Gödel Incompleteness Theorems: Solutions to sheet 2

A.

1. Show that the set $\{n : \mathrm{PAE} \vdash E_n[\overline{n}]\}$ is expressible in complexity Σ_1.

The set is defined by the conjunction of the following statements, which are Σ_1.
(a) n *is the Gödel number of a formula.*
(b) There exists m *such that* $m = \ulcorner \forall v_1 (v_1 = \urcorner ^\frown \ulcorner \overline{n} \urcorner ^\frown \ulcorner \rightarrow \urcorner ^\frown n ^\frown \ulcorner) \urcorner$, *and* $\mathrm{Pr}_{\mathrm{PAE}}(\overline{m})$.

2. Show that for any two formulae $F(v_1)$ and $G(v_1)$ in $\mathscr{L}_E$ with one free variable, there exist sentences X and Y such that the sentences $(X \leftrightarrow G(\overline{\ulcorner Y \urcorner}))$ and $(Y \leftrightarrow F(\overline{\ulcorner X \urcorner}))$ are both true.

Define $\psi(n)$ *to be* $G(\overline{\ulcorner F(\overline{\ulcorner d(n) \urcorner}) \urcorner})$.
Let $k = \ulcorner \psi \urcorner$.
Let $X = \psi[k]$, *and let* $Y = F(\overline{\ulcorner X \urcorner})$.
Then Y *is equal to* $F(\overline{\ulcorner X \urcorner})$, *so is certainly equivalent to it.*
Also, X *is* $\psi[k]$. *But* $k = \ulcorner \psi \urcorner$. *So* X *is* $E_k[k]$; *thus* X *is* $d(k)$. *Also,* X *is* $\psi[k]$ *which is equivalent to* $\psi(k)$ *which is* $G(\overline{\ulcorner F(\overline{\ulcorner d(k) \urcorner}) \urcorner})$, *which is* $G(\overline{\ulcorner F(\overline{\ulcorner X \urcorner}) \urcorner})$ *which is* $G(\overline{\ulcorner Y \urcorner})$.

B.

3. Show that if S is a definable set of sentences in $\mathscr{L}_E$, and Pr_S is an associated proof predicate, and X and Y are any formulae, then

$$\mathrm{PAE} \vdash (\mathrm{Pr}_S(\overline{\ulcorner X \rightarrow Y \urcorner}) \rightarrow (\mathrm{Pr}_S(\overline{\ulcorner X \urcorner}) \rightarrow \mathrm{Pr}_S(\overline{\ulcorner Y \urcorner}))).$$

Because, if x *is a proof for* X *and* z *is a proof for* $X \rightarrow Y$, *then if we concatenate* x *with its last # removed,* z, $\ulcorner Y \urcorner$, *and #, then we get a proof for* Y; *and all of this stuff can be said in the language.*

4. Show that the following functions are primitive recursive.

(i) $P(n)$, which is $n - 1$ if $n > 0$ and 0 if $n = 0$.
Let $h(n, k) = n$. *Then* $P(0) = 0$, *and* $P(n + 1) = h(n, P(n))$.

(ii) $S(m, n)$, which is $m - n$ if $m \geq n$, and 0 if $m < n$.
Let $g(m) = m$, *and* $h(m, n, k) = P(k)$.
Then $S(m, 0) = g(m)$ *for all* m, $S(m, n + 1) = h(m, n, S(m, n))$ *for all* m *and* n.

(iii) $M(m, n) = m.n$.
Let $g(m) = 0$ *for all* m, *and* $h(m, n, k) = k + m$ *(addition was shown to be PR in lectures).*
Then $M(m, 0) = g(m)$ *for all* m, $M(m, n + 1) = h(m, n, M(m, n))$ *for all* m *and* n.

(iv) $E(m, n) = m^n$.
Let $g(m) = 1$ *(which is the successor function composed with the function constant at zero, and so is PR), and* $h(m, n, k) = k.m$.
Then $E(m, 0) = g(m)$, *and* $E(m, n + 1) = h(m, n, E(m, n))$.

(v) $L(m, n) = \min(m, n)$, and $U(m, n) = \max(m, n)$.

$L(m,n) = S(m,n).n + S(n,m).m$, and $U(m,n) = S(m,n).m + S(n,m).n$.

(vi) $G(n) = \min_{m \le n} F(m)$ and $H(n) = \max_{m \le n} F(m)$, where F is primitive recursive.
For G, let $h(n,k) = \min(F(n),k)$; this is PR.
Then $G(0) = F(0)$, and $G(n+1) = h(n, G(n))$.
H is similar.

5. (i) Show that every true, quantifier-free sentence is provable from PAE.

The true atomic formulae are $\overline{n} = \overline{n}$ for all n, all of which are logically valid and therefore provable from PAE, *and $\overline{m} \le \overline{n}$ where $n \ge m$, each instance of which can be proved using the axiom $\forall v_i \forall v_j\, (v_i \le v_j \leftrightarrow (v_i = v_j \vee {v_i}^+ \le v_j))$.*

The true negated atomic formulae are as follows. Firstly, $\neg \overline{m} = \overline{n}$, where $m \ne n$. For the case $m = 0$, this follows from the axiom $\forall v_i\, \neg {v_i}^+ = \overline{0}$ (which is incorrect in the current version of the notes, I need to correct it); for other cases it can be deduced from this by a finite number of applications of the axiom $\forall v_i \forall v_j\, ({v_i}^+ = {v_j}^+ \rightarrow v_i = v_j)$. And secondly, $\neg \overline{m} \le \overline{n}$, where $m > n$. This can be proved from the true atomic statement $\overline{n} \le \overline{m}$, the true statement $\neg \overline{m} = \overline{n}$, and the axiom $\forall v_i \forall v_j\, (((v_i \le v_j) \wedge (v_j \le v_i)) \rightarrow v_i = v_j)$.

Now, it follows that any conjunction of true atomic and negated-atomic sentences is provable; and hence that any true quantifier-free statement in disjunctive normal form is provable. The result now readily follows.

(ii) Prove that if ϕ is quantifier-free, and $\exists v_i \le \overline{n}\, \phi$ is a sentence, then there is a quantifier-free sentence ϕ' which is true if and only if ϕ is true. [*Note that n here is a fixed natural number, and the choice of ϕ' will depend on the choice of n.*]
Define ϕ' to be $(\phi(\overline{0}) \vee \phi(\overline{1}) \vee \cdots \vee \phi(\overline{n}))$.

(iii) Prove that every true Σ_0 sentence is provable from PAE.
Straightforward induction on the complexity of a Σ_0 formula.

(iv) Deduce that every true Σ_1 sentence is provable from PAE.
Suppose that $\phi(v_1)$ is Σ_0, and $\exists v_1\, \phi(v_1)$ is true in $\mathbb{N}$. Then this existential statement has a witness, n say. Then $\phi(\overline{n})$ is true, Σ_0, and hence provable. The result follows.

6. Let $F(\overline{n})$ be the statement "there exists a Σ_1 formula ϕ such that $n = \ulcorner \phi \urcorner$". [*Assume that this is expressible in complexity Σ_0.*]

This is justified, because all we need to say is "$\overline{n}$ is the Gödel number of a formula, $\overline{n}$ begins with $\neg\forall$, and any other instance of $\forall$ in $\overline{n}$ is immediately followed by a variable letter v_i (for some i), then by $(\overline{v}_i \le \sigma \rightarrow$, where σ is a variable letter or a numeral term", and all of this can be said in Σ_0, with all quantifiers bounded by n.

If ϕ is any formula, and n and k are natural numbers, write $\phi(\overline{n}, \overline{k}, \mathbf{0})$ for the result of substituting $\overline{n}$ for all free occurrences of v_1 in ϕ, $\overline{k}$ for all free occurrences of v_2, and $\overline{0}$ for all other free variables. [*Assume that the statement $G(\overline{m}, \overline{m'}, \overline{n}, \overline{k})$ which we define as "If ϕ is such that $m = \ulcorner \phi \urcorner$, then $m' = \ulcorner \phi(\overline{n}, \overline{k}, \mathbf{0}) \urcorner$" can be expressed in Σ_0.*]

We're interested in the case when $\phi(\overline{m}, \overline{n})$ defines a function. The previous version of the sheet attempted to define a function f_ϕ which would coincide with it, but that function wasn't reliably recursive.

(i) Show that the statement $H(\overline{m}, \overline{n}, \overline{k})$, which we define as "$F(\overline{m})$ is true, and if ϕ satisfies $m = \ulcorner \phi \urcorner$, then $\phi(\overline{n}, \overline{k}, \mathbf{0})$" is expressible in complexity Σ_1.

Using the result that any recursively enumerable set is Σ_1 definable, we argue that the set of values (m,n,k) for which $H(\overline{m},\overline{n},\overline{k})$ is recursively enumerable. We do that by arguing that there is an algorithm that will terminate with the answer "yes" on input (m,n,k) if and only if $H(\overline{m},\overline{n},\overline{k})$ is true.

Begin by finding out whether $F(\overline{m})$ is true. If not, then enter an infinite loop (or carry out some behaviour other than outputting "yes" and stopping).

We can now effectively (I don't assume people know the word "effectively") read the formula ϕ whose Gödel number is m, and what we do next depends on the complexity of ϕ. If $\phi(\overline{n},\overline{k})$ is Σ_0, well $\phi(\overline{n},\overline{k})$ is true if and only if it is provable, and so one of $\phi(\overline{n},\overline{k})$ and $\neg\phi(\overline{n},\overline{k})$ will be provable. Look for proofs of those two in parallel, and when you find one, stop and output the appropriate answer.

If $\phi(\overline{n},\overline{k},\mathbf{0}) = \exists w\, \psi(\overline{n},\overline{k},y,\mathbf{0},w)$, where $\psi(\overline{n},\overline{k},\mathbf{z},w)$ is Σ_0, then for each w in turn, decide by the above method whether $\psi(\overline{n},\overline{k},\mathbf{0},w)$ is true or not. As soon as we find one that is true, stop and output "yes".

(ii) Prove that the statement $K(\overline{m},\overline{n})$ which we define as "$F(\overline{m})$ is true, and if ϕ is such that $m = \ulcorner\phi\urcorner$, then there exists k such that $\phi(\overline{n},k,\mathbf{0})$" is expressible in complexity Σ_1.

Modify the above answer as follows.

If $\phi(\overline{n},\overline{k},\mathbf{0}) = \exists w\, \psi(\overline{n},\overline{k},y,\mathbf{0},w)$, where $\psi(\overline{n},\overline{k},\mathbf{z},w)$ is Σ_0, then for each pair $[y,w]$ in turn, decide whether $\psi(\overline{n},y,\mathbf{0},w)$ is true or not. As soon as we find one that is true, stop and output "yes".

C.

7. We use the same notation as in the previous question.

(i) Show that $\neg K(\overline{n},\overline{n})$ is not expressible in complexity Σ_1.

This is another statement of the Halting Problem.

Suppose that $\neg K(\overline{n},\overline{n})$ is expressed by a Σ_1 formula $\phi(\overline{n})$.

Let $\phi'(n,k)$ be the statement "$\neg K(\overline{n},\overline{n})$ holds, and $k=0$". Then $F(\overline{\ulcorner\phi'\urcorner})$ holds. Let $m = \overline{\ulcorner\phi'\urcorner}$.

Then $K(\overline{m},\overline{m})$ holds if and only if $F(\overline{m})$ and $f_{\phi'}(m)$ is defined, if and only if $f_{\phi'}(m)$ is defined, if and only if $\neg K(\overline{m},\overline{m})$", giving us a contradiction.

(ii) (Optional: hard) Let Γ be the smallest set of partial functions with the following properties.

(α) Every recursive partial function belongs to Γ.

(β) The characteristic function of the set $\{(m,n) : K(\overline{m},\overline{n}) \text{ is false}\}$ belongs to Γ.

(γ) Γ is closed under substitution, primitive recursion, and minimalisation.

Here the minimalisation operator, as applied to partial functions f, is defined as follows. g is defined from minimalisation from f iff, for all n, $g(n_1,\ldots,n_k,n)$ is the least m such that for all $l \le m$, $f(n_1,\ldots,n_k,l)$ is defined, and such that $f(n_1,\ldots,n_k,m)=0$, if such an m exists; otherwise $g(n_1,\ldots,n_k,n)$ is undefined.

Sketch an argument that the elements of Γ are precisely the partial functions that can be defined in complexity Σ_2.

This is the Turing jump, applied to the Turing degree of computable functions.

The graph of any Π_1 *partial function* f *(one dimension up) is* Π_1*, as is the set* $A \subseteq \mathbb{N}$ *of* $[n_1, \ldots, n_k]$ *such that* $(n_1, \ldots, n_k)$ *belongs to the graph of* f*.*

Hence the complement of A *is* Σ_1*, and so is the partial function* $\pi_{\mathbb{N}\setminus A}$*.*

Hence there exists m *such that for all* n*,* $K(m,n)$ *holds if and only if* $n \notin A$*.*

We can now define f *in terms of* K *thus:* $f(n_1, \ldots, n_{k-1}) = n_k$ *if and only if* $\chi_{\{(m,n):\neg K(m,n)\}}([n_1, \ldots, n_k]) = 1$*; this can be done using the operations mentioned in (*γ*).*

So any Π_1 *partial function belongs to* Γ*.*

If now f *is a* Σ_2 *function, suppose that* $f(n_1, \ldots, n_k) = n$ *if and only if* $\exists x\, \forall y\, \phi(n_1, \ldots, n_k, n, x, y)$*.*

Let $A = \{(n_1, \ldots, n_k, n, x) : \forall y\, \phi(n_1, \ldots, n_k, n, x)\}$*.*

Then π_A *is* Π_1*, so belongs to* Γ*, and* f *can be derived recursively from* π_A *by running, in parallel, processes checking whether* $\pi_A(n_1, \ldots, n_k, n, x)$ *is defined. (I.e. every unit of time numbered by a number which is a multiple of* 2^x *but not* 2^{x+1}*, run one step of an algorithm to try to compute* $\pi_A(n_1, \ldots, n_k, n, x)$*.)*

Hence f *is in* A*.*

So, any Σ_2 *function belongs to* Γ*.*

Now for the converse. Suppose that $\exists x\, \forall y\, \phi(x, y, \overline{m}_1, \ldots, m_k, n)$ *defines the set* $\{(m_1, \ldots, m_k, n) : g(m_1, \ldots, m_k) = n\}$*, where* g *is some partial function on* $\mathbb{N}^k$*, and* ϕ *is* Σ_0 *and has no free variables other than the ones shown.*

Then the set $A = \{(l, y) : \neg\phi(\overline{l_1}, y, \overline{l_2}, \ldots, \overline{l_{k+1}}, \overline{l_{k+2}})\}$*, where* $l_1, \ldots, k_{k+2}$ *are such that* $l = [l_1, [l_2, \ldots, [l_{k+1}, l_{k+2}] \ldots]]$*, is* Σ_0*-definable, so the set* $A' = \{l : \exists y\, (l, y) \in A\}$ *is* Σ_1*-definable, so* $\pi_{A'}$ *is recursive, so is* $l \mapsto K(k, l)$ *for some* k*.*

Now the function K *belongs to our class, so* $l \mapsto K(k, l)$ *belongs to it also, so* $\pi_{A'}$ *belongs to the class.*

Now $g(m_1, \ldots, m_k)$ *can be derived as follows. Let* l *be least such that* $1 - \pi_{A'}(l) = 0$*, where (for each* i*,* $l_{1+i} = m_i$*). Output* l_{k+2}*.*