Gödel Incompleteness Theorems: Problem sheet 0

This sheet contains revision of some material that might be contained in an introductory course in logic.

In more detail, questions 1. and 2., and the first few parts of 3., are revision of a first course in first order predicate calculus; the later parts of question 3. may or may not have fallen within such a course. Question 4. is not revision, and is not, strictly speaking, on the syllabus of this course, but is intended to help set the scene for it.

This sheet is not to be handed in and will not be marked.

Accordingly, all questions could be regarded as optional, regardless of whether I have marked them thus. I would suggest choosing those questions that look moderately difficult to you: that is, don't bother doing ones that you can see at once how to do without any difficulty, but do anything which looks interesting to you, or which will help you to revise things that you knew once but need a refresher on.

I will be assuming in this course that you are familiar with the Completeness Theorem for some deductive system in first order predicate calculus (the Completeness Theorem is sometimes packaged in two parts as the Soundness Theorem and the Adequacy Theorem), and with everything that that entails: that you understand what is meant by a formal proof, a tautology, a contradiction, and a structure for a language, a model of a theory, satisfaction of a formula in a model, and logical or semantic entailment and logical validity.

A book like *Propositional and Predicate Calculus: A Model of Argument* by Derek Goldrei, or *Logic for Mathematicians* by Hamilton, should cover all the necessary material.

1. (i) Show that the following is logically valid in any non-empty domain, where x is not free in G.

$$((\forall x\, F(x) \to G) \leftrightarrow \exists x\, (F(x) \to G)).$$

(ii) Show that the following is logically valid in any non-empty domain.

$$(\exists x\, (F(x) \vee G(x)) \leftrightarrow (\exists x\, F(x) \vee \exists x\, G(x))).$$

(iii) Show that

$$(\forall x\, (F(x) \vee G(x)) \leftrightarrow (\forall x\, F(x) \vee \forall x\, G(x)))$$

is not logically valid.

(iv) Show that

$$((\forall x\, F(x) \to G) \leftrightarrow \forall x\, (F(x) \to G)),$$

is not logically valid, where x is not free in G.

2. (i) Using the axiom schemata

$$(F \to (G \to F)) \qquad \text{(A1)}$$

and

$$((F \to (G \to H)) \to ((F \to G) \to (F \to H))) \qquad \text{(A2)}$$

and the rule of inference Modus Ponens, show that for Γ a set of formulae none of which contain any free variables, and for F and G formulae containing no free variables, that if $\Gamma \cup \{F\} \vdash G$, then $\Gamma \vdash (F \to G)$.

(ii) Show that $\forall x\, A(x)$ is a logical consequence of $A(x)$ (that is, in any interpretation in which $A(x)$ is true, $\forall x\, A(x)$ is true also), but that $(A(x) \to \forall x\, A(x))$ is not logically valid (that is, there is an interpretation in which it is not true).

(iii) Show that if F is a sentence (that is, a formula with no free variables) and $\mathfrak{M}$ is a structure, then either $\mathfrak{M} \models F$ or $\mathfrak{M} \models \neg F$. Give an example of a formula $F(x)$ with a free variable, and a structure $\mathfrak{M}$, such that it is not true either that $\mathfrak{M} \models F(x)$ or that $\mathfrak{M} \models \neg F(x)$.

3. (i) Write down a set of axioms and rules of inference for first order predicate calculus.

(ii) State the Completeness Theorem for your system, and sketch a proof of it.

(iii) State the Compactness Theorem, and deduce it from the Completeness Theorem.

(iv) Use the Compactness Theorem, or some other method, to construct a countably infinite structure $\mathfrak{N}$ of which the natural numbers $\mathbb{N}$ (equipped with a constant symbol to refer to 0, a unary function to refer to the function $n \mapsto n+1$, and binary functions to refer to additional and multiplication) is a proper subset and an *elementary substructure*, which is to say, if $a_1, \ldots, a_n$ are elements of $\mathbb{N}$ and $\phi(x_1, \ldots, x_n)$ is a formula in which only $x_1, \ldots, x_n$ are free, then $\mathbb{N} \models \phi(a_1, \ldots, a_n)$ if and only if $\mathfrak{N} \models \phi(a_1, \ldots, a_n)$.

Deduce that the theory of $\mathbb{N}$ is not $\aleph_0$-categorical, that is, that it has distinct, non-isomorphic countable models. (The *theory* of a structure $\mathfrak{M}$ is the set of all sentences that are true in $\mathfrak{M}$.)

(v) (Optional, and fiddly, but worth knowing): Let $\mathfrak{M}$ be an infinite model of some theory T in a countable language L of first order predicate calculus. Show that there is a countably infinite substructure $\mathfrak{N}$ of $\mathfrak{M}$ such that $\mathfrak{N}$ is an elementary substructure of $\mathfrak{M}$. Deduce the *Countable Downward Löwenheim-Skolem Theorem*: that if T has an infinite model, then it has a countable model.

(vi) (Optional, and only for those who know enough set theory): Assume that T is a theory in a countable language L of first order predicate calculus, which has arbitrarily large infinite models. Show that T has a model of every infinite size. (Your answer to the previous part will have assumed the Axiom of Choice in some form. Your answer to this one is likely to use it much more heavily.)

(vii) (Definitely optional, and requires set theory): Prove that a theory T in a countable first order language that has an infinite model has arbitrarily large infinite models.

(viii) (Optional: requires set theory) Deduce the *Löwenheim-Skolem Theorem*: if a theory T in a countable first order language has an infinite model, then it has a model of every infinite cardinality.

It follows, of course, that the theory of $\mathbb{N}$ has models of every infinite size.

4. (Optional: uses set theory and model theory) (i) Let $\mathfrak{N}$ be a countable model of the theory of $\mathbb{N}$ which is not isomorphic to $\mathbb{N}$. Prove that $\mathfrak{N}$ is totally ordered, and that it has an initial segment isomorphic to $\mathbb{N}$.

We refer to the elements of $\mathfrak{N}$ belonging to this initial segment as *standard*, and to all the other elements as *non-standard*.

(ii) Show that every non-standard element of $\mathfrak{N}$ belongs to an interval which is order-isomorphic to $\mathbb{Z}$.

(iii) Show that there is no initial such interval order-isomorphic to $\mathbb{Z}$, no final one, and that between any two, there is another.

(iv) Deduce that $\mathfrak{N}$ is order-isomorphic to $\mathbb{N} \oplus (\mathbb{Q} \otimes \mathbb{Z})$, where the operators $\oplus$ and $\otimes$ have the following meanings. If $(P, \leq_P)$ and $(Q, \leq_Q)$ are total orders, with P and Q being disjoint sets, then $P \oplus Q$ is the set $P \cup Q$ equipped with the order $\leq_{P \oplus Q}$ in which Q comes after P; that is, $x \leq y$ iff $x, y \in P$ and $x \leq_P y$, or $x, y \in Q$ and $x \leq_Q y$, or $x \in P$ and $y \in Q$; and $P \otimes Q$ is the cartesian product $P \times Q$ equipped with the lexigraphic order, in which $(p, q) \leq (p', q')$ iff $p < p'$ or $p = p'$ and $q \leq q'$.

Deduce that all countable models of the theory of $\mathbb{N}$ which are not isomorphic to $\mathbb{N}$, are order-isomorphic to each other.

(v) Let $A \subseteq \mathbb{N}$ be the set of all primes. If $f : A \to \{0, 1\}$, let Σ_f be the following set of formulae of an appropriate first-order language of arithmetic, with an additional constant symbol c:

$$\Sigma_f = \{(\overline{p} \mid c) : f(p) = 1\} \cup \{(\overline{p} \nmid c) : f(p) = 0\},$$

where $\overline{p}$ is some term referring to p, and $m \mid n$ means "m is a factor of n".

Prove that there is a countable model $\mathfrak{M}_f$ of the theory of $\mathbb{N}$ in which Σ_f is true.

(vi) Deduce that there are $2^{\aleph_0}$ different non-isomorphic countable models of the theory of $\mathbb{N}$.