta\rangle$ *is a* β*-sequence of cardinals (ie.* σ *is a function with domain* β *and* $\sigma(\alpha) = \kappa_\alpha$ *for all* $\alpha < \beta$*). Then we define*

1. $\sum_{\alpha<\beta} = \left|\bigcup_{\alpha<\beta}(\kappa_\alpha \times \{\alpha\})\right|$
2. $\prod_{\alpha<\beta} = \left|\{f : f : \beta \to \bigcup_{\alpha<\beta} \kappa_\alpha,\ \forall\alpha < \beta(f(\alpha) \in \kappa_\alpha)\}\right|$.

Lemma 10.1.10 *These definitions agree with the previous ones for* $\beta = 2$*. Further, if* κ, λ *are cardinals, then* $\kappa^\lambda = \prod_{\alpha<\lambda} \kappa$.

Proof. Easy exercise. □

Lemma 10.1.11 *(1) Suppose γ, δ are non-zero ordinals and $\langle \kappa_{\alpha,\beta} : \alpha < \gamma, \beta < \delta \rangle$ is a sequence of cardinals (indexed by $\gamma \times \delta$). Then*

$$\prod_{\alpha<\gamma} \sum_{\beta<\delta} \kappa_{\alpha,\beta} = \sum_{f \in {}^{\gamma}\delta} \prod_{\alpha<\gamma} \kappa_{\alpha,f(\alpha)}.$$

(ie. $\prod$ distributes over $\sum$.)

(2) Suppose β is a non-zero ordinal and $\langle \kappa_\alpha : \alpha < \beta \rangle$ is a β-sequence of cardinals and κ is any cardinal. Then

(a) $\kappa . \sum_{\alpha<\beta} \kappa_\alpha = \sum_{\alpha<\beta} (\kappa . \kappa_\alpha)$.

(b) If $\kappa_\alpha = \kappa$ for all $\alpha < \beta$, then $\sum_{\alpha<\beta} \kappa_\alpha = \sum_{\alpha<\beta} \kappa = |\beta| . \kappa$.

(3) $\sum$, $\prod$ are (weakly) order-preserving.

Proof. Exercises. □

Theorem 10.1.12 *("The König Inequality") Suppose $\kappa_\alpha < \lambda_\alpha$ for all $\alpha < \beta$. Then*

$$\sum_{\alpha<\beta} \kappa_\alpha < \prod_{\alpha<\beta} \lambda_\alpha.$$

Proof. Define $f : \bigcup_{\alpha<\beta} (\kappa_\alpha \times \{\alpha\}) \to \prod_{\alpha<\beta} \lambda_\alpha$ by

$$(f(\langle \eta, \alpha \rangle))(v) = \begin{cases} 1+\eta & \text{if } v = \alpha \\ 0 & \text{if } v \neq \alpha \end{cases}$$

Clearly f is injective, so $\sum_{\alpha<\beta} \kappa_\alpha \leq \prod_{\alpha<\beta} \lambda_\alpha$.

Now suppose that $h : \bigcup_{\alpha<\beta} (\kappa_\alpha \times \{\alpha\}) \to \prod_{\alpha<\beta} \lambda_\alpha$. We show that h is not onto.

For $\gamma < \beta$, define $h_\gamma : \bigcup_{\alpha<\beta} (\kappa_\alpha \times \{\alpha\}) \to \lambda_\gamma$ by

$$h_\gamma(\langle \eta, \alpha \rangle) = (h(\langle \eta, \alpha \rangle))(\gamma) \qquad (*)$$

Since $\kappa_\gamma < \lambda_\gamma$, $h_\gamma \restriction \kappa_\gamma \times \{\gamma\}$ cannot map onto λ_γ so there is an $a_\gamma \in \lambda_\gamma \setminus h_\gamma[\kappa_\gamma \times \{\gamma\}]$ (**).

Define $g \in \prod_{\alpha<\beta} \lambda_\alpha$ by $g(\gamma) = a_\gamma$ (for $\gamma < \beta$).

Then $g \notin \operatorname{ran} h$, since if $h(\langle \gamma, \alpha \rangle) = g$, then $h(\langle \gamma, \alpha \rangle)(\gamma) = g(\gamma)$ for all $\gamma < \beta$, so $h(\langle \gamma, \alpha \rangle)(\alpha) = g(\alpha) = a_\alpha$, ie $h_\alpha(\langle \gamma, \alpha \rangle) = a_\alpha$, so $a_\alpha \in h_\alpha[\kappa_\alpha \times \{\alpha\}]$, contradicting (**). □

Definition 10.1.13 *(1) Let α be a limit ordinal and suppose $S \subseteq \alpha$. Then S is* unbounded *in α if $\forall \beta < \alpha \, \exists \gamma \in S \, (\gamma > \beta)$.*

(2) Let κ be a cardinal. Then $\operatorname{cof}(\kappa)$ is the least ordinal α such that there exists a function $f : \alpha \to \kappa$ such that $\operatorname{ran} f$ is unbounded in κ.

Remark 10.1.14 *Suppose* $\mathrm{cof}(\kappa) = \alpha$ *and* $\gamma < \alpha$, $\gamma \sim \alpha$. *Say* $p : \gamma \to \alpha$ *is a bijection. Let* $f : \alpha \to \kappa$ *be such that* $\mathrm{ran} f$ *is unbounded in* κ. *Now clearly* $\mathrm{ran} f = \mathrm{ran}(fp)$, *so* $fp : \gamma \to \kappa$ *is a function whose range is unbounded in* κ. *Since* $\gamma < \alpha$ *this contradicts the definition of* $\mathrm{cof}(\kappa)$. *Hence no such* γ *exists, ie.* $\mathrm{cof}(\kappa)$ *is always a cardinal. Clearly* $\mathrm{cof}(\kappa) \leq \kappa$.

Definition 10.1.15 *An infinite cardinal* κ *is called* regular *if* $\mathrm{cof}(\kappa) = \kappa$.

Examples 10.1.16 *(a)* $\mathrm{cof}(\aleph_0) = \aleph_0$ *(obvious).*

(b) $\mathrm{cof}(\aleph_1) = \aleph_1$, *since if* $\mathrm{cof}(\aleph_1) < \aleph_1$, *then* $\mathrm{cof}(\aleph_1) = \aleph_0$. *Say* $f : \aleph_0 \to \aleph_1$ *is unbounded. Then* $\aleph_1 = \bigcup_{n<\aleph_0} f(n)$, *and is a countable union of countable sets, and thus (in ZFC) countable, which is impossible.*

(c) $\mathrm{cof}(\aleph_\omega) = \aleph_0$. $\geq$ *is clear. Consider* $f : \aleph_0 \to \aleph_\omega$ *defined so that* $f(n) = \aleph_n$.

Theorem 10.1.17 *For any infinite cardinal* κ, $\mathrm{cof}(\kappa)$ *is the least ordinal* β *such that there is a* β*-sequence* $\langle \kappa_\alpha : \alpha < \beta \rangle$ *of cardinals such that*

1. $\kappa_\alpha < \kappa$ *for all* $\alpha < \beta$,

2. $\sum_{\alpha<\beta} \kappa_\alpha = \kappa$.

Proof. Exercise. □

Theorem 10.1.18 *For any infinite cardinal* κ,

1. κ^+ *is regular,*

2. $\mathrm{cof}(2^\kappa) > \kappa$.

Proof. (1) Let $\beta = \mathrm{cof}(\kappa^+)$ and suppose $\beta < \kappa^+$. Then $\beta \leq \kappa$. By 10.1.17, there are $\kappa_\alpha < \kappa^+$ (for $\alpha < \beta$) such that $\sum_{\alpha<\beta} \kappa_\alpha = \kappa^+$. Then $\kappa_\alpha \leq \kappa$ for all α. But $\sum_{\alpha<\beta} \kappa_\alpha \leq \sum_{\alpha<\beta} \kappa \leq \kappa.\kappa = \kappa^2 = \kappa$—a contradiction.

(2) Suppose $\mu = \mathrm{cof}(2^\kappa)$, and $\mu \leq \kappa$. Choose $\langle \kappa_\alpha : \alpha < \mu \rangle$ such that $\kappa_\alpha < 2^\kappa$ for all $\alpha < \mu$ and such that $\sum_{\alpha<\mu} \kappa_\alpha = 2^\kappa$.

By König, $\sum_{\alpha<\mu} \kappa_\alpha < \prod_{\alpha<\mu} 2^\kappa$, ie. $2^\kappa < \prod_{\alpha<\mu} 2^\kappa$.

But $\prod_{\alpha<\mu} 2^\kappa = (2^\mu)^\mu = 2^{\kappa \cdot \mu} = 2^\kappa$ (since $\mu < \kappa$). This is a contradiction. □

Examples 10.1.19 $\mathrm{cof}(2^{\aleph_0}) > \aleph_0$*; and this is the only provable constraint on the value of* $2^{\aleph_0}$. *—So, for example,* $2^{\aleph_0} \neq \aleph_\omega$.

Theorem 10.1.20 *Suppose* α *is an infinite ordinal. Then* $|L_\alpha| = \alpha$.

Proof. Induction on α.

For $\alpha = \omega$, $L_\omega = \bigcup_{n\in\omega} L_n$. Since each L_n is finite, and $\omega \subseteq L_\omega$ (so L_ω is not finite), $|L_\omega| = \aleph_0 = |\omega|$.

Suppose $|L_\alpha| = |\alpha|$.

Now $L_{\alpha+1} = \{G(m, a, s) : m \in \omega,\ s \in {}^{<\omega}L_\alpha\}$.

However, for x infinite, $|{}^{<\omega}x| = |x|$.

So $|L_{\alpha+1}| \leq \aleph_0 . |{}^{<\omega}L_\alpha| = \aleph_0 . |L_\alpha| = \aleph_0 . |\alpha| = |\alpha| = |\alpha + 1|$.

Also $L_\alpha \subseteq L_{\alpha+1}$, so $|L_{\alpha+1}| \geq |L_\alpha| = |\alpha| = |\alpha + 1|$.

For δ a limit, $|L_\delta| = \left|\bigcup_{\alpha<\delta} L_\alpha\right| \leq \sum_{\alpha<\delta} |L_\alpha| \leq \aleph_0 + \sum_{\omega\leq\alpha<\delta} |L_\alpha| = \aleph_0 + \sum_{\omega\leq\alpha<\delta} |\alpha|$ (IH) $\leq \aleph_0 + \sum_{\omega\leq\alpha<\delta} |\delta| = \aleph_0 + |\delta|^2 = |\delta|$ (since δ is infinite).

—and the other way round too: $\delta \subseteq L_\delta$, so that works. $\square$

Chapter 11

The Mostowski-Shepherdson Collapsing Lemma

11.1

Lemma 11.1.1 *Suppose X is a set and M_1, M_2 are transitive sets. Suppose $\pi_i : X \to M_i$ are $\in$-isomorphisms (ie. $\forall x, y \in X(x \in y \leftrightarrow \pi_i(x) \in \pi_i(y))$). Then $\pi_1 = \pi_2$ (and hence $M_1 = M_2$).*

Proof. Define $\phi(x) \Leftrightarrow x \notin X \vee \pi_1(x) = \pi_2(x)$.

We prove $\forall x\, \phi(x)$ by $\in$-induction (see 3.2.5).

Suppose x is any set, and $\phi(y)$ holds for all $y \in x$. If $x \notin X$ we are done. Hence suppose $x \in X$, and $\pi_1(x) \neq \pi_2(x)$. Then there is z such that (say) $z \in \pi_1(x)$ and $z \notin \pi_2(x)$. Since M_1 is transitive and $pi_1(x) \in M_1$, we have $z \in M_1$. Hence (since π_1 is onto), $\exists y \in X$ such that $\pi_1(y) = z$. Since $\pi_1(y) \in \pi_1(x)$, we have $y \in x$, and hence (by IH), $z = \pi_1(y) = \pi_2(y)$ and $\pi_2(y) \in \pi_2(x)$. So $z \in \pi_2(x)$—a contradiction.

Thus $\phi(x)$ holds, hence result by 3.2.5. □

Theorem 11.1.2 *Suppose X is any set such that $\langle X, \in \rangle \vDash$ Extensionality. (ie. if $a, b \in X$ and $a \neq b$, then $\exists x \in X$ such that $x \in a \wedge x \notin b$ or vice versa.) Then there is a unique transitive set M and a unique function π such that π is an $\in$-isomorphism from X to M.*

Proof.

Uniqueness is by 11.1.1. For existence, we prove by induction on $\alpha \in On$, that $\exists \pi_\alpha : X \cap V_\alpha \sim M_\alpha$ for some transitive set M_α. (Since $X \subseteq V_\alpha$ for some α, this is sufficient.

Note that $\forall\alpha \in On$, $\langle X \cap V_\alpha, \in\rangle \vDash$ Extensionality (since V_α is transitive). Now suppose π_α, M_α exist for all $\alpha < \beta$. It's easy to show (by 11.1.1) that they are unique and $\forall\alpha < \alpha' < \beta$ $M_\alpha \subseteq M_{\alpha'}$, and $\pi_\alpha = \pi_{\alpha'}{\restriction}M_\alpha$. Hence if β is a limit ordinal, then take $M_\beta = \bigcup_{\alpha<\beta} M_\alpha$ and $\pi_\beta = \bigcup_{\alpha<\beta} \pi_\alpha$.

So suppose $\beta = \gamma + 1$. We have $\pi_\gamma : X \cap V_\gamma \sim M_\gamma$. For $x \in X \cap V_{\gamma+1}$, note that $y \in x \cap X \to y \in X \cap V_\gamma$, so we may define

$$\pi_{\gamma+1}(x) = \{\pi_\gamma(y) : y \in x \cap X\}.$$

Let $M_{\gamma+1} = \pi_{\gamma+1}[X \cap V_{\gamma+1}]$. Then $\pi_{\gamma+1} : X \cap V_{\gamma+1} \to M_{\gamma+1}$ is surjective.

Suppose $a, b \in X \cap V_{\gamma+1}$, $a \neq b$. Since $\langle X \cap V_{\gamma+1}, \in\rangle \vDash$ Extensionality, $\exists c \in X \cap V_{\gamma+1}$ such that (say) $c \in a \wedge c \notin b$.

Then $\pi_{\gamma+1}(a) = \{\pi_\gamma(y) : y \in a \cap X\} \ni \pi_\gamma(c)$.

Suppose $\pi_\gamma(c) \in \pi_{\gamma+1}(b)$. Then $\pi_\gamma(c) = \pi_\gamma(t)$ for some $t \in b \cap X$. Since $c \notin b \cap X$, we have $c \neq t$, so π_γ is not injective—contradiction.

Thus $\pi_\gamma(c) \notin \pi_{\gamma+1}(b)$, so $pi_{\gamma+1}(a) \neq \pi_{\gamma+1}(b)$ and so $\pi_{\gamma+1}$ is injective.

We now show that if $x \in X \cap V_\gamma$ ($\subseteq X \cap V_{\gamma+1}$), then $\pi_\gamma(x) = \pi_{\gamma+1}(x)$ (*)

For, $y \in \pi_\gamma(x)$ implies $y \in \pi_\gamma(x) \in M_\gamma$ implies $y \in M_\gamma$ (since M_γ is transitive), say $\pi_\gamma(t) = y$ ($t \in X \cap V_\gamma$).

Then $\pi_\gamma(t) \in \pi_\gamma(x)$, so $t \in x$, hence $t \in x \cap X$.

Thus $\pi_{\gamma+1}(x) = \{\pi_\gamma(z) : z \in x \cap X\} \ni \pi_\gamma(t) = y$.

This shows $\pi_\gamma(x) \subseteq \pi_{\gamma+1}(x)$.

Conversely, suppose $y \in \pi_{\gamma+1}(x)$. Then $y = \pi_\gamma(t)$ for some $t \in x \cap X$. Since $t \in x \in X \cap V_\gamma$, we have $\pi_\gamma(t) \in \pi_\gamma(x)$ (since π_γ is an $\in$-isomorphism). Ie. $y \in \pi_\gamma(x)$. So $\pi_{\gamma+1}(x) \subseteq \pi_\gamma(x)$, and we have (*).

Now suppose $a, b \in X \cap V_{\gamma+1}$, and $a \in b$ (so $a \in X \cap V_\gamma$).

Then $\pi_{\gamma+1}(b) = \{\pi_\gamma(y) : y \in b \cap X\}$. But $a \in b \cap X$, so $\pi_\gamma(a) \in \pi_{\gamma+1}(b)$. Hence by (*) $\pi_{\gamma+1}(a) \in \pi_{\gamma+1}(b)$.

Finally, $M_{\gamma+1}$ is transitive, since if $a \in b \in M_{\gamma+1}$, then $b = \pi_{\gamma+1}(x)$ for some $x \in X \cap V_{\gamma+1}$, and hence $a = \pi_\gamma(y)$ for some $y \in x \cap X$. Since $y \in X \cap V_\gamma$, we have, by (*), $\pi_\gamma(y) = \pi_{\gamma+1}(y)$, so $a \in \operatorname{ran} \pi_{\gamma+1} = M_{\gamma+1}$, as required. $\square$

Chapter 12

The Condensation Lemma and GCH

12.1

Theorem 12.1.1 *(The Condensation Lemma) Let α be a limit ordinal and suppose $X \preceq L_\alpha$ (ie. $\forall a_1, \ldots, a_n \in X$, and formulas $\phi(v_1, \ldots, v_n)$ of LST, $\langle X, \in \rangle \models \phi(a_1, \ldots, a_n)$ iff $\langle L_\alpha, \in \rangle \models \phi(a_1, \ldots, a_n)$, although we only need this when ϕ is a Σ_1 formula). Then there is unique π and β such that $\beta \leq \alpha$ and $\pi : X \sim L_\beta$ is an $\in$-isomorphism. Further if $Y \subseteq X$ and Y is transitive, then $\pi(y) = y$ for all $y \in Y$.*

We prove this in stages.

Lemma 12.1.2 *$\forall m \in \omega$, $L_m \subseteq X$.*

Proof. Clear for $m = 0$. Suppose $L_m \subseteq X$ and let $a \in L_{m+1}$, so $a = \{a_1, \ldots, a_n\} \subseteq L_m$. Then $L_\alpha \models \exists x((a_1 \in x \wedge \ldots \wedge a_n \in x) \wedge \forall y \in x(y = a_1 \vee \ldots \vee y = a_n))$. Hence $X \models \exists x((a_1 \in x \wedge \ldots \wedge a_n \in x) \wedge \forall y \in x(y = a_1 \vee \ldots \vee y = a_n))$. Clearly such an x must be a, so $a \in X$. Hence $L_{m+1} \subseteq X$. Hence the result follows by induction. □

Lemma 12.1.3 *$X \models$ Extensionality.*

Proof. For suppose $a, b \in X$ and $a \neq b$. Then $\exists c, c \in a \wedge c \notin b$ (say), and $c \in L_\alpha$ since L_α is transitive. Thus $L_\alpha \models \exists x(x \in a \wedge x \notin b)$, so $X \models \exists x(x \in a \wedge x \notin b)$, as required. □

By 11.1.2 there is transitive M and $\pi : X \sim M$. Now since M is transitive, $M \cap On$ is a transitive set of ordinals so is an ordinal, β, say. Then $\beta \leq \alpha$ (exercise—suppose $\beta > \alpha$, so $\pi^{-1}(\alpha) \in X$. Show $\pi^{-1}(\alpha) = \alpha$ to get contradiction). We show $M = L_\beta$.

An admission! For this proof we need the fact that most of the formulas that we have proven Δ_1^{ZF} are in fact absolute between transitive classes satisfying much weaker axioms than ZF—in fact BS—basic Set Theory (see Devlin). BS is such that $L_\alpha \vDash \mathrm{BS}$ for any limit ordinal $\alpha > \omega$. In particular, the formula $\mathbf{On}(x)$, and $\Phi(x,y) \ := \mathbf{On}(x) \wedge y = L_x$, is Δ_1^{ZF} and hence absolute between V and L_α and between V and M. (Since M is transitive.) As an application, suppose $\beta = \gamma \cup \{\gamma\}$. Since $\beta \notin M$, and $\gamma \in M$, and $M \vDash On(\gamma)$ (since $On(\gamma)$ really is Σ_0 and M is transitive), we have $M \vDash \exists x(On(x) \wedge \forall y y \neq x \cup \{x\})$. Now $X \sim M$, so $X \vDash \exists x(On(x) \wedge \forall y y \neq x \cup \{x\})$, hence $L_\alpha \vDash \ \vDash \exists x(On(x) \wedge \forall y y \neq x \cup \{x\})$, which is a contradiction, since α is a limit ordinal. Hence, we have shown:

Lemma 12.1.4 *β is a limit ordinal.*

Lemma 12.1.5 *$L_\beta \subseteq M$.*

Proof. Since β is a limit, $L_\beta = \bigcup_{\gamma<\beta} L_\gamma$, so fix $\gamma < \beta$. Sufficient to show $L_\gamma \subseteq M$.

Now for any $\eta < \alpha$, $L_\eta \in L_\alpha$. Since $L_\alpha \cap On = \alpha$, we have $L_\alpha \vDash \underbrace{\forall x(On(x) \rightarrow \exists y \Phi(x,y))}_{\sigma}$.

Hence $X \vDash \sigma$, since $X \preceq L_\alpha$, so $M \vDash \sigma$, since $X \sim M$.

Since $\forall x \in M$, $M \vDash On(u) \Leftrightarrow u \in On \wedge u < \beta$, we have in particular $M \vDash \exists y \Phi(\gamma, y)$—say $a \in M$ and $M \vDash \Phi(\gamma, a)$. By absoluteness $a = L_\gamma$, so $L_\gamma \in M$, so $L_\gamma \subseteq M$ since M is transitive. $\square$

Lemma 12.1.6 *$M \subseteq L_\beta$.*

Proof. Since $L_\alpha = \bigcup_{\gamma<\alpha} L_\gamma$, we have $L_\alpha \vDash \underbrace{\forall x \exists y \exists z(On(y) \wedge \Phi(y,z) \wedge x \in z)}_{\tau}$.

Hence $X \vDash \tau$ (since $X \preceq L_\alpha$), hence $M \vDash \tau$ (since $X \sim M$.

Let $a \in M$. Then for some $c, d \in M$,

$$M \vDash On(c) \wedge \Phi(c,d) \wedge a \in d.$$

By absoluteness, $c \in On$, and hence $c < \beta$, and $d = L_c$ and $a \in L_c$. Hence $a \in \bigcup_{\gamma<\beta} L_\gamma = L_\beta$, as required. $\square$

Lemma 12.1.7 *Suppose $Y \subseteq X$, Y transitive. Then $\forall y \in Y$ $\pi(y) = y$.*

Proof. It's easy to show $\pi[Y]$ is transitive and $\pi : Y \sim \pi[Y]$. However, $id{\restriction}Y \sim Y$. Hence by 11.1.1, $\pi = id{\restriction}Y$. $\square$

We have now completed the proof of 12.1.1.

Lemma 12.1.8 *(ZFC) Let A be any set and $Y \subseteq A$. Then there is a set X such that $Y \subseteq X \subseteq A$ and $\langle X, \in\rangle \preceq \langle A, \in\rangle$, and $|X| = \max(\aleph_0, |X|)$.*

Proof. This is the downward Löwenheim-Skolem Theorem. $\square$

Theorem 12.1.9 *(ZF+V=L) Let κ be a cardinal, and suppose x is a bounded subset of κ. Then $x \in L_\kappa$.*

Proof. Clear if $\kappa \leq \omega$, so assume $\kappa > \omega$. Now $x \subseteq \alpha$ for some $\omega \leq \alpha < \kappa$, so $x \subseteq L_\alpha$. Then $L_\alpha \cup \{x\}$ is transitive.

Using V=L, let λ be a limit ordinal such that $\lambda \geq \kappa$, and $L_\alpha \cup \{x\} \subseteq L_\lambda$. By 12.1.8, with $A = L_\lambda$ and $Y = L_\alpha \cup \{x\}$, let X be such that $L_\alpha \cup \{x\} \subseteq X$ and $X \preceq L_\lambda$, with $|X| \leq |L_\alpha \cup \{x\}| = |\alpha|$. Let $\pi : X \sim L_\beta$ be as in 12.1.1. Then $|\beta| = |L_\beta| = |X| \leq |\alpha| < \kappa$, so $\beta < \kappa$. But $L_\alpha \cup \{x\}$ is transitive so, in particular, $\pi(x) = x$, so $x \in L_\beta \subseteq L_\kappa$, as required. □

Corollary 12.1.10 *ZF+V=L$\vdash$GCH. Hence if ZF is consistent, so is ZFC+GCH.*

Proof. By 12.1.9. ZF+V=L$\vdash$ for all infinite κ, $\mathbb{P}\kappa \subseteq L_{\kappa^+}$. But ZF$\vdash$ for all infinite κ, $|L_{\kappa^+}| = \kappa^+$, hence ZF+V=L$\vdash$ for all infinite κ, $|\mathbb{P}\kappa| \leq \kappa^+$. So $2^\kappa \leq \kappa^+$, and $\geq$ is obvious. □