Axiomatic Set Theory: Problem sheet 1

A.

1. Write the following as formulas of LST:
(a) $x = \langle y, z \rangle$;
(b) $x = y \times z$;
(c) $x = y \cup \{y\}$;
(d) "x is a successor set";
(e) $x = \omega$.

2. Deduce the Axiom of Pairs from the other axioms of ZF*.

3. Assuming ZF, show that if a is a non-empty transitive set then $\varnothing \in a$.

B.

4. Which of the Axiom of Extensionality, the Empty Set Axiom, the Powerset Axiom, and the Axiom of Infinity hold in the structure $\langle \mathbb{Q}, < \rangle$? Also, find an instance of the Separation Schema that is true in $\langle \mathbb{Q}, < \rangle$ and one that is false.

5. Assuming ZF*, show that there exists a *transitive* set M such that
(a) $\varnothing \in M$, and
(b) if $x \in M$ and $y \in M$, then $\{x, y\} \in M$, and
(c) every element of M contains at most two elements.
Show further that if σ is an axiom of ZF*+AC other than the Axioms of Infinity, Unions and Powerset, then $\langle M, \in \rangle \vDash \sigma$. (It follows that if ZF* is consistent then so is this reduced set of axioms, together with the Axiom of Choice.)

C.

6. (a) Assuming ZF (ie. ZF*+Foundation) prove that the following two definitions of "ordinal" are equivalent:
(i) An ordinal is a transitive set well-ordered by $\in$.
(ii) An ordinal is a transitive set totally ordered by $\in$.
(b) Prove the principle of induction for **On** using only ZF*.

7. (ZF) Let H_ω denote the class of *hereditarily finite sets*, ie. $H_\omega = \{x : TC(x) \text{ is finite}\}$. Prove that $H_\omega = V_\omega$ (and hence that H_ω is a set). Prove that $\langle V_\omega, \in \rangle \vDash$ the axiom of foundation, and $\langle V_\omega, \in \rangle \vDash \neg$ the axiom of infinity.

[It is easy, but tedious, to check that $\langle V_\omega, \in \rangle \vDash$ the other axioms of ZF. This shows that the other axioms of ZF do not imply the axiom of infinity.]