Axiomatic Set Theory: Problem sheet 2

A.

1. (ZF*) Define a "natural" ordinal exponentiation using the recursion theorem for ordinals, and show that for all ordinals α, β and γ, $\alpha^{(\beta+\gamma)} = \alpha^\beta\alpha^\gamma$, and $\alpha^{(\beta\cdot\gamma)} = (\alpha^\beta)^\gamma$. Show also that $2^\omega = \omega$.

We define $\alpha^0 = 1$, and $\alpha^{\beta+1}$ to be $\alpha^\beta.\alpha$. Defining α^β when $\alpha = 0$ makes the limit case, annoyingly, more complicated: if λ is a limit, then $\alpha^\lambda = \sup_{0<\beta<\lambda} \alpha^\beta$.

We demonstrate the required properties of exponentiation by induction.

$\alpha^{\beta+0} = \alpha^\beta = \alpha^\beta.1 = \alpha^\beta.\alpha^0$; $\alpha^{\beta+(\gamma+1)} = \alpha^{(\beta+\gamma)+1} = \alpha^{\beta+\gamma}.\alpha = \alpha^\beta.\alpha^\gamma.\alpha = \alpha^\beta.\alpha^{\gamma+1}$; for limit λ, $\alpha^{\beta+\lambda} = \alpha^{\sup_{\gamma<\lambda}(\beta+\gamma)} = \sup_{0<\delta<\beta+\lambda}\alpha^\delta = \sup_{0<\gamma<\lambda}\alpha^{\beta+\gamma} = \alpha^\beta.\alpha^\lambda$.

$\alpha^{(\beta.0)} = \alpha^0 = 1 = (\alpha^\beta)^0$; $\alpha^{(\beta.(\gamma+1))} = \alpha^{\beta.\gamma+\beta} = \alpha^{\beta.\gamma}.\alpha^\beta = (\alpha^\beta)^\gamma.\alpha^\beta = \alpha^{\beta^\gamma.\beta} = (\alpha^\beta)^{\gamma+1}$. If λ is a limit, then $\alpha^{\beta.\lambda} = \alpha^{\sup_{\gamma<\lambda}\beta.\gamma} = \sup_{0<\delta<\beta.\lambda}\alpha^\delta = \sup_{0<\gamma<\lambda}\alpha^{\beta.\gamma} = \sup_{0<\gamma<\lambda}(\alpha^\beta)^\gamma = (\alpha^\beta)^\lambda$.

$2^\omega = \sup_{0<n<\omega} 2^n = 2^\omega$.

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2. (ZF*) Prove that $(V, \in)$ satisfies the Axiom of Unions and the Axiom of Infinity.

The statements "$x = \bigcup y$" and "$x = \omega$" are both absolute between transitive classes. Also, $\omega \in V$, and if $x \in V$ then $\bigcup x \in V$ also. The Axioms of Unions and Infinity for V now follow.

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3. (ZF*) Let $\alpha \in \mathbf{On}$ and suppose that $a \in V_\alpha$ and $b \subseteq a$. Prove that $b \in V_\alpha$.

Suppose that $a \in V_\alpha$. Then for some $\beta < \alpha$, $a \subseteq V_\beta$ (this can be proved by induction on α). Then $b \subseteq V_\beta$ also, and so $b \in V_{\beta+1} \subseteq V_\alpha$.

B.

4. (ZF*) Suppose $F : \mathbf{On} \to \mathbf{On}$ is a class term satisfying:

(1) $\alpha < \beta \to F(\alpha) < F(\beta)$ (for $\alpha, \beta \in On$)

(2) $F(\delta) = \bigcup_{\alpha<\delta} F(\alpha)$ (for limit ordinals δ).

Prove that for all $\alpha \in \mathbf{On}$ there exists $\beta \in \mathbf{On}$ such that $\beta > \alpha$ and $F(\beta) = \beta$ (ie. F has arbitrarily large fixed points). What is the smallest non-zero fixed point of the term $F : On \to \mathbf{On}$ defined by $F(x) = \omega.x$ (for $x \in \mathbf{On}$)?

Define G by recursion on the ordinals so that $G(0) = \alpha + 1$, $G(\beta + 1) = F(G(\beta))$, $G(\lambda) = \sup_{\beta<\lambda} F(\beta)$.

Then $G(\omega)$ is a fixed point for F (and indeed so is $G(\lambda)$ for any limit λ).

For, we prove by induction on γ that if $\beta \le \gamma$, then $G(\beta) \le G(\gamma)$; and now $G(\omega) = \sup_{n\in\omega} G(n) = \sup_{n\in\omega} G(n+1) = \sup_{n\in\omega} F(G(n)) = F(G(\omega))$, as required.

The first fixed point of the given function F is ω^2.

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5. (ZF*) Prove that the axiom of foundation is equivalent to $\forall x(x \in V)$.

$\Rightarrow$) Let x be a set that does not belong to V. Then x is not empty. Let $y = \mathrm{TC}(\{x\})$. Let $z = y \setminus V$. Then $x \in z$, so z is non-empty. Suppose $m \in z$. Then $m \notin V$. If $m \subseteq V$, then by Replacement, there exists an ordinal α such that $m \subseteq V_\alpha$. Then $m \in V_{\alpha+1} \subseteq V$, contradiction. So $m \not\subseteq V$. Let $w \in m \setminus V$. Then $w \in z \cap m$, and this contradicts Foundation.

$\Leftarrow$) Suppose that for all x, $x \in V$. Suppose x is not empty. Then since V is transitive, $x \subseteq V$. Let α be least such that $x \cap V_\alpha \neq \varnothing$, and let $m \in x \cap V_\alpha$. Then for all $y \in m$, there exists $\beta < \alpha$ such that $y \in V_\beta$. Hence $m \cap x = \varnothing$, verifying Foundation for x.

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6. (ZF*) Later in the course we shall be concerned with those formulas whose truth does not depend on which transitive class they are interpreted in. More precisely, let A be a transitive class. A formula $\phi(v_1, \ldots, v_n)$ (*without* parameters) of LST is called A-*absolute* if for any $a_1, \ldots, a_n \in A$, $\phi(a_1, \ldots, a_n)$ holds (ie. $(V^*, \in) \vDash \phi(a_1, \ldots, a_n)$) iff $\phi(a_1, \ldots, a_n)$ holds in A (ie. $(A, \in) \vDash \phi(a_1, \ldots, a_n)$). Prove that the following statements (or the natural formulas of LST which these translate) are A-absolute, for any transitive class A:

(i) $v_1 \subseteq v_2$

This is equivalent to $\forall x \in v_1\, v_1 \in v_2$, which is Σ_0.

(ii) $v_1 = \bigcup v_2$

This is equivalent to

$$(\forall x \in v_1\, \forall y \in x\, x \in v_2) \wedge (\forall x \in v_2\, \exists y \in v_1\, y \in x),$$

which is Σ_0.

(iii) $v_1 = \{v_2, v_3\}$

This is equivalent to

$$\big(\forall x \in v_2\, (x = v_2 \vee x = v_3)\big) \wedge v_2 \in v_1 \wedge v_3 \in v_1,$$

which is Σ_0.

(iv) $v_1 = v_2 \cup \{v_2\}$.

This is equivalent to $v_1 = \bigcup\{v_2, \{v_2, v_2\}\}$; we now appeal to parts (ii) and (iii).

C.

7. Show that "$v_1 = \wp v_2$" is not ω-absolute. (Note that ω is a transitive class.)

ω is a transitive class, and in $\langle \omega, \in \rangle$, $m \subseteq n$ iff $m \leq n$. So in $\langle \omega, \in \rangle$, the set of subsets of n is $\{m \in \omega : m \leq n\}$, or $n + 1$. So for all n, $\langle \omega, \in \rangle \vDash n + 1 = \wp(n)$.
