Axiomatic Set Theory: Problem sheet 3

A.

1. Assuming (as was shown in the lectures), that $a \in L \to \bigcup a \in L$ and $a \in L \to \wp a \cap L \in L$, verify carefully that $\langle L, \in \rangle \models$ Union, Powerset.

2. The *rank* of a set A, $rk(A)$, is defined to be the least $\alpha \in On$ such that $A \subseteq V_\alpha$. Prove that $\forall \alpha \in On(rk(L_\alpha) = \alpha)$.

3. Let E denote the set of even natural numbers. Prove that $E \in L_{\omega+1}$.

B.

4. For $\phi(\mathbf{v})$ a formula of LST (without parameters) and a any set, let $\phi_a(\mathbf{v})$ denote the formula (with parameter a) obtained by relativizing $\phi(\mathbf{v})$ to the class a. Prove that for any transitive class A and $a, \mathbf{b} \in A$, $(A, \in) \models \phi_a(\mathbf{b})$ iff $\phi_a(\mathbf{b})$ (ie. $\phi_a(\mathbf{v})$ is A-absolute).

5. Prove that $\forall \alpha, \beta \in On$, (i) $V_\alpha \cap On = \alpha$, and (ii) if $\alpha \in V_\beta$, then $V_\alpha \in V_\beta$.

6. A *club* is, by definition, a closed, unbounded class of ordinals. Prove that if U_1 and U_2 are clubs then so is $U_1 \cap U_2$. More generally, suppose that X is a class such that $X \subseteq \omega \times On$. For $i \in \omega$, let $X_i = \{\alpha \in On : \langle i, \alpha \rangle \in X\}$. Suppose that for all $i \in \omega$, X_i is a club. Prove that $\bigcap_{i \in \omega} X_i$ is a club.

C.

7. (i) It is known that there is a formula $\phi(x)$ of LST (without parameters) such that (in ZF one can prove that) for any set a, $\phi(a)$ iff "$\langle a, \in \rangle \models$ ZF and a is transitive". Further, this formula is A-absolute for any transitive class A (see previous sheet). Show that one cannot prove the sentence $\exists x \phi(x)$ from ZF. [Hint: Consider the least $\alpha \in On$ such that $\exists x \in V_\alpha(\phi(x))$.]

(ii) As formulated in the lectures, ZF is a countably infinite collection of axioms (since there is one separation and replacement axiom for each formula of LST, and there are clearly a countably infinite number of such formulas). Prove that there is no finite subcollection, T, say, of ZF, such that $T \vdash$ ZF.

8. * What is wrong with the following argument:

Let $\{\sigma_i : i \in \omega\}$ be an enumeration of all the axioms of ZF. By Lévy's Reflection Principle, for each $i \in \omega$, the class $\{\alpha \in On : \langle V_\alpha, \in \rangle \models \sigma_i\}$ (call it X_i) is a club (since $(V, \in) \models \sigma_i$). By question (3) above, $\bigcap_{i \in \omega} X_i$ is a club (we are using question (3) by setting $X = \{\langle i, \alpha \rangle : \alpha \in X_i\}$). In particular, $\bigcap_{i \in \omega} X_i$ is non-empty. Let $\beta \in \bigcap_{i \in \omega} X_i$. Then $\beta \in X_i$ for all $i \in \omega$, so $\langle V_\beta, \in \rangle \models \sigma_i$ for all $i \in \omega$, so $\langle V_\beta, \in \rangle \models$ ZF. Hence $\phi(V_\beta)$ holds, so $\exists x \phi(x)$ (where $\phi(x)$ is the formula in (4)(i)). Since $(V, \in)$ is an arbitrary model of ZF, we have ZF $\vdash \exists x \phi(x)$!

9. Suppose $F : V \to V$ is a term definable without parameters (ie. the formula defining "$F(x) = y$" has no parameters). Suppose further that it is an *elementary map*, ie. for any formula $\phi(v_0, \ldots, v_{n-1})$ of LST (without parameters), and any $a_0, \ldots, a_{n-1} \in V$,

$$\phi(a_0, \ldots, a_{n-1}) \Leftrightarrow \phi(F(a_0), \ldots, F(a_{n-1})).$$

Prove that F is the identity. [Hint: first show that for all ordinals α, $F(\alpha) = \alpha$, by considering the first β for which $F(\beta) \neq \beta$.]

[Remark: Assuming only ZF, it is not known whether such an elementary map definable *with* parameters can exist other than the identity, although if ZFC is assumed it is known that there is no such.]