

C3.1 Algebraic Topology

Please be aware there are

likely typos in these notes:

comments/corrections are welcome!

Course Book

- **Hatcher, Algebraic Topology** — Chp. 2 & 3

This is also freely available from the author's website.

Expectations

- You are expected to read chapters 2 & 3 of Hatcher
- You should read the technical remarks about orientation signs in these notes: we will likely not have time for those in lectures.
- This course will not discuss intersection numbers rigorously. The notes often mention these in order to develop your intuition.

The books by Bott & Tu and Guillemin & Pollack discuss these ideas rigorously

Other references

- Ulrike Tillmann's C3.1 notes — see course page
- The discussion of orientations in my B3.2 Geometry of Surfaces notes may be helpful.

Other books

Massey, A basic course in Algebraic Topology

(Not "Algebraic Topology: An Introduction" ← does not treat homology)

James W. Vick, Homology Theory

MORE BASIC but full of ideas:

Fulton, Algebraic Topology: a first course.

MORE ADVANCED:

May, A concise course in Algebraic Topology

Davis & Kirk, Lecture Notes in Algebraic Topology

Bredon, Topology and Geometry

Classics by **Spanier**, **Dold**, also see references in May's book

Bott & Tu, Differential forms in Algebraic Topology

Guillemin & Pollack, Differential Topology

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0. OVERVIEW OF THE COURSE

Motivation, category theory, functors H_* and H^* : some computations why functors are useful: Invariance of dimension, Brouwer fixed pt thm

1. ALGEBRAIC PRELIMINARIES: CHAIN COMPLEXES

Graded abelian groups

chain complexes, chain maps, subcomplex, quotient complex

chain map induces map on homology

exact sequence, snake lemma: SES induces LES on H_* , naturality of LES

5-Lemma, SES splits \Leftrightarrow direct sum

2. Δ -COMPLEXES AND SIMPLICIAL HOMOLOGY

Δ^n , n -simplices, Δ -complex (structure), simplicial cx, triangulation

simplicial chain complex, $H_*(S^n)$, $H_*(T^2)$, remark about orientations

$H_*^{\Delta}(\sqcup \text{conn.comp.}) \cong \bigoplus H_*^{\Delta}(\text{conn.comp.})$, $H_0^{\Delta}(X) \cong \mathbb{Z}^{\# \text{conn.comp}}$

3. SINGULAR HOMOLOGY

Motivation, singular chain cx

naturality / functoriality, H_* (point)

4. CHAIN HOMOTOPIES AND HOMOTOPY INVARIANCE

chain homotopy, prism operator

homotopic maps $f \simeq g$ (relative A), homotopy equivalent spaces $X \simeq Y$

contractible space, homotopic maps induce equal maps on homology

homotopy equivalence induces iso on H_* , $H_*(\mathbb{R}^n) = H_*(D^n) = H_*(pt)$

pairs of spaces, relative homology $H_*(X, A)$, LES in H_* for pair

reduced homology $\tilde{H}_*(X)$, LES for \tilde{H}_* , $H_*(D^n, S^{n-1}) \cong \tilde{H}_{*-1}(S^{n-1})$

naturality of LES for pairs

5. EXCISION THEOREM AND QUOTIENTS

retractions, deformation retractions, excision thm, quotients,

good pairs $\Rightarrow H^*(X/A) \cong \tilde{H}^*(X/A)$, generator of $H_n(S^n) \cong H_n(D^n, S^{n-1}) \cong \mathbb{Z}$

another remark about orientations

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6. MAYER-VIETORIS SEQUENCE

MV LES, $H_*(S^n)$

wedge sum $X \vee Y$, cone CX , suspension ΣX , connected sum $X \# Y$

7. DEGREE OF MAPS OF SPHERES

degree, application to tangent vector-fields on sphere, hairy ball theorem
local degree, proof of fundamental thm of algebra

8. CELLULAR HOMOLOGY

CW complexes, cellular complex, rank $H_n^{CW} \leq \#n\text{-cells}$
 $H_*^{CW}(D^1 \times D^1)$, $H_*^{CW}(\mathbb{R}P^n)$, $H_*^{CW}(S^n)$, $H_*^{CW}(\Sigma g)$

$\Delta\text{-cx} \Rightarrow CW\text{ cx}$, $H_*^{CW}(X) \cong H_*^{\Delta}(X) \cong H_*^{\Delta}(X)$, Axioms for homology

9. COHOMOLOGY

cochains, cohomology, $H^*(X)$, $H_{CW}^*(X)$, $H_{\Delta}^*(X)$, $H^*(\mathbb{R}P^3)$
functoriality, homotopy invariance, cochain homotopy, dual of a SES
excision, LES, Mayer-Vietoris for H^* , axioms for cohomology

10. CUP PRODUCT

Cup product, $H^*(T^2)$, $H^*(\Sigma_2)$, remarks about intersection theory
examples: $H^*(T^2)$, $H^*(\Sigma_2)$, remarks about intersection theory

11. KÜNNETH FORMULA AND PRODUCT SPACES

Tensor products of R -mods, tensor product of chain cxes,
algebraic Künneth thm, product spaces $X \times Y$, Euler characteristic χ
CW-cx for product space, Künneth thm, $H^*(S^n \times S^m)$, $H^*(T^n)$

12. UNIVERSAL COEFFICIENTS THEOREM

Universal coeff. thm, Background on Ext groups and free resolutions
(Co)homology with coefficients in a ring/field/module, $H^*(\mathbb{R}P^2; \mathbb{Z}/2)$

Univ. coeff. thm for PID R , Duality $H^*(X; \mathbb{F}) \cong H_*(X; \mathbb{F})$ over fields

Structure thm for f.g. mods M over PID R , $\text{Ext}_R^1(M; R)$, torsion shift H_* to H^{*+1}

13. MANIFOLDS: POINCARÉ-LEFSCHETZ DUALITY

Local orientation, orientation, fundamental class, degree, P. duality, L. duality,
Locally finite homology H_*^{lf} , cohomology with compact supports H_c^* , Cap product and P.D.,
Alexander duality, knot complements, Jordan curve thm

O. OVERVIEW OF THE COURSE

Motivation

Space X associate \implies Algebraic object $A(X)$
like numbers, groups, rings, ...

Isomorphism of spaces $X \cong Y \implies$ Isomorphism $A(X) \cong A(Y)$

PAY-OFF Can tell spaces apart:

compute $A(X), A(Y) \rightsquigarrow$ if $A(X) \not\cong A(Y)$ then $X \neq Y$

Examples

1) Set $X \longrightarrow A(X) = \#X \in \mathbb{N} \cup \{\infty\}$
(bijection $X \rightarrow Y$) \implies same size

2) Vector space $X \longrightarrow A(X) = \dim X \in \mathbb{N} \cup \{\infty\}$
(linear iso $X \rightarrow Y$) \implies same dim

3) Topological Space $X \longrightarrow \# \pi_0(X) = \# \text{path components} \in \mathbb{N} \cup \{\infty\}$
 $\longrightarrow \# \text{Connected components}$

$\chi(X) = \text{Euler characteristic} \in \mathbb{Z}$

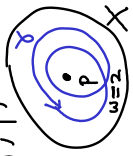
$\leftarrow \text{Loops} = C^0(S^1, X)$

Function $X \times \mathbb{Z}X \longrightarrow \mathbb{Z} \cup \{\infty\}$

$(P, \gamma) \mapsto w(\gamma; P)$

Winding number of γ around P .

(Homeomorphism $X \rightarrow Y$) $\longrightarrow A(X) = A(Y)$



CONVENTION for this course:

Space = topological space

Map of spaces = continuous map = cts map

For spaces, " \cong " means homeomorphism

"id" = identity map

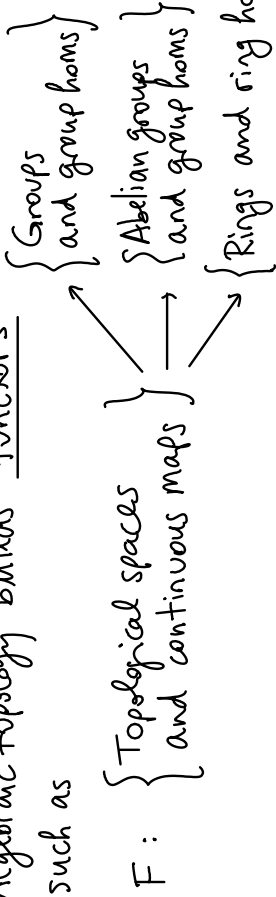
All diagrams commute unless we say otherwise, e.g.

$A \xrightarrow{\alpha} B$ means $\delta \downarrow \delta \downarrow \beta \circ \alpha = \delta \circ \beta$

Category theory is the best language to phrase all this

Algebraic topology builds functors

such as



We will not use much category theory, just basic terminology:

Def A category C consists of the data:

Ob(C) = a collection of objects

Hom(A, B) = a set of morphisms between any $A, B \in \text{Ob } C$ ("arrows")

- with composition rule $\text{Hom}(B, C) \times \text{Hom}(A, B) \rightarrow \text{Hom}(A, C)$
 $A \xrightarrow{f} B \xrightarrow{g} C$

- with identity morphs $\text{id}_A \in \text{Hom}(A, A)$ s.t. $f \circ \text{id}_A = \text{id}_B \circ f = f$

Example Sets = { sets with all maps between sets }

Top = { topological spaces with continuous maps }

Gps = { groups with group homs }

Def A (covariant) functor $F: C_1 \rightarrow C_2$ is the data:

- an assignment $(A \in \text{Ob } C_1) \mapsto (F(A) \in \text{Ob } C_2)$
- an assignment $(A \xrightarrow{f} B) \mapsto (F(A) \xrightarrow{F(f)} F(B))$

Compatible with identities and compositions.

$F(\text{id}_A) = \text{id}_{F(A)}$ $F(g \circ f) = F(g) \circ F(f)$

A contravariant functor is defined similarly except it reverses the direction of arrows: $(F(A) \xleftarrow{F(f)} F(B)) \in \text{Hom}(F(B), F(A))$

(so $F(g \circ f) = F(f) \circ F(g)$ reverses order of compositions)

Examples

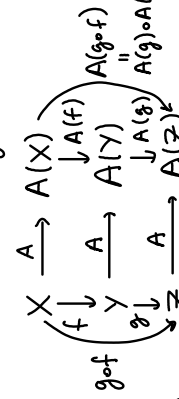
1) $F: \text{Top} \rightarrow \text{Sets}, A \mapsto A, f \mapsto f$ "forget the topology and continuity"

2) $F: \text{Sets} \rightarrow \text{Gps}, A \mapsto$ free abelian group generated by A

$$Z\langle A \rangle = \left\{ \sum_{\text{finite}} n_i \cdot a_i : a_i \in A, n_i \in \mathbb{Z} \right\}$$

$$(A \xrightarrow{f} B) \mapsto (F(f) : Z\langle A \rangle \rightarrow Z\langle B \rangle)$$

When we say a construction is natural we mean functorial:



A: (a category of spaces) \rightarrow (a cat. of algebraic objects)

The algebraic objects we assigned

are assigned compatibly with maps of spaces,

and the compatibility maps $A(f)$ are also compatible w.r.t. composition.

So we made compatible choices in constructing A.

Not to be confused with natural transformations of functors (later) which is about relating two such constructions A_1, A_2 in a compatible way

Example of a functor in algebraic topology (see B.3.5 Topology and Groups course)

$$\pi_1(X, p) = \text{Fundamental group} = \left\{ \gamma \in \mathcal{L}X : \gamma(1) = p \right\}$$

continuous deformations of loops based at p

Group multiplication: concatenate loops $\gamma_1 * \gamma_2$ (each travelling twice as fast)

Examples $\pi_1(\mathbb{R}^n) = 0$ (for basepoint $0 \in \mathbb{R}^n$: deform $h: S^1 \times [0, 1] \rightarrow \mathbb{R}^n, h(t, s) = (1-s)\gamma(t)$)

$\pi_1(S^1) \cong \mathbb{Z}$ (total # times wind around circle)

$\pi_1(S^n) \cong 0$ ($n \geq 2$ not obvious)

$\pi_1(\text{torus}) \cong \mathbb{Z}^2$

Those loops generate π_1

Based Top = { Topological spaces with choice of base point, and continuous basepoint-preserving maps } $\xrightarrow{\pi_1} \text{Gps}$

$(X, p) \mapsto \pi_1(X, p)$

$((X, x) \xrightarrow{f} (Y, y)) \mapsto (\pi_1(X, x) \xrightarrow{f_{\text{phom.}}} \pi_1(Y, y))$

$\gamma \mapsto f \circ \gamma$



Lemma Functors map isomorphisms to isomorphisms (iso. means \exists inverse w.r.t. composition)
 Pf $A \xrightarrow{f} B \xrightarrow{g} A \Rightarrow FA \xrightarrow{Ff} FB \xrightarrow{Fg} FA$, similarly for $B \xrightarrow{g} A \xrightarrow{f} B$. \square

Def **Natural transformation** $\alpha: F \rightarrow G$ between functors $C \xrightarrow{F} G$
 is an association $(A \in \text{Ob } C) \mapsto (\alpha_A: F(A) \rightarrow G(A))$

such that $(A \xrightarrow{f} B) \Rightarrow \begin{matrix} F(A) \xrightarrow{\alpha_A} G(A) \\ \downarrow F(f) \quad \downarrow G(f) \\ F(B) \xrightarrow{\alpha_B} G(B) \end{matrix}$ (commutes)

It is called a **natural isomorphism** if each α_A is an isomorphism in C_2

Example of a natural transformation in algebraic topology

Let $H_1(X, P) =$ abelianisation of $\pi_1(X, P)$ (want to identify $ab=ba$)
 \Rightarrow natural trans. $(\text{Based Top } \pi_1 \rightarrow \text{Gps}) \xrightarrow{\alpha} (\text{Based Top } H_1 \rightarrow \text{Gps})$ **Commutators**
 which associates $(X, P) \mapsto (\alpha_{(X,P)}: \pi_1(X, P) \xrightarrow{\text{quotient}} H_1(X, P))$

Cultural link higher homotopy groups $\pi_n(X, P) = \left\{ \begin{matrix} S^n \xrightarrow{\text{cts}} X \\ \text{basept } \mapsto P \end{matrix} \right\} / \text{deform}$

FACT abelian for $n \geq 2$, but hard: e.g. $\pi_k(S^n)$ not all known.
 We will not study these in this course.
 We will study simpler invariants called HOMOLOGY groups $H_n(X)$

FACT (Hurewicz) \exists natural transformation $\pi_n \rightarrow H_n$
 which will make sense at the end of course:
 $f: S^n \xrightarrow{\text{cts}} X$ gives rise to a class $f_*[S^n] \in H_n(X)$.

Exercise to practice these notions from category theory:

- Summarise your undergraduate linear algebra as follows:
- \exists functor $F: \left\{ \begin{matrix} \mathbb{R}^n \text{ for } n \in \mathbb{N} \\ \text{Hom}(\mathbb{R}^m, \mathbb{R}^n) = \begin{matrix} m \times n \\ \text{[matrices]} \end{matrix} \end{matrix} \right\} \rightarrow \left\{ \begin{matrix} \text{finite dimensional real vector spaces} \\ \text{and linear maps} \end{matrix} \right\}$
 - A choice of basis for each vector space V determines a functor $G: \text{Vect} \rightarrow \text{Mat}$
 - Construct natural isomorphisms $G \circ F \xrightarrow{\alpha} \text{Id Mat}$, $F \circ G \xrightarrow{\beta} \text{Id Vect}$
- When functors satisfying such natural isos exist, the categories are called **equivalent** (not isomorphic). So Mat, Vect are equivalent categories.

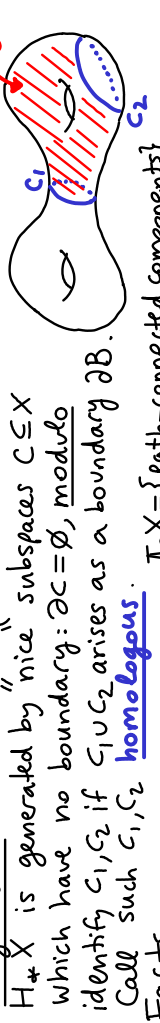
Aim of the course: build a functor

HOMOLOGY $H_*: \text{Top} \rightarrow \text{Graded abelian groups}$
 $(X \rightarrow Y) \mapsto (H_*(X) \rightarrow H_*(Y))$
 (grading preserving hom)

and a contravariant functor

COHOMOLOGY $H^*: \text{Top} \rightarrow \text{Graded rings}$
 $(X \rightarrow Y) \mapsto (H^*(X) \leftarrow H^*(Y))$

Rough idea:



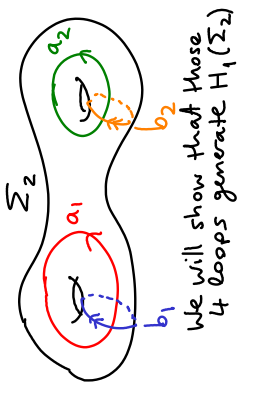
H_*X is generated by "nice" subspaces $C \subseteq X$ which have no boundary: $\partial C = \emptyset$, modulo identify C_1, C_2 if $C_1 \cup C_2$ arises as a boundary ∂B .
 Call such C_1, C_2 **homologous**.
Facts
 • $H_0(X) \cong \bigoplus_{\text{Top } X} \mathbb{Z} \leftarrow \pi_0 X = \{\text{path-connected components}\}$
 • $X = \sqcup X_i$; path-components $\Rightarrow H_*(X) \cong \bigoplus H_*(X_i)$
 • $\chi(X) = \sum_{d \geq 0} (-1)^d \text{rank } H_d(X)$
 generated by a point in each path-comp. (two points connected by a path are homologous)

Euler characteristic

Example: compact surfaces

$H_*(\Sigma_g) \cong \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z}^{2g} & * = 1 \\ \mathbb{Z} & * = 2 \\ 0 & \text{else} \end{cases}$
 orientable surface
 genus g
 $\chi = 2 - 2g$

$H_*(N_k) \cong \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z}_2 \oplus \mathbb{Z}^{k-1} & * = 1 \\ 0 & \text{else} \end{cases}$
 non-orientable surface
 S^2 with k Möbius bands attached
 $\chi = 2 - k$



We will show that those 4 loops generate $H_1(\Sigma_2)$
 $N_1 = \mathbb{R}P^2 = S^2 / \pm \text{Id}$
 Notice γ is a loop. It generates $H_1(N_1) = \mathbb{Z}_2$
 (notice " 2γ " is homologous to 0:)

Examples of homology calculations

$$H_*(\mathbb{R}^n) \cong H_*(D^n) \cong \begin{cases} \mathbb{Z} & * = 0 \\ 0 & \text{else} \end{cases}$$

$$H_*(S^n) \cong \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z} & * = n \\ 0 & \text{else} \end{cases}$$

$\{x \in \mathbb{R}^{n+1} : \|x\|=1\}$ n-dim sphere

$$H_*(X) = \begin{cases} 0 & \text{for } * < 0 \\ 0 & \text{for } * > n \\ \mathbb{Z} & \text{for } * = n \text{ for connected orientable compact manifold} \\ 0 & \text{for } * = n \text{ for non-orientable} \\ 0 & \text{for } * = n \text{ for non-compact} \end{cases}$$

boundary point has an open nbhd homeo to open nbhd of $0 \in \text{half-space} : \{x \in \mathbb{R}^n : x_n \geq 0\}$

Rmk M compact connected n-mfd

$$\Rightarrow H_{n-1}(M) \cong \begin{cases} \mathbb{Z}^k & \text{some } k \geq 0 \text{ if orientable} \\ \mathbb{Z}^k \oplus \mathbb{Z}/2 & \text{non-orientable} \end{cases}$$

$$H_*(\mathbb{R}P^n) \cong \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z}/2 & \text{odd } * = 1, 3, 5, \dots < n \\ \mathbb{Z} & * = n \text{ if } n \text{ odd} \\ 0 & \text{else} \end{cases}$$

$\mathbb{R}P^n$ orientable $\Leftrightarrow n$ odd (e.g. $\mathbb{R}P^1 \cong S^1$)

(equivalently: spaced real lines through 0 in \mathbb{R}^{n+1})

$$H_*(\mathbb{C}P^n) \cong \begin{cases} \mathbb{Z} & \text{even } * = 0, 2, 4, \dots, 2n \\ 0 & \text{else} \end{cases}$$

e.g. $\mathbb{C}P^1 \cong S^2$

space of complex lines through $0 \in \mathbb{C}^{n+1}$

$$\text{Complex projective space} \cong (\mathbb{C}^{n+1} \setminus 0) / \mathbb{C}^* \text{-rescaling} = \{ [z_0, z_1, \dots, z_n] : z_j \in \mathbb{C} \text{ not all } 0 \} / \text{for } \lambda \in \mathbb{C}^*$$



stereographic projection

Examples of cohomology calculations

$$H^0(X) = \prod_{\pi_0 X} \mathbb{Z} \leftarrow \text{if } \pi_0 X \text{ finite, then } \cong \bigoplus_{\pi_0 X} \mathbb{Z} \cong H_0 X$$

but if infinite then not: here allow only finite sums

$$H^*(X) \cong \prod H^*(X_i) \leftarrow X_i \text{ path-components of } X$$

FACT If $H_n(X)$ finitely generated abelian gp, so

$$H_n(X) \cong \mathbb{Z}^r \oplus T_n \leftarrow T_n = \text{torsion elements} = \text{elements of finite order}$$

Then $H^n(X) \cong \mathbb{Z}^r \oplus T_{n-1}$ as abelian groups

$\Rightarrow H^*(\Sigma_g), H^*(\mathbb{R}^n), H^*(D^n), H^*(S^n), H^*(\mathbb{C}P^n)$ same as for H_* , but:

$$H^*(N_k) = \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z}^{h-1} & * = 1 \\ \mathbb{Z}/2 & * = 2 \\ 0 & \text{else} \end{cases} \quad H^*(\mathbb{R}P^2) = \begin{cases} \mathbb{Z} & * = 0 \\ 0 & * = 1 \\ \mathbb{Z}/2 & * = 2 \\ 0 & \text{else} \end{cases} \quad H^*(\mathbb{R}P^n) \cong \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z}/2 & * \text{ even} = 2, 4, \dots \leq n \\ 0 & * = n \text{ if } n \text{ odd} \\ 0 & \text{else} \end{cases}$$

and H^n (non-orientable compact n-mfd) $\cong \mathbb{Z}/2$.

\Rightarrow The interesting feature is the ring structure:

$$H^*(\mathbb{C}P^n) \cong \mathbb{Z}[x] / x^{n+1}$$

$\mathbb{Z}[x]$ = polynomials in x with \mathbb{Z} -coefficients

$$H^*(S^n) \cong \mathbb{Z}[x] / x^2$$

grading: $|x|=2$

$$H^*(T^n) \cong \wedge[x_1, \dots, x_n]$$

$|x_i|=1$

$S^1 \times \dots \times S^1$ n-torus

exterior algebra generated by symbols $x_i \wedge \dots \wedge x_k$ with $i_1 < \dots < i_k$

product given by \wedge using relations $x_i \wedge x_j = -x_j \wedge x_i$ and $x_i \wedge x_i = 0$.

$$H^*(\mathbb{R}P^{2n}) \cong \mathbb{Z}[x] / (2x, x^{n+1})$$

$$H^*(\mathbb{R}P^{2n+1}) \cong \mathbb{Z}[x, y] / (2x, x^{n+1}, y^2, xy)$$

where $|x|=2$

$$H^*(\Sigma_g) \cong \wedge[a_1, \dots, a_g, b_1, \dots, b_g]$$

$|a_i|=|b_i|=1$

exterior alg. instead of poly. alg. since $a_i \cdot b_i = -b_i \cdot a_i$

Why more information? connected sum: remove a ball in each, glue along ∂ ball

$S^2 \times S^2$ and $\mathbb{C}P^2 \# \mathbb{C}P^2$ have same $H_* = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & * = 0 \\ \mathbb{Z} & * = 2 \end{cases}$

but the rings H^* are not iso, hence $S^2 \times S^2 \not\cong \mathbb{C}P^2 \# \mathbb{C}P^2$.

Example of why such functors are useful

Suppose $\exists F_*: \text{Top} \rightarrow \text{Gps}$ functors s.t.

① $F_*(S^n) \neq 0 \iff * = n$ and ② $F_*(D^n) = 0$ all $*$

Rmk we'll build such an F_* : $\text{reduced homology } \tilde{H}_*$
 s.t. $\tilde{H}_* = H_*$ for $* \neq 0$, and $\tilde{H}_0 \cong \mathbb{Z}^{\#(\text{path-components})-1}$

Theorem Invariance of dimension

$$\begin{matrix} S^n \cong S^m & \iff & n=m \\ \mathbb{R}^n \cong \mathbb{R}^m & \iff & n=m \end{matrix}$$

by ①

Pf Lemma $\Rightarrow F_n(S^n \cong S^m)$ is iso $F_n(S^n) \cong F_n(S^m)$ of gps.

If $\mathbb{R}^n \xrightarrow{\cong} \mathbb{R}^m$, then can extend $\times 0$ if $n \neq m$ ✓

φ to the one-point compactifications: $S^n \cong \mathbb{R}^n \cup \{\infty\} \xrightarrow{\cong} \mathbb{R}^m \cup \{\infty\} \cong S^m, \infty \mapsto \infty. \square$
 ("Alexandroff extension") $\xrightarrow{\text{stereographic projection } (x_0, \dots, x_n) \mapsto \frac{(x_1, \dots, x_n)}{1-x_0}}$

Rmk new open neighbourhoods at ∞ are $\{\infty\} \cup (\mathbb{R}^n \setminus C)$ where C is (closed B) compact.
 The extended map is cts since $\varphi^{-1}(C)$ is (closed B) compact since φ^{-1} is homeo.

Theorem Brouwer fixed point thm by ① & ②

$f: D^n \rightarrow D^n$ continuous $\Rightarrow f$ has a fixed point ($f(p) = p$ some p)

Proof Suppose not. Let $r(x) = (\text{ray from } f(x) \text{ to } x) \cap \partial D^n$

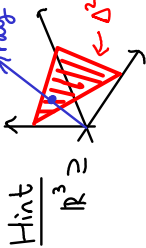
notice: $\cdot r: D^n \rightarrow \partial D^n = S^{n-1}$ continuous

$$\cdot r|_{\partial D^n} = \text{id}_{S^{n-1}} \quad \text{continuous}$$

$$S^{n-1} = \partial D^n \xrightarrow{\text{inclusion } i} D^n \xrightarrow{r} S^{n-1}$$

apply F_{n-1} $F_{n-1}(r) \circ F_{n-1}(i) = \text{id} \Rightarrow F_{n-1}(i)$ injective $F_{n-1}(S^{n-1}) \rightarrow F_{n-1}(D^n) \xrightarrow{\cong} 0$

Example $A = n \times n$ matrix, $A_{ij} > 0$ real $\Rightarrow \exists$ real value $\lambda > 0$ with real vector (v_1, \dots, v_n) with $v_i > 0$



Hint $\mathbb{R}^3 \cong X = \{\text{rays in positive octant}\} \xrightarrow{\text{notice } AX \subseteq X}$

notice $AX \subseteq X$
 notice $X \cong \Delta^n = \{x \in \text{octant} : \sum x_i = 1\} \cong D^n$
 $\text{ray} \mapsto \text{ray} \cap \Delta^n$

1. ALGEBRAIC PRELIMINARIES: CHAIN COMPLEXES

Graded abelian groups

Def A \mathbb{Z} -graded abelian group C is an abelian group together

with a direct sum decomposition

$$C = \bigoplus_{n \in \mathbb{Z}} C_n$$

abelian group

Convention: always grade by \mathbb{Z} unless say otherwise.

Example $C = \mathbb{Z}[x] = \text{integer polynomials in } x, C_n = \mathbb{Z} \cdot x^n \leftarrow \text{so grading by degree}$

A graded ab. gp. A is a graded subgp of C if $\cdot \text{subgp}$

$\cdot A_n \subseteq C_n$

A homomorphism $h: C \rightarrow D$ of gr.ab.gps is hom of gps s.t.

$$h(C_n) \subseteq D_n$$

A hom of degree k is hom with

$$h(C_n) \subseteq D_{n+k}$$

Shift by k: \mathbb{Z} -gr.ab.gp. $C[k]$ with

$$C[k]_n = C_{k+n}$$

Notice:

$C[k]_0 = C_k$
 is now in degree zero, so shifted down by k

\Rightarrow Can view gr. hom of deg k as a gr. hom

$$h: C \rightarrow D[k]$$

Abelian groups which are finitely generated

FACT Finitely generated abelian groups are classified:

$$G \cong \mathbb{Z}^r \oplus \mathbb{Z}/p_1^{n_1} \oplus \dots \oplus \mathbb{Z}/p_k^{n_k}$$

free part \rightarrow called rank G

Compare finite dimensional vector spaces/field \mathbb{F} : $V \cong \mathbb{F}^r, r = \dim V$

"homeomorphisms preserve dimension"

Non-trivial result because there are space-filling curves.

e.g. Peano (1890)

\exists cts surjection $[0,1] \rightarrow [0,1]^2$

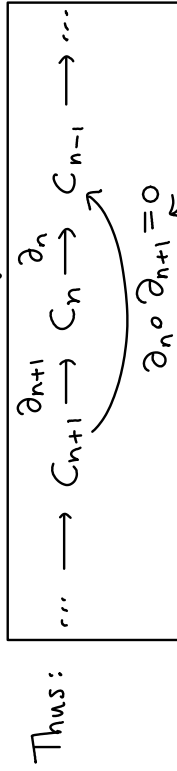
interval square

The theorem implies this is not injective.

(cts, bij, compact \rightarrow Hausdorff) \Rightarrow homeo

Chain complexes

Def A chain complex (C_*, ∂_*) is a gr. ab. gp. C together with a hom ∂ of degree -1 such that $\partial \circ \partial = 0$.



n-chains = elements of C_n

hence $\text{Im } \partial_{n+1} \subseteq \text{Ker } \partial_n$

$B_n \subseteq Z_n$ n-boundaries n-cycles

Now consider "cycles modulo boundaries":

Def The homology of (C_*, ∂_*) is the gr. ab. gp.

$$H_n(C_*, \partial_*) = \frac{\text{Ker } \partial_n}{\text{Im } \partial_{n+1}}$$

often abbreviate by $H_*(C)$

Two cycles are called homologous if they differ by a boundary

Def A chain map $h : (C_*, \partial_*) \rightarrow (\tilde{C}_*, \tilde{\partial}_*)$ is a graded hom such that $h \circ \partial_* = \tilde{\partial}_* \circ h$

Example A chain subcomplex $C_* \subseteq \tilde{C}_*$ is a graded subgp with $\partial_* = \text{restriction of } \tilde{\partial}_* \text{ to } C_*$ (so require $\tilde{\partial}_*(C_*) \subseteq C_*$)
 So the inclusion $\text{incl} : (C_*, \partial_*) \rightarrow (\tilde{C}_*, \tilde{\partial}_*)$ is a chain map.
 Also get quotient complex $\tilde{C}_*/C_* \leftarrow (\text{so cosets } [\tilde{c}] = \{\tilde{c} + c : c \in C_*\})$
 with $\tilde{\partial}_*[\tilde{c}] = [\tilde{\partial}_*\tilde{c}]$ (well-defined: $\tilde{\partial}_*C_* = \partial_*C_* \subseteq C_*$)

Lemma A chain map induces a hom on homology

$$H_*(h) = h_* : H_*(C_*, \partial_*) \longrightarrow H_*(\tilde{C}_*, \tilde{\partial}_*)$$

$$[x] \longmapsto [h(x)]$$

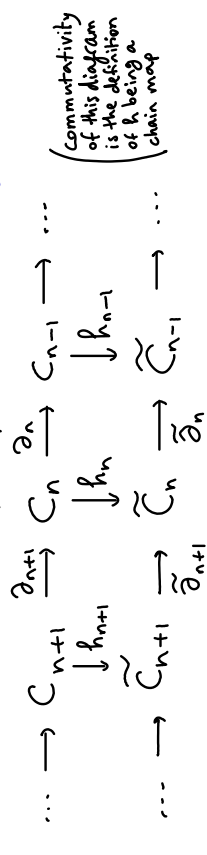
Proof $\text{Ker } \partial_n \xrightarrow{\text{hom}} \text{Ker } \tilde{\partial}_n$ since $\tilde{\partial}(h(x)) = h(\partial x) = 0$

Need $\text{Im } \partial_n \longrightarrow \text{Im } \tilde{\partial}_n$ to get well-defined hom

$$H_n(C) = \text{Ker } \partial_n / \text{Im } \partial_{n+1} \longrightarrow \text{Ker } \tilde{\partial}_n / \text{Im } \tilde{\partial}_{n+1} = H_n(\tilde{C})$$

Proof: $h(b) = h(\partial c) = \tilde{\partial} h(c) \in \text{Im } (\tilde{\partial})$. \square

The last step was a very simple example of a proof by "diagram chasing"



(Commutativity of this diagram is the definition of h being a chain map)

$$c \xrightarrow{\partial} \partial c = b$$

$$h \downarrow \quad \downarrow h$$

$$h(c) \xrightarrow{\tilde{\partial}} \tilde{\partial}(h(c)) = h(\partial c) = h(b)$$

Curiosity (Non-examinable) If C_n free abelian gp $\forall n$, then every graded hom $H_*(C) \rightarrow H_*(\tilde{C})$ arises from a chain map. [see Dold, Proposition II.4.6]

Def (C_*, ∂_*) is exact (or acyclic) if $H_*(C) = 0$

so $\text{Im } \partial_{n+1} = \text{Ker } \partial_n$

In general exact sequence means "Im(previous map) = Ker(next map)"

Example If $0 \rightarrow A \xrightarrow{i} B \rightarrow 0$ is exact then $A \cong B$ via i .

A short exact sequence (SES) is an exact sequence $0 \rightarrow A \xrightarrow{i} B \xrightarrow{\pi} C \rightarrow 0$

Easy exercise

$$\begin{aligned}
 & \left(0 \rightarrow A \xrightarrow{i} B \xrightarrow{\pi} C \rightarrow 0 \right) \Leftrightarrow \begin{cases} i & \text{injective} \\ \pi & \text{surjective} \end{cases} \\
 & \text{exact} \\
 & B/i(A) \cong C \text{ via } [b] \mapsto \pi(b)
 \end{aligned}$$

Examples

$$\begin{aligned}
 0 & \rightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\text{mod } 2} \mathbb{Z}/2 \rightarrow 0 \\
 0 & \rightarrow \mathbb{Z} \xrightarrow{\text{inclusion}} \mathbb{Z} \oplus \mathbb{Z}/2 \xrightarrow{\text{project}} \mathbb{Z}/2 \rightarrow 0
 \end{aligned}$$

Note A, C do not determine B.

Snake Lemma A SES of chain complexes and chain maps yields a long exact sequence (LES) on homology:

$$\dots \rightarrow H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{\pi_*} H_n(C) \xrightarrow{\delta} H_{n-1}(A) \xrightarrow{i_*[-1]} \dots$$

(So exact triangle: $H_*(A) \rightarrow H_*(B) \rightarrow H_*(C)$)
 degree -1 map $H_*(C) \rightarrow H_*(A)[-1]$ called connecting map

Pf Simplify notation by identifying A with $i(A) \subseteq B$:
 $\epsilon_A \equiv i(A) \in B$
 $\partial a \equiv i \partial a = \partial i a$
 \Rightarrow now $A_* \subseteq B_*$ inclusion of subcomplex:

$$\begin{aligned}
 0 & \rightarrow (A_*, \partial_*) \xrightarrow{\text{incl}} (B_*, \partial_*) \xrightarrow{\pi} (C_*, \tilde{\partial}_*) \rightarrow 0 \\
 0 & \rightarrow A_n \rightarrow B_n \rightarrow C_n \rightarrow 0 \\
 0 & \rightarrow A_{n-1} \rightarrow B_{n-1} \rightarrow C_{n-1} \rightarrow 0
 \end{aligned}$$

$$\exists b \xrightarrow{\text{surj.}} C \xrightarrow{\text{cycle}} \pi(b) = 0$$

$$\begin{aligned}
 \partial b & \rightarrow \partial b \rightarrow \tilde{\partial} c = 0 \\
 & \text{lifts to } A \text{ by exactness}
 \end{aligned}$$

Define $\delta: H_*(C) \rightarrow H_*(A)[-1]$ (typically b is not in A, so ∂b need not be a bdy in A)

Well-defined? $\pi^{-1}(c) = \{b+a: a \in A\}$ where $b \in \pi^{-1}(c)$

- cycle \rightarrow cycle: $\partial(\partial b) = 0$ ✓
- boundary \rightarrow boundary: $\exists \beta \xrightarrow{\text{surj.}} c \in C_{n+1}$ boundary in A ✓

\Rightarrow can pick $b = \partial \beta$

$\Rightarrow \partial b = \partial \partial \beta = 0$ ✓

because if $b' \in \pi^{-1}(c)$ then $\pi(b-b') = c-c = 0$ so $b-b' \in A$ by exactness

Exactness at $H_n(C)$ (exercise: check exactness at H_*A, H_*B):

Need $\text{Im } \pi_* = \text{Ker } \delta$:

$$\leq: \delta(\pi_* b) = \partial b = 0 \text{ ✓}$$

$\geq: \exists a \rightarrow b \rightarrow c = \pi_* b$ not necessarily cycle!

$$\begin{aligned}
 & \exists a \rightarrow b \rightarrow c = \pi_* b \quad \pi_* A = 0 \\
 & \partial a = \delta c = \partial b \rightarrow \partial b = 0 \quad \pi_*(b-a) = c \\
 & \text{assumption } \delta c = 0 \in H_*A \quad \partial(b-a) = \partial b - \partial a = 0 \text{ thus cycle!} \\
 & \Rightarrow c = \pi_*(b-a) \in \text{Im } \pi_* \quad \square
 \end{aligned}$$

Rmk $0 \rightarrow A \xrightarrow{i} B \xrightarrow{\pi} C \rightarrow 0$ SES \Rightarrow the connecting map of LES is

$$\boxed{\delta: H_*(C) \rightarrow H_*(A)[-1]}$$

Lemma The construction of δ is natural (i.e. functorial)

$$\begin{aligned}
 \text{chain maps } 0 \rightarrow A \xrightarrow{i} B \xrightarrow{\pi} C \rightarrow 0 & \xrightarrow{\delta} 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \Rightarrow \delta h c = \tilde{i}^{-1} \tilde{\partial} g b \\
 \begin{matrix} f \downarrow & g \downarrow & h \downarrow \\ 0 \rightarrow \tilde{A} \xrightarrow{\tilde{i}} \tilde{B} \xrightarrow{\tilde{\pi}} \tilde{C} \rightarrow 0 \end{matrix} & \begin{matrix} f \downarrow & g \downarrow & h \downarrow \\ 0 \rightarrow \tilde{A} \xrightarrow{\tilde{i}} \tilde{B} \xrightarrow{\tilde{\pi}} \tilde{C} \rightarrow 0 \end{matrix} & \begin{matrix} f \downarrow & g \downarrow & h \downarrow \\ 0 \rightarrow \tilde{A} \xrightarrow{\tilde{i}} \tilde{B} \xrightarrow{\tilde{\pi}} \tilde{C} \rightarrow 0 \end{matrix} \\
 \text{maps } & \text{maps } & \text{maps } \\
 & \tilde{g} b = \delta h c & = f a \\
 & & = f \delta c \quad \square
 \end{aligned}$$

Exercise Deduce the LES is natural, so

$$\begin{aligned}
 \dots \rightarrow H_*A \xrightarrow{i_*} H_*B \xrightarrow{\pi_*} H_*C \xrightarrow{\delta} H_{*-1}(A) \rightarrow \dots \\
 \begin{matrix} f_* \downarrow & g_* \downarrow & h_* \downarrow \\ \dots \rightarrow H_*\tilde{A} \xrightarrow{\tilde{i}_*} H_*\tilde{B} \xrightarrow{\tilde{\pi}_*} H_*\tilde{C} \xrightarrow{\tilde{\delta}} H_{*-1}(\tilde{C}) \rightarrow \dots \end{matrix}
 \end{aligned}$$

5-Lemma

$$\begin{array}{c}
 A \rightarrow B \rightarrow C \rightarrow D \rightarrow E \\
 \cong \downarrow \alpha \cong \downarrow \beta \quad \downarrow \gamma \quad \cong \downarrow \delta \cong \downarrow \epsilon \\
 A' \rightarrow B' \rightarrow C' \rightarrow D' \rightarrow E'
 \end{array}$$

exact rows $\implies \gamma$ also iso.

Pf exercise (diagram chase) \square

Splitting Lemma

Cor $0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0$ SES of abelian gps

If $B \xrightarrow{\beta} C$ s.t. $\beta \circ \gamma = \text{id}_C$ then the SES splits: $B \cong A \oplus C$

Pf $0 \rightarrow A \rightarrow A \oplus C \rightarrow C \rightarrow 0$
 $\parallel \downarrow \alpha \downarrow \gamma \parallel \parallel$
 $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$

converse: if $B \cong A \oplus C$
 then define $\gamma: C \rightarrow B$

Exercise If $A \xrightarrow{\alpha} B$ s.t. $\mu \circ \alpha = \text{id}_A$ then it splits: $B \cong A \oplus C$
 (and equivalent to existence of γ above)

Exercise If C is a free abelian group ($C \cong \bigoplus_{i \in I} \mathbb{Z}$) then the SES splits.

Rank A free \nRightarrow splits, e.g. $0 \rightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \rightarrow \mathbb{Z}/2 \rightarrow 0$

Cultural Rank Splitting Lemma generalises the rank-nullity theorem from

linear algebra: $V \xrightarrow{\beta} W$ linear map of vector spaces $\implies \text{Im } \beta \oplus \text{Ker } \beta \cong V$

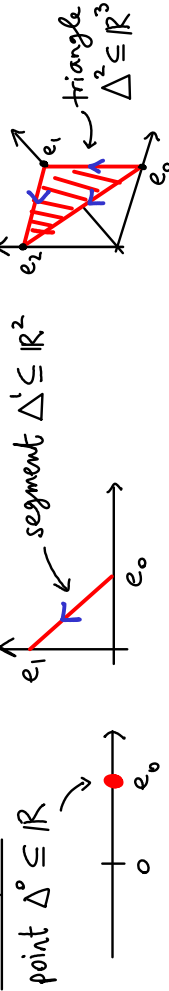
Pf $0 \rightarrow \text{Ker } \beta \xrightarrow{\text{incl}} V \xrightarrow{\beta} \text{Im } \beta \rightarrow 0$ is SES, and splits since $\text{Im } \beta$ free.

2. Δ -COMPLEXES AND SIMPLICIAL HOMOLOGY

standard n-simplex $\Delta^n = \left\{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} : t_i \geq 0, \sum t_i = 1 \right\}$

standard basis of \mathbb{R}^{n+1}
 e_0, \dots, e_n ($e_0 = (1, 0, \dots, 0)$, ...)

Examples



Def For $\{v_0, \dots, v_n\} \subseteq \mathbb{R}^{n+k}$ s.t. any $k \geq 0$

v_1, \dots, v_n \mathbb{R} -linearly independent

$[v_0, \dots, v_n] = n$ -Simplex spanned by v_0, \dots, v_n

= convex hull of v_0, \dots, v_n

= $\{ \sum t_i v_i : t_i \geq 0 \text{ and } \sum t_i = 1 \}$

= Image of \mathbb{R} -linear homeo $\sigma: \Delta^n \rightarrow \mathbb{R}^{n+k}$
 $\sigma(e_i) = v_i$

canonical homeomorphism (homeo onto the image, not onto \mathbb{R}^{n+k})

Will often blur the distinction between map σ and its image,

$$\sigma \equiv [\sigma e_0, \dots, \sigma e_n]$$

but the ordering of the v_i will be important (so the map σ is more precise)

We encode this extra data by orienting the edges $v_i \rightarrow v_j$ if $i < j$

Def d-dimensional faces $[v_{i_0}, \dots, v_{i_d}]$ for $i_0 < \dots < i_d$

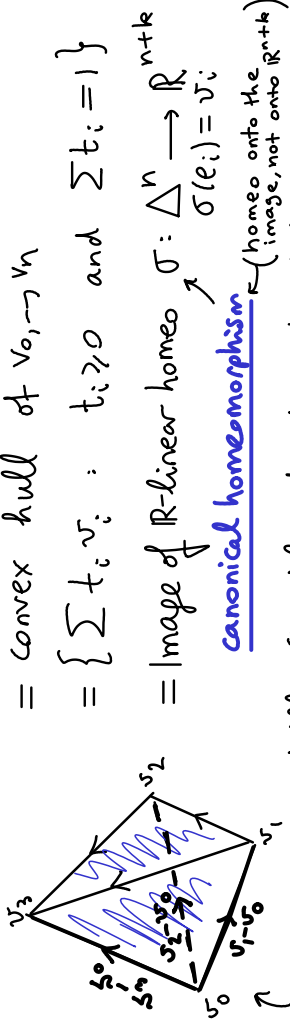
Example 0-dim faces are the vertices v_0, \dots, v_n

facets = $(n-1)$ -dimensional faces

= $[v_0, \dots, \hat{v}_i, \dots, v_n]$ where we omit v_i

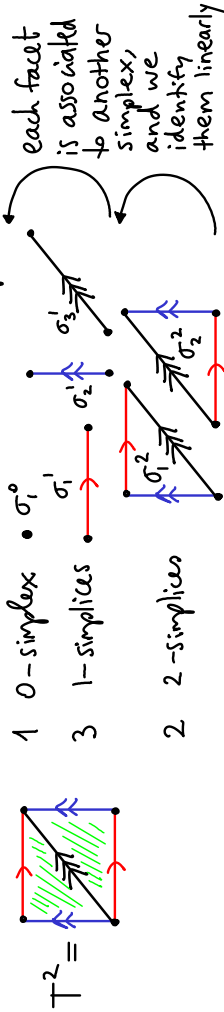
= $\{ \sum t_i v_i \in [v_0, \dots, v_n] : t_i = 0 \}$

= Image $\sigma|_{\Delta_i^{n-1}}: \Delta_i^{n-1} \rightarrow \mathbb{R}^{n+k}$
 $\{ t \in \Delta^n : t_i = 0 \}$



(Solid prism: includes inside)

Example Can build a torus out of simplices:



$T^2 = \text{quotient space } \bigsqcup \sigma_i^n / \text{canonical homeos associated to the facets}$
 for example identify facet σ_1^2 with σ_2^2 via linear homeo (orientation-preserving)

- indexing set I_n , for each $n \in \mathbb{N}$
- choice of n -simplex σ_α^n (not necessarily standard) for each $\alpha \in I_n$
- gluing data: for each $\alpha \in I_n$, $0 \leq i \leq n$, associate some $(\beta, i) \in I_{n-1}$
- consistency condition (see later)

The Δ -complex is the quotient space

$$X = \bigsqcup_{\substack{\alpha \in I_n \\ n \in \mathbb{N}}} \sigma_\alpha^n / \begin{matrix} i\text{-th facet of } \sigma_\alpha^n \text{ is identified with } \sigma_{\beta(\alpha, i)}^{n-1} \\ \text{via the order-preserving canonical linear homeo} \end{matrix}$$

(quotient topology: $U \subseteq X$ is open $\Leftrightarrow U$ intersects σ_α^n in an open set, $\forall \alpha, n$)

A Δ -complex structure on a top. space Y is a homeo from a Δ -cx $X \cong Y$.

Explicit description of the facet identification

$$\left\{ \sum s_j v_j \right\} = [w_0, \dots, w_{n-1}] \longrightarrow [\sigma_0, \dots, \sigma_n] = \left\{ \sum t_j v_j \right\} \cup \left\{ s_0 v_0 + \dots + s_{i-1} v_{i-1} + s_i v_{i+1} + \dots + s_n v_n \right\}$$

$$\sigma_{\beta(\alpha, i)}^{n-1} \xrightarrow{\sigma_\alpha^n} \sigma_\alpha^n \xrightarrow{\sigma_\alpha^n} \Delta_i^{n-1} = [\sigma_0, \dots, \hat{\sigma}_i, \dots, \sigma_n]$$

$$\Delta^{n-1} \longrightarrow \Delta_i^{n-1} \subseteq \Delta^n$$

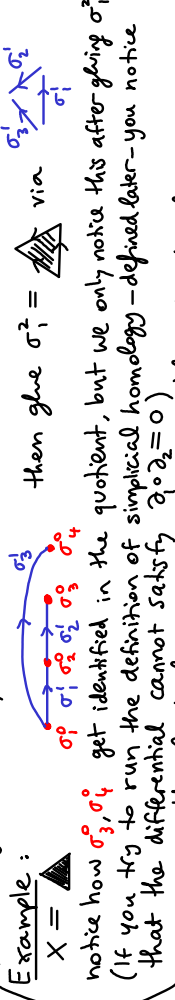
$$(\sigma_0, \dots, \sigma_{n-1}) \mapsto (\sigma_0, \dots, \hat{\sigma}_i, \dots, \sigma_{n-1})$$

Non-example

This decomposition for T^2 is not a Δ -complex.

Consistency condition

We want to additionally ensure that each point of X lies in the interior of exactly one σ_α^n , because we want to avoid unexpected identifications.



Equivalently: the facet gluing maps are compatible under double restriction: $\forall i, j$
 $[\sigma_0, \dots, v_n] \xrightarrow{\text{facet}} [\sigma_0, \dots, \hat{v}_i, \dots, v_n] \xrightarrow{\text{identity}} [\omega_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_{n-1}] \xrightarrow{\text{identity}} [\sigma_0, \dots, \hat{v}_j, \dots, v_n]$
 $[\sigma_0, \dots, \hat{v}_j, \dots, v_n] \xrightarrow{\text{identity}} [\sigma_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_{n-1}] \xrightarrow{\text{identity}} [\sigma_0, \dots, \hat{v}_i, \dots, v_n]$

this ensures that $[\sigma_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_n]$ is identified with the same $[\sigma_0, \dots, \hat{v}_j, \dots, \hat{v}_i, \dots, v_n]$ whether we first restrict to $t_i = 0$ (omit v_i) or first restrict to $t_j = 0$ (omit v_j).

Another equivalent condition: can define the k -th skeleton of Δ -cx X ,

$X^k = \text{quotient space you get by gluing all simplices of dimensions } \leq k$. Consistency is the condition that the boundary of each σ_α^n should map continuously into X^{n-1} (in the above Example consider the vertex $\Delta = \partial \sigma^2$) (more precisely, the "topological realization" of a simple complex)

Rmk (see part A) A simplicial complex is a Δ -complex in which

each d -dim face is uniquely determined by d distinct vertices.

A homeo from such a complex to X is a triangulation of X .

Non-example both 2-simplices have vertices v, v, v

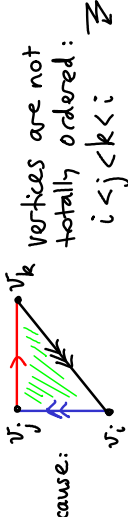


Harder exercise (Non-examinable) If you apply barycentric subdivision twice to a Δ -cx then you get a simplicial complex. (Hint: After the first subdivision, any given simplex has distinct vertices)

Simplicial chain complex

Def For a Δ -complex X , let $X_n = \text{set of } n\text{-simplices of } X$

$$C_n(X) = \text{free abelian group generated by the set } X_n = \left\{ \sum_{\alpha \in I_n} c_\alpha \cdot \sigma_\alpha^n : c_\alpha \in \mathbb{Z} \text{ and only finitely many } c_\alpha \neq 0 \right\}$$



vertices are not totally ordered: $i < j < k < i$

differential: $\partial_n \sigma_\alpha = \sum (-1)^i \cdot \sigma_{\beta(\alpha, i)}$

so: $\partial_n [v_0, \dots, v_n] = \sum (-1)^i \cdot [v_0, \dots, \hat{v}_i, \dots, v_n]$

and extend linearly

will show $\partial \circ \partial = 0$, so get simplicial homology: $H_*^\Delta(X) = H_*(C_*^\Delta, \partial_*)$

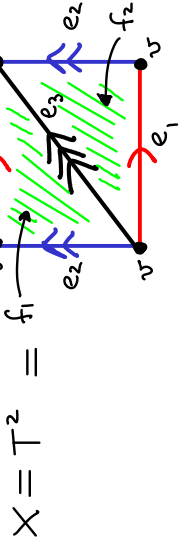
Examples
 $\partial_1 \begin{pmatrix} v_0 \\ \rightarrow \\ v_1 \end{pmatrix} = \begin{pmatrix} \bullet \\ -v_0 \\ +v_1 \end{pmatrix}$
 $\partial_2 \begin{pmatrix} v_0 \\ \nearrow \\ v_1 \\ \searrow \\ v_2 \end{pmatrix} = \begin{pmatrix} \bullet \\ +v_1 \\ -v_2 \end{pmatrix} + \begin{pmatrix} \bullet \\ -v_0 \\ +v_1 \end{pmatrix}$
 $\partial_1 \circ \partial_2$ (this) $= + (v_2 - v_1) - (v_2 - v_0) + (v_1 - v_0) = 0$
 $\partial \circ \partial = 0$ fails for Δ (not Δ -complex), try!

Lemma $\partial \circ \partial = 0$
 Pf $\partial_{n-1} (\partial_n [v_0, \dots, v_n]) = \sum (-1)^i \partial_{n-1} [v_0, \dots, \hat{v}_i, \dots, v_n]$
 $= \sum (-1)^i (-1)^{j-1} [v_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_n]$
 $+ \sum (-1)^i (-1)^{j-1} [v_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_n]$
 $= 0 \square$
 antisymmetric if swap v_i, v_j

Example $S^1 = \text{circle}$
 Δ -cx: $X_0 = 1$ 0-simplex \bullet $e_i^0 = e_{\beta(i,0)} = e_{\beta(i,1)}$
 $X_1 = 1$ 1-simplex $\rightarrow e_i^1$
 $0 \rightarrow C_1^\Delta \rightarrow C_0^\Delta \rightarrow 0$
 $\parallel \quad \parallel$
 $\mathbb{Z}e \quad \mathbb{Z}v$
 $e \mapsto v - v = 0$
 $\Rightarrow H_*^\Delta(S^1) = \begin{cases} \mathbb{Z} & * = 0, 1 \\ 0 & \text{else} \end{cases}$

Example Δ -cx structure on S^n :
 $S^n = \Delta^n \cup \Delta^n$ / glue along $\partial \Delta^n$
 call this Δ_1 this Δ_0
 One can deduce: $H_* S^n = \begin{cases} \mathbb{Z} \langle \text{point} \rangle \cong \mathbb{Z} & * = 0 \\ \mathbb{Z} \langle \Delta_1 - \Delta_0 \rangle \cong \mathbb{Z} & * = n \\ 0 & \text{else} \end{cases}$
 but messy! pick any vertex

Example



$$0 \rightarrow C_2^\Delta \xrightarrow{\parallel} C_1^\Delta \xrightarrow{\parallel} C_0^\Delta \rightarrow 0$$

$$\mathbb{Z}f_1 + \mathbb{Z}f_2 \xrightarrow{\parallel} \mathbb{Z}e_1 + \mathbb{Z}e_2 + \mathbb{Z}e_3 \xrightarrow{\parallel} \mathbb{Z}v$$

$$f_1 \mapsto e_1 - e_3 + e_2 \quad e_1, e_2, e_3 \mapsto v - v = 0$$

$$f_2 \mapsto e_2 - e_3 + e_1$$

$$H_*^\Delta(T^2) = \begin{cases} \mathbb{Z}v & * = 0 \\ (\mathbb{Z}e_1 + \mathbb{Z}e_2 + \mathbb{Z}e_3) / \mathbb{Z}(e_1 - e_3 + e_2) & * = 1 \\ \mathbb{Z}(f_1, -f_2) & * = 2 \\ 0 & \text{else} \end{cases}$$

freely generated by e_1, e_2

Alternative useful method using a trick from algebra:
 Smith normal form of ∂_2 :
 $\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \xrightarrow{\text{row}} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \xrightarrow{\text{col}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
 hence: $H_2 = \ker \partial_2 \cong \mathbb{Z}$
 $H_1 = \text{coker } \partial_2 \cong \mathbb{Z}^2$
 so after \mathbb{Z} -isos of C_2, C_1 , we get $\mathbb{Z}^2 \rightarrow \mathbb{Z}^3 \xrightarrow{(a, b)} (a, 0, 0)$

Remark about orientations (see also my B3.2 Geometry of Surfaces notes)
 For a vector space an orientation is a choice of basis modulo linear endomorphisms of $\det > 0$

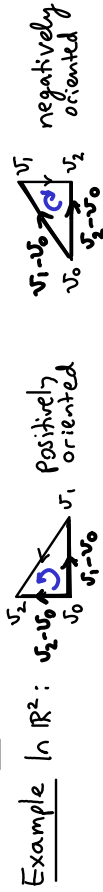
Example \mathbb{R}^2

$$\begin{matrix} e_2 & \uparrow & \text{right-hand} & \text{orientation} & \text{left-hand} & \text{orientation} \\ \uparrow & \text{positive} & \text{orientation} & \text{(negative)} & \text{orientation} & \text{(negative)} \\ \mathbb{R}^2 & \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} & \mathbb{R}^2 & \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} & \mathbb{R}^2 & \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} \end{matrix}$$

Fact $GL(n, \mathbb{R})$ has 2 path-components $\langle A : \det A > 0 \rangle$ so can always continuously deform a basis to another within same orientation
Canonical orientation on \mathbb{R}^n : e_1, \dots, e_n standard basis \leftarrow "positive orientation"

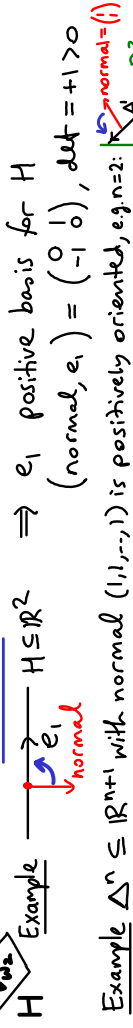
Example $[v_0, \dots, v_n]$ simplex $\Rightarrow v_1 - v_0, \dots, v_n - v_0$ is a basis of vector subspace $V = \{ \sum a_i v_i : \sum a_i = 0 \} \subseteq \mathbb{R}^{n+1}$
 hence a choice of orientation of V , and each transposition of vertices v_0, \dots, v_n switches the orientation class.
 (if swap v_i, v_j consider the reflection in the hyperplane $(e_i = e_j)$ in V)

If $v_0, v_1, v_2 \in \mathbb{R}^n$ then $V = \mathbb{R}^n$ so simplex's orientation can be compared with \mathbb{R}^n -orient.



No canonical choice of orientation for an abstract vector space. Need choose basis v_1, \dots, v_n then declare another basis positively oriented if the change of basis matrix has $\det > 0$.

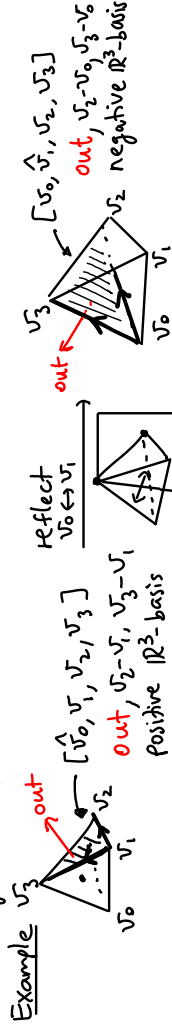
For hyperplane $H \subseteq \mathbb{R}^n$ with choice of normal can declare orientation of basis w_1, \dots, w_{n-1} of H positive if $\langle w_1, \dots, w_{n-1}, n \rangle$ is positive \mathbb{R}^n -basis convention "outward normal first"



UPSHOT For an n -simplex $[v_0, \dots, v_n]$ in \mathbb{R}^n , each facet lies in a hyperplane and have canonical choice of normal: outward normal. Hence facets are canonically oriented.



Any reflection of \mathbb{R}^n will swap orientation: after $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ get clockwise



UPSHOT $(-1)^i$ in $(-1)^i [v_0, \dots, \hat{v}_i, \dots, v_k]$ in definition of simplicial ∂ secretly keeps track of whether the orientation of the simplex agrees or not with the orientation induced geometrically by the above conventions. From this point of view, the equation $\partial \circ \partial = 0$ holds because a codimension 2 face γ of a simplex σ arises as the facet of exactly two facets f_1, f_2 of σ , and the geometric orientations of f_1, f_2 induce opposite geometric orientations on γ (therefore if we keep track of orientation signs we count $+\gamma - \gamma = 0$). (checking that they are opposite requires some thought, one approach is to say that we can deform f_1, f_2 until they make a flat angle, and) Picture: f_1 get \uparrow from ∂f_1 , then their outward normals will be opposite.

Lemma $H_*^\Delta(X) \cong \bigoplus H_*^\Delta(X_i)$ where X_i are the path-components of X . Pf $\bigoplus C_*^\Delta(X_i) \rightarrow C_*^\Delta(X), \bigoplus c_i \mapsto \sum c_i$ since Δ^k path-com. is chain isomorphism since any simplex $\sigma: \Delta^k \rightarrow X$ has path-connected image, so $\subseteq X_i$ some i . \square

Rmk X top space \Rightarrow path com. component \subseteq connected component since path-com. \Rightarrow connected. For Δ -cx, these are same (since connected + locally path-com. \Rightarrow path-connected).

However for top space they need not be the same, e.g. Topologist's sine curve $\{(x, \sin \frac{1}{x}) : x \in (0, 1]\} \cup \{0\} \times [0, 1] \subseteq \mathbb{R}^2$

Lemma $X \Delta$ -cx, X connected $\Rightarrow \forall$ vertices v_1, v_2, \exists path consisting of edges from v_1 to v_2 Proof Pick a vertex v_0 . Let $A = \{\text{simplices } \sigma \text{ in } X \text{ such that } v_0 \in \sigma\}$ vertices of σ can be connected to v_0 by a path in X consisting of edges. Let $B = \{\text{simplices } \sigma \text{ of } X \text{ not in } A\}$. Observe that if $\sigma \in A$ then the facets of σ are in A , similarly for B . After gluing simplices as prescribed in X , the set A and B define subspaces of X (using the quotient topology). Exercise: A, B are closed subspaces Note $A \cap B = \emptyset$ (otherwise they have a vertex in common and we contradict the definition of A) $\Rightarrow X = A \sqcup B$ is a disjoint union of closed sets, so X connected forces $B = \emptyset$. \square


Theorem X has Δ -cx structure $\Rightarrow H_0(X) \cong \bigoplus_{\text{path-com. components}} \mathbb{Z}$

Pf By lemma, $\text{wlog } X$ path-connected • vertex $v \Rightarrow v \in C_0(X) \xrightarrow{\partial} C_{-1}(X) = 0 \Rightarrow [v] \in H_0(X)$ • vertices $v_0, v_1 \in X \xrightarrow{\text{lemma}} \exists$ path γ from v_0 to v_1 , consisting of edges $\Rightarrow \gamma$ is sum of 1-chains s.t. $\partial \gamma = v_1 - v_0$, so $[v_0] = [v_1] \in H_0(X)$. • $H_0(X) = \langle [v] \rangle \subseteq \mathbb{Z}$ is injective? $n \nu \leftarrow n$ suppose $n \nu = \partial c$ some $c \in C_1(X)$

consider augmentation hom $\epsilon: C_0(X) \rightarrow \mathbb{Z}, \sum n_i \sigma_i \mapsto \sum n_i$ \Rightarrow composition $C_1(X) \xrightarrow{\partial} C_0(X) \xrightarrow{\epsilon} \mathbb{Z}$ is 0 as $\partial(\text{1-simplex}) = \sigma_1 - \sigma_0 \mapsto 1 - 1 = 0$. \square $\Rightarrow n = \epsilon(n \nu) = \epsilon \partial c = 0$. \square

Harder exercise (non-examinable) If $\delta: [0, 1] \rightarrow X$ path from vertex v_0 to v_1 , then we can homotope (=continuously deform) it into a path consisting of edges. (Hints: first show that a Δ -cx is locally path-connected. Then use the fact that the domain $[0, 1]$ of δ is compact in order to approximate δ arbitrarily well by a piecewise linear path. Finally prove the result for piecewise linear paths.)

3. SINGULAR HOMOLOGY

Motivation Not obvious that H_*^Δ is functorial: $\Delta^k \xrightarrow{\sigma} X \xrightarrow{f} Y$
 then $f \circ \sigma$ typically not a simplex: $\Delta \xrightarrow{\sigma} \Delta \xrightarrow{f} X$  $\xrightarrow{\text{continuous map}}$ Y

Solution 1: only allow simplifical maps $f: X \rightarrow Y$ (so $f \circ \sigma$ simplex $\forall \sigma$)

Solution 2: show that any cts map $f: X \rightarrow Y$ can be approximated arbitrarily well by a simplicial map after having performed barycentric subdivision on X, Y enough times. Also any two such approximations induce the same map $H_*^\Delta(X) \rightarrow H_*^\Delta(Y)$

Solution 3: develop new homology $H_*(X)$ which allows any cts map $\Delta^n \rightarrow X$ will do THIS. and prove $H_*^\Delta(X) \cong H_*(X)$ for Δ -complexes X .

Def Singular n-simplex is any continuous map $\sigma: \Delta^n \rightarrow X$
 X is any top. space

Singular n-chains $C_n(X) =$ free abelian group generated by $\left\{ \sum_{\text{singular } n\text{-simplices } \sigma} c_\sigma \cdot \sigma : c_\sigma \in \mathbb{Z} \right\}$
 only finitely many $c_\sigma \neq 0$

Lemma Here $\Delta_i^{n-1} = [e_0, \dots, \hat{e}_i, \dots, e_n]$ is identified canonically with Δ^{n-1} (send $e_k \rightarrow e_{k-1}$ for $k > i$)
 (and extend linearly)

Proof $\partial_{n+1}(\partial_n \sigma) = \partial_{n+1} \left(\sum_{j < i} (-1)^j \sigma|_{\Delta_j^{n-1}} \right) = \sum_{j < i} (-1)^j \sigma|_{[e_0, \dots, \hat{e}_j, \dots, \hat{e}_i, \dots, e_n]} + \sum_{j > i} (-1)^j \sigma|_{[e_0, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_n]}$
 antisymmetric if swap i, j
 $= 0$

Singular homology: $H_*(X) = H_*(C_*, \partial_*)$

For Δ -complex X have inclusion of subcomplex $C_*^\Delta \hookrightarrow C_*$
 \Rightarrow induces $H_*^\Delta(X) \rightarrow H_*(X)$ Fact: isomorphism (proof later, see cellular $H_*^{CW} \cong H_*$)

Corollary $H_*^\Delta(X)$ is independent of choice of Δ -cx structure on X

Example $X = \text{point} \Rightarrow C_n(X) = \mathbb{Z} \cdot (\sigma_n: \Delta^n \rightarrow X \text{ the constant map})$
 $\partial \sigma_n = \sum (-1)^i \sigma_n|_{\Delta_i^{n-1}} = \sum (-1)^i \sigma_{n-1} \Rightarrow \dots \Rightarrow \mathbb{Z} \xrightarrow{\partial} \mathbb{Z} \xrightarrow{\partial} \mathbb{Z} \xrightarrow{\partial} \dots \xrightarrow{\partial} \mathbb{Z} \xrightarrow{\partial} \mathbb{Z} \xrightarrow{\partial} \mathbb{Z} \xrightarrow{\partial} \dots$
 $\begin{cases} 1 & \text{if } n \text{ even} \\ 0 & \text{if } n \text{ odd} \end{cases} \Rightarrow H_*(\text{pt}) \cong \begin{cases} \mathbb{Z} & * = 0 \\ 0 & \text{else} \end{cases}$

Lemma $H_*(X) \cong \bigoplus H_*(X_i)$ where X_i are path-components of X
Pf Image of cts map $\Delta^n \rightarrow X$ is path conn. so lies in some X_i . \square

Cor $H_0(X) = \bigoplus \mathbb{Z}_{X_i}$ \leftarrow generators of $C_0(X)$

Pf By Lemma, wlog X path-connected. $\Delta^0 = \text{pt} \rightarrow X$ is cycle since $C_{-1}(X) = 0$
 Given 2 points $x, y \in X$, a path $\Delta^1 = [0, 1] \xrightarrow{\gamma} X, \gamma(0) = x, \gamma(1) = y$ is also a 1-chain!
 So $y - x = \partial \gamma$, so x, y are homologous. Finally if $n \cdot [x] = 0 \in H_0(X)$ then $n x = \partial c$ some $c \in C_1(X)$ generated by paths. Now run the augmentation hom-trick like we did for H_0^{Δ} : $n = \varepsilon(n x) = \varepsilon \partial c = 0$ as $\varepsilon \partial = 0$. \square

Naturality (i.e. functoriality)

Lemma $f: X \rightarrow Y$ continuous

$\Rightarrow H_*(f) = f_*: H_*(X) \rightarrow H_*(Y)$ induced by chain map
 $f_*: C_*(X) \rightarrow C_*(Y)$ and extend linearly
 $f_*(\sigma) = f \circ \sigma$

Pf $\partial_n(f_* \sigma) = \sum (-1)^i f \circ \sigma|_{\Delta_i^{n-1}} = f_* \left(\sum (-1)^i \sigma|_{\Delta_i^{n-1}} \right) = f_* (\partial_n \sigma) \square$
Properties 1) $X \xrightarrow{f} Y \xrightarrow{g} Z \Rightarrow (g \circ f)_* = g_* \circ f_*$
 2) $\text{id}_X = \text{id} \Rightarrow \text{id}_*(\sigma) = \sigma$

Pf 1) $(g \circ f)_* \sigma = g_* \circ f_* \sigma = g_* (f_* \sigma) = g_* (\sigma) = \sigma$ \checkmark
 2) $\text{id}_*(\sigma) = \text{id} \circ \sigma = \sigma$ \checkmark \square

Cor $H_*: \left\{ \begin{array}{l} \text{topological spaces} \\ \& \text{cts maps} \end{array} \right\} \rightarrow \left\{ \begin{array}{l} \text{graded abelian groups} \\ \& \text{graded homs} \end{array} \right\}$ is a functor

Cor $X \cong Y \Rightarrow H_*(X) \cong H_*(Y)$

4. CHAIN HOMOPIES AND HOMOPIY INVARIANCE

Algebra: chain homotopies

$$f_*, g_* : (C_*, \partial_*) \rightarrow (C_*, \tilde{\partial}_*) \text{ chain maps}$$

Def f_*, g_* are chain homotopic if \exists (degree +1)

$$\text{hom } h : C_* \rightarrow \tilde{C}_*[1] \text{ s.t.}$$

$$\tilde{\partial} \circ h + h \circ \partial = f_* - g_*$$

h is called a chain homotopy

$$\text{Consequence } f_* = g_* : H_*(C_*, \partial_*) \rightarrow H_*(\tilde{C}_*, \tilde{\partial}_*) \text{ on homology}$$

Pf $C_{n+1} \rightarrow C_n \xrightarrow{\partial_n} C_{n-1}$ For c cycle $\in C_n$:

$$\begin{aligned} \Rightarrow f_n(c) - g_n(c) &= \tilde{\partial}_{n+1} \circ h_n(c) + h_{n-1} \circ \partial_n(c) \\ &= \underbrace{\tilde{\partial}_{n+1} \circ h_n(c)}_{\text{boundary}} + \underbrace{h_{n-1} \circ \partial_n(c)}_{=0} \end{aligned}$$

so $0 \in H_n(C_*) \cdot \square$

Harder exercise (Non-examinable)

If chain maps $f_*, g_* : C_* \rightarrow \tilde{C}_*$ induce the same map on homology and C_n, \tilde{C}_n are free abelian groups $\forall n$, then f_*, g_* are chain homotopic.

Hints Let $B_{n-1} = \text{Im } \partial_n, K_n = \text{Ker } \partial_n$. Use fact that subgroups of free groups are free to show that B_n, K_n are free. Deduce that $C_n \cong B_{n-1} \oplus K_n$ (show $0 \rightarrow K_n \xrightarrow{\partial_n} B_{n-1} \rightarrow 0$ splits) More precisely deduce: the chain complex (C_*, ∂_*) is isomorphic to $D_* = \bigoplus_{n \geq 0} (D_n \rightarrow D_{n-1} \rightarrow \dots)$ via that iso, $f_* - g_*$ determines a chain map $\alpha : D_* \rightarrow \tilde{D}_*$ such that on homology $\alpha = 0 : H_n(D) \rightarrow H_n(\tilde{D})$. Consider $\beta : D_n \rightarrow D_{n+1}, \beta : K_n \rightarrow B_n$ is α . (hence $\alpha(K_n) \subseteq B_n \subseteq K_n$)

Theorem

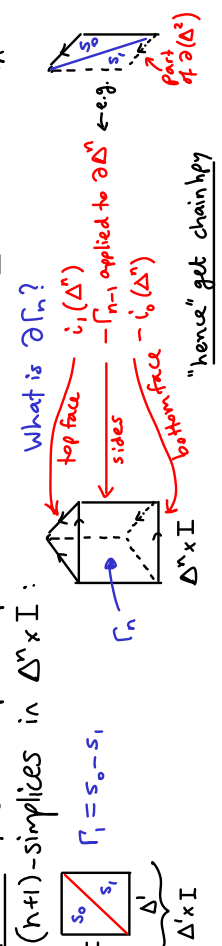
$$\text{Let } i_0 : X \rightarrow X \times I, i_1(x) = (x, 0) \text{ where } I = [0, 1]$$

$$i_1 : X \rightarrow X \times I, i_1(x) = (x, 1)$$

$$\Rightarrow i_0^*, i_1^* : C_*(X) \rightarrow C_*(X \times I) \text{ are chain hpic.}$$

Key idea Need the "prism operator" which cuts $\Delta^n \times I$ into a sum Γ_n of

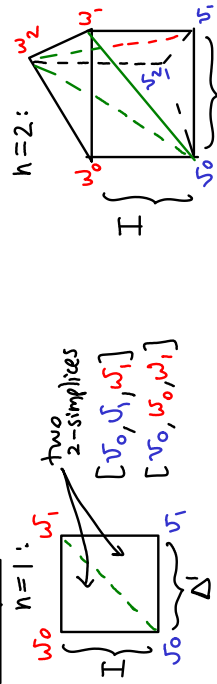
$(n+1)$ -simplices in $\Delta^n \times I$:



Non-examinable

$$\text{Pf } \left. \begin{aligned} \text{bottom facet } \Delta^n \times 0 &= [v_0, \dots, v_n] \\ \text{top facet } \Delta^n \times 1 &= [w_0, \dots, w_n] \end{aligned} \right\} \leftarrow v_i = e_i \times 0 \subseteq \Delta^n \times [0, 1] \subseteq \mathbb{R}^{n+1}$$

Examples $n=1$: $w_0 \leftarrow v_0, w_1 \leftarrow v_1$
 $n=2$: $w_0 \leftarrow v_0, w_1 \leftarrow v_1, w_2 \leftarrow v_2$



three 3-simplices:

- $[v_0, v_1, v_2, w_2]$
- $[v_0, v_1, w_1, w_2]$
- $[v_0, w_0, w_1, w_2]$

Let $s_i = [v_0, \dots, v_i, w_i, \dots, w_n]$

Claim The s_i cover $\Delta^n \times [0, 1]$ and give Δ -cx structure on $\Delta^n \times I$

$$\text{Pf } \sum_{k \leq i} t_k v_k + \sum_{k > i} t_k w_k = (t_0, \dots, t_{i-1}, t_i + r_i, r_{i+1}, \dots, r_n) \underbrace{(r_i, \dots, r_n)}_{\in [0, 1]^n} = I$$

So given $(x_0, \dots, x_n) \in \Delta^n \times I$, equate and solve:

$$t_0 = x_0, \dots, t_{i-1} = x_{i-1}, r_{i+1} = x_{i+1}, \dots, r_n = x_n, \text{ and } \begin{cases} r_i = a - x_{i+1} - \dots - x_n \\ t_i = x_i - r_i \end{cases}$$

Note $x_k \geq 0, \sum x_k = 1, a \in [0, 1]$ hence $\sum t_k + \sum r_k = 1 \checkmark$
 $\begin{cases} t_k \geq 0 \text{ for } k < i \\ r_k \geq 0 \text{ for } k > i \end{cases}$

but $r_i \geq 0 \Leftrightarrow \begin{cases} a \geq x_{i+1} + \dots + x_n \\ x_i + x_{i+1} + \dots + x_n \geq a \end{cases}$. Thus a solution exists if we pick $i = \min \{k : a \geq x_{k+1} + \dots + x_n\}$

(There are multiple solutions if $a = x_{i+1} + \dots + x_n$ since then also "i+1" works) since $x_{i+1} + x_{i+2} + \dots + x_n \geq a \geq x_{i+2} + \dots + x_n$. This is expected since s_i, s_{i+1} meet along a facet. Example: $n=1$ picture above: s_0, s_1 meet along facet $[v_0, w_1]$ which arises for $i=0, r_0=0$ but also $i=1, t_1=0$.

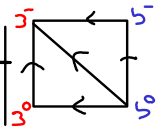
Def

$$\Gamma_n = \sum_i (-1)^i s_i \in C_{n+1}(\Delta^n \times [0, 1]) \leftarrow \text{geometrically this "represents" } \Delta^n \times I \text{ as a simplicial chain}$$

$$\begin{aligned} \Rightarrow \partial \Gamma_n &= \sum_{j \leq i} (-1)^j (-1)^i [v_0, \dots, v_j, \dots, v_i, w_i, \dots, w_n] \\ &\quad + \sum_{j > i} (-1)^j (-1)^i [v_0, \dots, v_i, w_i, \dots, w_j, \dots, w_n] \end{aligned}$$

geometrically, this "represents" $\partial(\Delta^n \times I) = (\partial \Delta^n \times I) \cup (\Delta^n \times \partial I)$

Example



$$\Gamma_1 = [v_0, w_0, w_1] - [v_0, v_1, w_1] \text{ "is the square"}$$

$$\partial \Gamma_1 = [w_0, w_1] - [v_0, w_1] + [v_0, w_0] - [v_1, w_1] + [v_0, v_1] - [v_0, v_1]$$

"inside facets" cancel

Prism operator $P : C_n(X) \rightarrow C_{n+1}(X \times [0,1])$

$$P(\sigma) = (\sigma \times \text{id})_* (\Gamma_n)$$

$\sigma : \Delta^n \rightarrow X$
 $\sigma \times \text{id} : \Delta^n \times [0,1] \rightarrow X \times [0,1]$
 $(\sigma \times \text{id})(x,t) = (\sigma(x), t)$

this abbreviated notation means the map
 $(t_0, \dots, t_n) \mapsto (\sigma(t_0 e_0 + \dots + t_j e_j + \dots + t_n e_n), t_0 + \dots + t_n)$
 $\in X \times I$

$$\begin{aligned} \partial P(\sigma) &= \partial(\sigma \times \text{id})_* (\Gamma_n) \\ &= (\sigma \times \text{id})_* (\partial \Gamma_n) \\ &= \sum_{i \leq n} (-1)^i (\sigma \times \text{id})_* [i_0 \sigma e_0, \dots, i_0 \sigma e_n] \\ &\quad + \sum_{j \leq n} (-1)^{j+1} [i_0 \sigma e_0, \dots, i_0 \sigma e_j, \dots, i_0 \sigma e_n] \end{aligned}$$

Example to clarify $n=2, k=0, i=1$
 $\Gamma = [v_0, v_1, v_2] = [v_0, v_1, w_1]$
 $(\sigma \times \text{id})_* \Gamma = (\sigma \times \text{id})_* [v_0, v_1, w_1]$
maps $t_0 v_0 + t_1 v_1 + t_2 w_1 = (t_0, t_1, t_2)$
to $(\sigma(t_0, t_1, t_2), t_2) = (\sigma(t_0 e_0 + t_1 e_1 + t_2 e_2), t_2)$

Homotopy invariance

Def $f_0, f_1 : X \rightarrow Y, f_0 \simeq f_1$ homotopic if \exists continuous map

$$F : X \times [0,1] \rightarrow Y \text{ called homotopy s.t. } \begin{cases} f_0 = F \circ i_0 \\ f_1 = F \circ i_1 \end{cases}$$

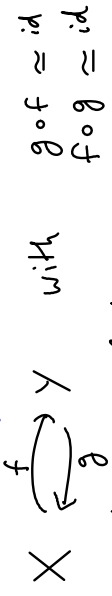
Idea Think of this as a continuous family of maps

$$f_t = F(\cdot, t) : X \rightarrow Y \text{ from } f_0 \text{ to } f_1.$$

Exercise \simeq is an equivalence relation.

Homotopic relative to $A \subseteq X$ if $F(a,t) = f_0(a) = f_1(a)$ all $a \in A$ all t .
 write "f $\simeq g$ rel A"

Def $X \simeq Y$ homotopy equivalent spaces if \exists maps



Rmk homeo \Rightarrow hpy equivalent

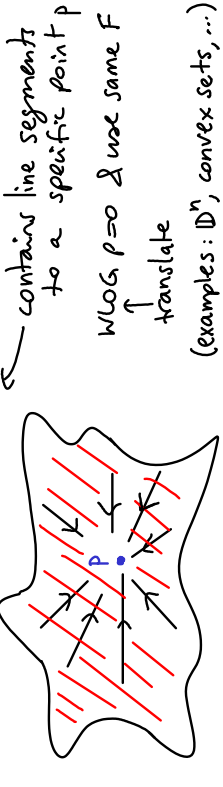
Def X contractible if $X \simeq \text{pt}$

equivalently $(X \xrightarrow{\text{id}} X) \simeq (X \xrightarrow{\text{const}} \text{point} \in X)$

Example. $\mathbb{R}^n \simeq \text{pt}$

$F(x,t) = tx$ then $f_0 \equiv 0, f_1 = \text{id}$.

(star-shaped subsets of $\mathbb{R}^n \simeq \text{pt}$)



Theorem $f_0 \simeq f_1 \Rightarrow f_{0*} = f_{1*}$ (examples: \mathbb{D}^n , convex sets, ...)

Pf $f_{1*} - f_{0*} = F_* i_{1*} - F_* i_{0*} = F_* (i_{1*} - i_{0*})$

previous thm $= F_* (\partial P + P \partial)$

F_* chain map $\Rightarrow \partial \circ (F_* P) + (F_* P) \circ \partial = 0$

$F_* P$ is chain hpy from f_{0*} to f_{1*} \square

Cor $X \simeq Y \Rightarrow H_* X \cong H_* Y$

Pf $f_* g_* = \text{id}_*$, $g_* f_* = \text{id}_*$ \square

Example X contractible $\Rightarrow H_* X \cong H_*(\text{pt}) \cong \begin{cases} \mathbb{Z} & * = 0 \\ 0 & \text{else} \end{cases}$

Cultural Rmk (Whitehead's theorem) For nice topological spaces \leftarrow (CW complexes - see later in) if X, Y are simply connected and $\exists f : X \rightarrow Y$ inducing isomorphisms on H_* then $X \simeq Y$ are homotopy equivalent.

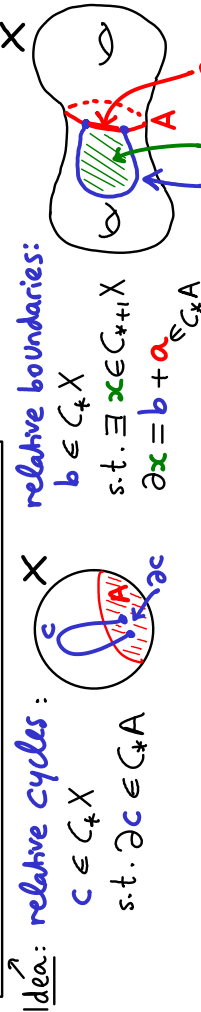
Relative homology

Def (X, A) pair of spaces if $A \subseteq X$ topological subspace

$\Rightarrow \hat{i} = \text{incl}: A \hookrightarrow X$ induces a subcx $\hat{i}_*: C_*A \rightarrow C_*X$

$\Rightarrow C_*X/C_*A$ quotient chain cx (recall $\partial[x] = [\partial x]$)

$$H_*(X, A) = H_*(C_*X/C_*A)$$



$$\Rightarrow 0 \rightarrow C_*A \rightarrow C_*X \rightarrow C_*X/C_*A \rightarrow 0 \text{ SES}$$

$$\text{Cor} \dots \rightarrow H_n(A) \xrightarrow{\hat{i}_*} H_n(X) \rightarrow H_n(X, A) \xrightarrow{\delta} H_{n-1}(A) \xrightarrow{\hat{i}_*} \dots$$

LES for the pair



Reduced homology

$$\tilde{H}_*X = \ker(H_*X \rightarrow H_*(pt))$$

\leftarrow induced by $X \rightarrow pt$

For $X \neq \emptyset$, $\tilde{H}_*X = H_*$ of augmented chain complex:

$$\dots \rightarrow C_n(X) \rightarrow \dots \rightarrow C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \rightarrow 0$$

$\epsilon(\sum \alpha_i \cdot p_i) = \sum \alpha_i$ points $\in X$

can view $C_{-1}(X) = \mathbb{Z}$ (map $\phi \rightarrow X$) where allow the empty simplex \emptyset

Example $\tilde{H}_*(\mathbb{R}^n) \cong \tilde{H}_*(\mathbb{D}^n) \cong \tilde{H}_*(pt) = 0$

Check: $H_*X = \tilde{H}_*X$ $*$ $\neq 0$, and $H_0X \cong \tilde{H}_0X \oplus \mathbb{Z}$ for $X \neq \emptyset$

\cdot $f: X \rightarrow Y \Rightarrow f_*: \tilde{H}_*X \rightarrow \tilde{H}_*Y$

Lemma (X, A) pair $\Rightarrow \exists$ LES

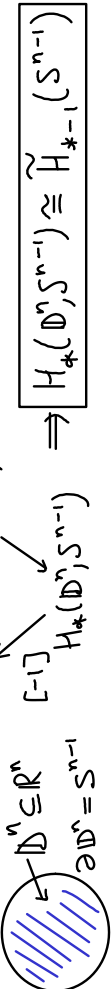
$$\dots \rightarrow \tilde{H}_n(A) \xrightarrow{\hat{i}_*} \tilde{H}_n(X) \rightarrow H_n(X, A) \rightarrow \tilde{H}_{n-1}(A) \xrightarrow{\hat{i}_*} \dots \rightarrow H_0(X, A) \rightarrow 0$$

Pf use augmented ch. cx. and $\mathbb{Z}/\mathbb{Z} = 0 \square$

Cor $H_*(X, pt) \cong \tilde{H}_*(X)$ for $X \neq \emptyset$

Pf $\tilde{H}_*(pt) = 0. \square$

Example LES: $\tilde{H}_*(S^{n-1}) \rightarrow \tilde{H}_*(\mathbb{D}^n) = 0$



Naturality of the LES for pairs

Def A map of pairs of spaces $(X, A) \xrightarrow{f} (Y, B)$

means $f: X \rightarrow Y$ and $f(A) \subseteq B$.

$$\text{Lemma} \dots \rightarrow H_*A \rightarrow H_*X \rightarrow H_*(X, A) \rightarrow H_{*-1}A \rightarrow \dots$$

$$\begin{matrix} f_* \downarrow & & f_* \downarrow & & f_* \downarrow \\ \dots \rightarrow & H_*B & \rightarrow & H_*Y & \rightarrow & H_*(Y, B) & \rightarrow & H_{*-1}B & \rightarrow & \dots \end{matrix}$$

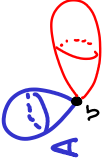
and similarly for \tilde{H}_* .

Pf $0 \rightarrow C_*A \rightarrow C_*X \rightarrow C_*X/C_*A \rightarrow 0 \Rightarrow$ claim follows by naturality of LES induced by SES of chain cxes. \square

5. EXCISION THEOREM AND QUOTIENTS

(X, A) pair

Def $r: X \rightarrow X$ retraction onto A if $\begin{cases} r(X) = A \\ r|_A = \text{id}_A \end{cases}$



Example $X = S^2 \vee S^2 =$ two spheres glued at one point v
 $r: X \rightarrow A$ map second sphere to v (wedge sum)

Example In Pf of Brouwer fixed pt thm we built a retraction r by contradiction

Cor r retraction $\Rightarrow r_*: H_*X \rightarrow H_*A$ surjective

$\text{incl}_*: H_*A \rightarrow H_*X$ injective

Pf $A \xrightarrow{\text{incl}} X \xrightarrow{r} A$ now use H_* functorial \square



Def $r: X \rightarrow X$ deformation retraction onto A if $\begin{cases} r \text{ retraction} \\ r = \text{id}_X \text{ rel } A \end{cases}$

Example $X = \text{Möbius strip}$, $A = \text{equator}$

Lemma $r \text{ def. retr.} \Rightarrow A \xrightarrow{\text{incl}} X$ is a homotopy equivalence.

incl_* and r_* are isos on H_* , so $H_*A \cong H_*X$.
 Pf $A \xrightarrow{\text{incl}} X \xrightarrow{r} A$ $\text{incl} \circ r = r \circ \text{id}_X$, $r \circ \text{incl} = r|_A = \text{id}_A \quad \square$

Example $S^n \setminus \text{pt}$ deforms to $\mathbb{D}^n \cong$ lower hemisphere.

 $S^n \setminus \{2 \text{ points}\} \cong \mathbb{D}^n \setminus \text{pt} \cong \mathbb{D}^n \setminus \{0\} \cong S^{n-1}$
 $S^n \setminus \{3 \text{ points}\} \cong \mathbb{D}^n \setminus \{0\} \cong S^{n-1}$
 (Warning: check these separately, not true that $X \approx Y$ implies $X \setminus \text{pt} \approx Y \setminus \text{pt}$. (e.g. consider $X = \mathbb{R}^2$, $Y = \mathbb{R}$, $X \setminus \text{pt} \cong S^1$, $Y \setminus \text{pt} \cong 2 \text{ points}$)

Excision theorem
 $E \subseteq A \subseteq X$ subspaces $\Rightarrow (X \setminus E, A \setminus E) \rightarrow (X, A)$ induces iso
 with $\bar{E} \subseteq A^o$
 Proof later.

Example $X = S^1 \vee S^1 = \infty \cong A = \bigcirc \cong E = \bigcirc \cong S^1$
 $\Rightarrow H_1(X, A) \cong H_1(\bigcirc, \bigcirc) \cong H_1(\mathbb{D}^1, \partial \mathbb{D}^1) \cong \tilde{H}_0(S^0) \cong \mathbb{Z}$ (2 points)
 (exc. thm.) \hookrightarrow hpy invce

Example Invariance of dimension from chapter 0 also holds if replace $\mathbb{R}^n, \mathbb{R}^m$ by non-empty open sets $U \subseteq \mathbb{R}^n, V \subseteq \mathbb{R}^m$ because for $p \in U$:
 $H_*(U, U-p) \cong H_*(\mathbb{R}^n, \mathbb{R}^n-p) \cong \tilde{H}_{*-1}(\mathbb{R}^n-p) \cong \tilde{H}_{*-1}(S^{n-1}) \cong \mathbb{Z}$ if $n = m$, 0 otherwise.

Rephrasing of Excision Thm
 $X = A^o \cup B^o \Rightarrow H_*(X, A) \cong H_*(B, A \cap B)$
 $(A, B \subseteq X \text{ subspaces})$
 induced by inclusion $(X, A) \hookrightarrow (B, A \cap B)$
 Check $\bar{E} \subseteq A^o$:
 $E = X \setminus B \subseteq X \setminus A^o \subseteq X \setminus B^o \subseteq A^o$ (closed in X)

Pf Take $E = X \setminus B$ so $X \setminus E = B$ and $A \cap B = A \setminus E$. \square $\Rightarrow \bar{E} \subseteq X \setminus B^o \subseteq A^o$
 Idea why excision holds: $C_*(A) + C_*(B) \rightarrow C_*(X)$ is a homotopy equivalence and $C_*(A) \cap C_*(B) = C_*(A \cap B)$. Idea: can subdivide chains in X many times, and small enough chains lie either in A or in B (or in both).

Good pairs and quotients For (X, A) pair:

- Quotient $X/A = X/\sim$ \leftarrow equivalence relation $x \sim y \Leftrightarrow \begin{cases} x, y \in A \\ \text{or} \\ x=y \end{cases}$
- (X, A) good pair if $\begin{cases} A \neq \emptyset \\ A \text{ closed} \end{cases}$
 A deformation retract of nbhd V of A

Example $X = S^1 \vee S^1 = \infty \supseteq V = \bigcirc \supseteq A = \bigcirc \cong S^1$
 $X/A \cong \bigcirc$ (all points of A are identified with the node)

Non-example Topologist's sine curve
 $\{(x, \sin \frac{1}{x}) : x \in (0, 1]\} \cup \{0\} \times [0, 1] \subseteq \mathbb{R}^2$
 Cultural Rmk Smooth submanifold \subseteq Smooth manifold is a good pair. (by tubular neighbourhood theorem)

Cor (X, A) good $\Rightarrow (X, A) \rightarrow (X/A, \text{pt})$ induces iso
 $H_*(X, A) \rightarrow H_*(X/A, \text{pt}) = \tilde{H}_*(X/A)$

Pf good $\Rightarrow \exists$ nbhd V of A , and $A \xrightarrow{\text{incl}} V$.
 LES for pair $H_n(X, A) \xrightarrow{\cong} H_n(X, V) \xrightarrow{\text{quot.}} H_n(X/A, V/A) \xrightarrow{\cong} H_n(X/A, \text{pt})$
 (see below Remark) \hookrightarrow "id" = identity

Remark $X/A \rightarrow X/A \setminus p$, $x \mapsto [x] = \{x\}$ is essentially an identity map but we need to check it is a homeomorphism. The composite $X/A \xrightarrow{\text{id}} X \xrightarrow{\text{ctr}} X/A$ is cts and lands in subspc $X/A \setminus p$, but also need $X/A \rightarrow X/A \setminus p$ to send open sets to open sets. Since A is closed, open subsets $U \subseteq X/A$ are open in X and yield an open set $U \cap X/A$ not containing p , hence "id" $(U) = U \cap X/A \subseteq X/A \setminus p$ open.

Example $\mathbb{D}^n \supseteq S^{n-1}$ good:
 $\Rightarrow H_*(\mathbb{D}^n, S^{n-1}) \cong \tilde{H}_*(\mathbb{D}^n/S^{n-1}) \cong \tilde{H}_*(S^n) \cong \mathbb{Z}$ (for a map $(X, A) \rightarrow (Y, B)$ of good pairs get Comm. diagram...)

Exercise Check that the iso in the Cor is natural (good pairs get Comm. diagram...)

Recall we proved $H_*(D^n, S^{n-1}) \cong \widetilde{H}_{*-1}(S^{n-1})$ (from LES & $\widetilde{H}_*(D^n) \cong 0$)
 $\Rightarrow \widetilde{H}_k(S^n) \cong \widetilde{H}_{k-1}(S^{n-1}) \cong \dots \cong \widetilde{H}_{k-n}(S^0) \cong \begin{cases} \mathbb{Z} & n=k \\ 0 & \text{else} \end{cases}$
 inductively, using Example 2 points

Generator of $H_n(S^n) \cong \widetilde{H}_n(D^n/S^{n-1}) \cong H_n(D^n, S^{n-1}) \cong \mathbb{Z}$

Observe \exists homeo $e^n: D^n \cong D^n$ (homework) inducing Δ -cx structure on S^{n-1} :
 $\partial D^n \cong \partial D^n = S^{n-1}$
 Example $D^2 \cong \begin{matrix} \Delta_1 \\ \Delta_2 \end{matrix} \xrightarrow{\partial} -\Delta_1^+ + \Delta_2^+ \cong S^1$
 stretch cktly outwards from barycentre (Δ^n)

Upshot ($n \geq 2$)
 $H_n(D^n, S^{n-1}) = \mathbb{Z} \cdot e^n$
 $H_{n-1}(S^{n-1}) = \mathbb{Z} \cdot \partial e^n$
 $\widetilde{H}_n(D^n/S^{n-1}) = \mathbb{Z} \cdot [e^n]$
 for $n-1 \geq 1$, so $n \geq 2$ by Cor $[e^n]$ really lives in $H_n(D^n, S^{n-1}) \cong H_n(D^{n-1}/S^{n-2}, S^{n-1})$

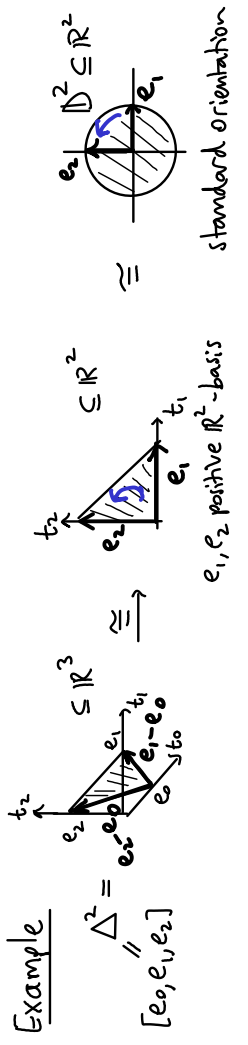
Exercise Recall another Δ -cx structure on S^n :
 $S^n = \Delta_1^+ \cup \Delta_2^- / \text{glue along } \partial \Delta_1^+ \cup \partial \Delta_2^-$
 call this Δ_1 this Δ_0

then $H_n(S^n) = \mathbb{Z} \cdot (\Delta_1 - \Delta_0)$ because $\Delta_1 - \Delta_0$ is a cycle and $H_n(S^n) \cong H_n(S^n, \Delta_0) \cong H_n(\Delta_1, \partial \Delta_1) \cong H_n(D^n, \partial D^n)$
 $\Delta_1 - \Delta_0 \xrightarrow{\partial} \Delta_1 \xrightarrow{\partial} e^n$ (by LES for (S^n, Δ_0) using $n \geq 1$)

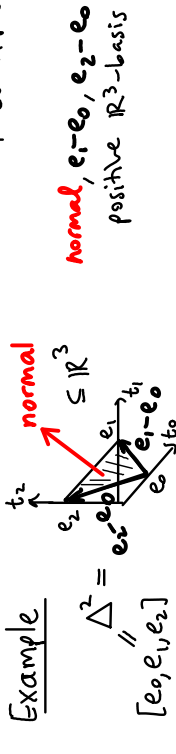
Another remark about orientations

Fact {homeos $\Delta^n \rightarrow D^n$ } has 2 path-components
 Above we chose a path-component by constructing e^n .
 If τ is any reflection in \mathbb{R}^{n+1} then $e^n \circ \tau$ is in the other path-component
 $H_n(S^n) \cong H_n(D^n, S^{n-1}) \xrightarrow{\cong} \mathbb{Z}$
 $e^n \circ \tau \xrightarrow{\partial} +1$
 $e^n \circ \tau \xrightarrow{\partial} -1$
 e.g. swap 2 coordinates in Δ^n

We will see later in the course that this corresponds to a choice of orientation of D^n and S^n .
 Our choice is consistent with the inclusion $D^n \subseteq \mathbb{R}^n$ (with the positive (canonical) orientation of \mathbb{R}^n) and the inclusion $(\Delta^n \subseteq \mathbb{R}^{n+1}) \cong \{(t_1, \dots, t_n) \in \mathbb{R}^n : t_i \geq 0, \sum_{i=1}^n t_i \leq 1\}$
 $(t_0, \dots, t_n) \mapsto (t_1, \dots, t_n)$
 $t_i \geq 0, \sum_{i=1}^n t_i = 1$



Our choice is also consistent with the "normal first" Convention for orienting hyperplanes with a given choice of normal:
 $\Delta^n \subseteq$ hyperplane $\{(t_0, \dots, t_n) : \sum_{j=1}^n t_j = 1\} \subseteq \mathbb{R}^{n+1}$ normal $(1, 1, \dots, 1)$ (so pointing to ∞ in positive quadrant)



Consistent also with the geometric boundary orientation (outward normal first)
 Geometric $\Delta^2 = [e_0, e_1, e_2] \cong [0, 1, 2]$

 $S^1 = \partial D^2 \subseteq \mathbb{R}^2$ standard orientation

Compare $\partial \Delta = +[e_0, e_1, e_2] - [e_0, \hat{e}_1, e_2] + [e_0, e_1, \hat{e}_2]$

This $-[e_0, e_2]$ is not equal to singular chain $[e_2, e_0]$ since they are different maps and we take free abelian group generated by maps. But $[e_0, e_2] + [e_2, e_0]$ is homologous to 0 (Homework).

Locality (or "small simplices theorem")

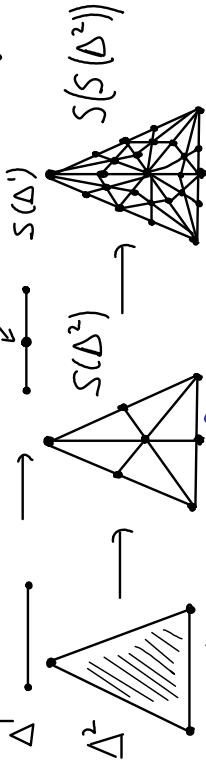
$\mathcal{U} = \{ \text{subspaces } U_i \subseteq X \}$ whose interior cover X :
 $X = \bigcup U_i$

Def $C_*^u(X) \subseteq C_*(X)$ subx generated by n -simplices σ with $\sigma(\Delta^n) \subseteq U_i$ some i

Theorem

$$H_* (C_*^u(X)) \cong H_* (C_*(X)) = H_* X$$

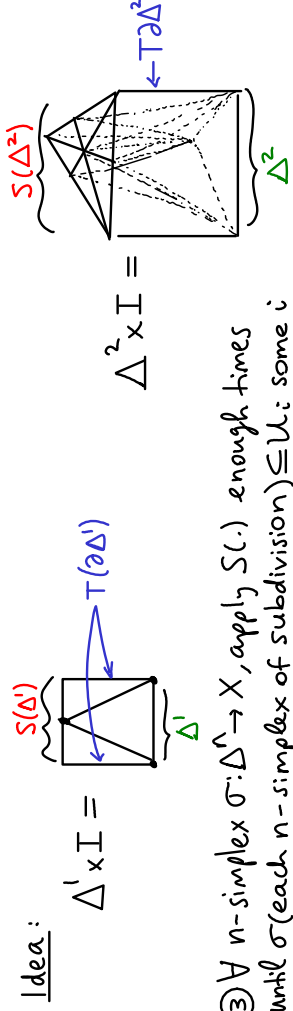
Sketch Pf ① Barycentric subdivision
 barycentre of $[v_0, \dots, v_n]$ is $\frac{1}{n+1}(v_0 + \dots + v_n)$
 barycentre divides edge in 2



\Rightarrow chain map $S: C_*(X) \rightarrow C_*(X)$
 and $S(C_*^u) \subseteq C_*^u$
 Construction of " $\sigma \circ S$ " is inductive:
 On linear simplices (them for maps σ you restrict to...)

$S[e_0] = [e_0]$
 $S[e_0, e_1] = [b, e_1] - [b, e_0]$ (geometrically $\vec{e_0} + \vec{e_1} = \vec{b}$)
 $S[e_0, e_1, e_2] = [b, S\partial[e_0, e_1, e_2]]$
 $= [b, S[e_1, e_2]] - [b, S[e_0, e_2]] + [b, S[e_0, e_1]]$
 $= ([b, b_{12}, e_2] - [b, b_{02}, e_2]) - ([b, b_{02}, e_1] - [b, b_{01}, e_1]) + ([b, b_{01}, e_0] - [b, b_{01}, e_0])$
 geometrically: $b_{02} = b_{01} + b_{12}$

so for $\sigma: \Delta^2 \rightarrow X$ you take $S(\sigma) = \sigma \circ S$
 ② S chain hpic to id:
 $T: C_n(X) \rightarrow C_{n+1}(X)$
 $T(\sigma) = \Delta^n \times I \xrightarrow{\text{project}} \Delta^n \xrightarrow{\sigma} X \Rightarrow S_*: H_*(X) \xrightarrow{\text{id}} H_*(X)$
exercise: $\partial T + T\partial = S - \text{id}$



③ $\forall n$ -simplex $\sigma: \Delta^n \rightarrow X$, apply $S(\cdot)$ enough times until σ (each n -simplex of subdivision) $\subseteq U_i$ some i
 \forall cycle $c, \exists n$ s.t. $S^n(c) \in C_*^u(X)$ cycle
 $\Rightarrow H_*^u(c) \rightarrow H_*(X)$ surjective
 $[S^n(c)] \rightarrow S_*^u[c] = [c]$ by ②

③ \forall bdry $c = \partial b, \exists n$ s.t. $S^n(b) \in C_*^u(X)$
claim: $H_*^u(c) \rightarrow H_*(X)$ injective
 suppose $[c] \mapsto 0$ then $c = \partial b$ for $b \in C_*(X)$
 now $S^n c, S^n b \in C_*^u(X)$ for large n
 $\Rightarrow \partial S^n b = S^n \partial b = S^n c$ in $C_*^u(X)$
 $\Rightarrow [c] \in S_*^u[c] = [S^n c] = [\partial S^n b] = 0$ in $H_*^u(X) \checkmark \square$

Proof of excision theorem

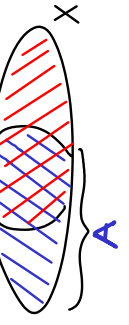
Let $B = X \setminus E$
 use $\mathcal{U} = \{A, B\}$
 so $C_*^u(X) = C_*(A) + C_*(B) \subseteq C_*(X)$

$$\Rightarrow C_*(X \setminus E) = C_*(B) \cong C_*(B) / C_*(A \cap B) \cong C_*^u(X) / C_*(A)$$

 Compare LES's:
 $H_*(X \setminus E, A \setminus E) \xrightarrow{\text{locality}} H_*(C_*^u X) \rightarrow H_*(C_* X / C_* A) \rightarrow H_{*-1}(A) \rightarrow H_{*-1}(C_* X)$
 $\parallel \text{locality} \cong \downarrow \text{iso by 5-lemma} \parallel \downarrow \text{locality} \cong$
 $H_*(A) \rightarrow H_*(X) \rightarrow H_*(C_* X / C_* A) \rightarrow H_{*-1}(A) \rightarrow H_{*-1}(X)$
 (we are using naturality of LES's induced by SESs)
 $H_*(X, A)$ \square

6. MAYER-VIETORIS SEQUENCE ← Key computational tool

$X = A \cup B$ s.t. $X = A \cup B^o$



any subspaces

MV Theorem \exists LES:

$$\dots \rightarrow H_*(A \cap B) \xrightarrow{i_*} H_*(A) \oplus H_*(B) \rightarrow H_*(X) \rightarrow H_{*-1}(A \cap B) \xrightarrow{i_*} \dots$$

& same holds for \tilde{H}_* provided $A \cap B \neq \emptyset$.

Pf SES $0 \rightarrow C_*(A \cap B) \xrightarrow{\sigma} C_*(A) \oplus C_*(B) \rightarrow C_*^u(X) \rightarrow 0$

$\sigma \mapsto (\sigma, -\sigma)$

$(\alpha, \beta) \mapsto \alpha + \beta$

\Rightarrow induces the LES (using locality $H_*^u X \cong H_* X$). D

Exercise connecting map is $\delta: H_*(X) \rightarrow H_{*-1}(A \cap B)$

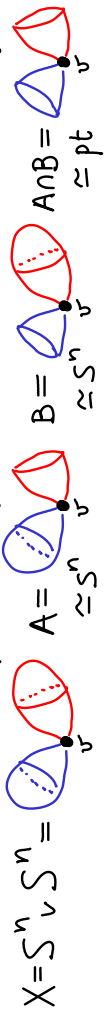
$[\alpha + \beta] \mapsto [\partial\alpha] = -[\partial\beta]$



$\dots \rightarrow H_2(\text{pt}) \oplus H_2(\text{pt}) \xrightarrow{\cong} H_2(S^2) \rightarrow H_1(S^1) \rightarrow H_1(\text{pt}) \oplus H_1(\text{pt}) \rightarrow \dots$

Exercise Compute $H_* S^n = \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z} & * = n \\ 0 & \text{else} \end{cases}$ using MV

Example wedge sum of X, Y with basepoints x, y



$X \vee Y = \frac{X \sqcup Y}{x \sim y}$

$0 \rightarrow \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\cong} H_n(X) \rightarrow 0 \rightarrow \dots \rightarrow 0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \oplus \mathbb{Z} \rightarrow H_0(X) \rightarrow 0$

Similarly $[H_*(X \vee Y) \cong H_*(X) \oplus H_*(Y)]$ for $* \neq 0$ if \exists contractible nbhds of $x \in X, y \in Y$.

Cones and suspensions

$CX = (X \times [0,1]) / \sim_{s=t=1}$

$\Sigma X = (X \times [0,1]) / \sim_{s=0, t=1}$

Example $CS^n \cong \mathbb{D}^{n+1}$, $\Sigma S^n \cong S^{n+1}$.
or $s=t=0$ or $s=t=1$

Lemma $\tilde{H}_*(\Sigma X) \cong \tilde{H}_{*-1}(X)$

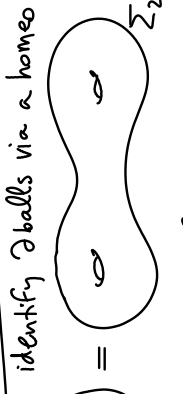


now apply MV. \square

Rmk $\phi \neq A \subseteq X \Rightarrow \tilde{H}_*(X \setminus A) \stackrel{\text{LES}}{\cong} H_*(X \setminus A, CA) \stackrel{\text{exc.}}{\cong} H_*(X, A)$

Connected sum

M, N n -manifolds $\Rightarrow M \# N = (M \setminus \text{open } n\text{-ball}) \cup (N \setminus \text{open } n\text{-ball})$



Fact compact connected orientable surfaces are homeo to S^2 or $T^2 \# \dots \# T^2$ $g = \#$ copies
and " " non-orientable ones: $\mathbb{R}P^2 \# \dots \# \mathbb{R}P^2$. $g=0$ called Σ_g

Exercise (Homework) For M, N compact connected n -mfd's:

By MV, $H_*(M \# N) \cong H_*(M) \oplus H_*(N)$ for $1 \leq * \leq n-2$

If M or N orientable: $* = n-1$ also works

If both non-orientable: $* = n-1$ one of $\mathbb{Z}/2$ summands becomes \mathbb{Z}

Cor 1) $\chi(M \# N) = \chi(M) + \chi(N) - \begin{cases} 2 & \text{for } n \text{ even} \\ 0 & \text{" " odd} \end{cases}$
2) $H_*(\Sigma_g) \cong \begin{cases} \mathbb{Z}^{2g} & * = 0 \\ \mathbb{Z} & * = 1 \\ \mathbb{Z} & * = 2 \end{cases} \cong \chi(S^n)$

$H_0(M \# N) \cong \mathbb{Z}$ since connected
fact: $H_n(M \# N)$ is \mathbb{Z} or 0
if M, N both orientable (see later in course)

7. DEGREE OF MAPS OF SPHERES

ASSUME $n \geq 1$ IN THIS CHAPTER

$$f: S^n \rightarrow S^n \implies f_n: H_n S^n \xrightarrow{\cong} H_n S^n$$

$$\mathbb{Z} \xrightarrow{\cong} \mathbb{Z}$$

$$1 \longmapsto \text{deg}(f) \in \mathbb{Z}$$

$\implies f_*: \tilde{H}_* S^n \rightarrow \tilde{H}_* S^n$ is $\text{deg}(f) \cdot \text{id}$

Properties

- 1) $\text{deg}(\text{id}) = 1$
- 2) $\text{deg}(f \circ g) = \text{deg} f \cdot \text{deg} g$
- 3) $f \simeq g \implies \text{deg} f = \text{deg} g$
- 4) $f \simeq \text{const} \implies \text{deg} f = 0$
- 5) f homeomorphism $\implies \text{deg} f = \pm 1$

sign depends on whether f is orientation-preserving or reversing

Pf $\text{id}_* = \text{id}$, $(f \circ g)_* = f_* \circ g_*$, $f \simeq g \implies f_* = g_*$, $\text{const}_* = 0$, f homeo $\implies f_n$ iso. \square
 since $S^n \xrightarrow{\text{pt}} S^n$ factors so $H_n S^n \xrightarrow{H_n(\text{pt})} H_n S^n$

Examples

- 1) $S^n = \Delta^n \times 1 \cup \Delta^n \times 0 \leftarrow \text{call this } \Delta_0$
 $(b, 1) \sim (b, 0)$ if $b \in \partial \Delta$
 recall $H_n S^n = \mathbb{Z} \cdot (\Delta_1, -\Delta_0)$
 reflection: $r: S^n \rightarrow S^n$, $r(x, t) = (x, 1-t)$
 so $\Delta_0 \leftrightarrow \Delta_1$ swapped by r , so $r_*(\Delta_1, -\Delta_0) = -(\Delta_1, -\Delta_0)$
 $\implies \text{deg}(r) = -1$



- 2) antipodal map $-\text{id}: S^n \rightarrow S^n$ viewing $S^n \subseteq \mathbb{R}^{n+1}$
 $\implies \text{deg}(-\text{id}) = (-1)^{n+1}$

Pf $-\text{id} = \begin{pmatrix} -1 & & \\ & \ddots & \\ & & -1 \end{pmatrix} \circ \begin{pmatrix} 1 & & \\ & \ddots & \\ & & -1 \end{pmatrix} \circ \dots \circ \begin{pmatrix} 1 & & \\ & \ddots & \\ & & -1 \end{pmatrix}$
 composition of $n+1$ reflections each homotopic to r . \square

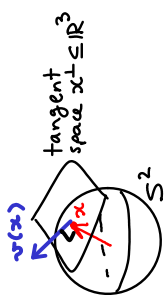
- 3) $A \in O(n) \implies A: S^{n-1} \rightarrow S^{n-1} \implies \text{deg} A = \det A = \pm 1$
Pf fact $SO(n)$ is path-connected so $A \in SO(n)$ is $\simeq \text{id}$ so $\text{deg} A = \det A = +1$
 The other path-component of $O(n)$ is $r \circ SO(n)$ where r is any reflection. \square

4) f not surjective $\implies \text{deg} f = 0$

Pf If $y \notin \text{Im} f \implies H_n(S^n) \xrightarrow{f_*} H_n(S^n) \rightarrow H_n(\mathbb{R}^n) = 0$

Application to vector fields on S^n

$v: S^n \rightarrow \mathbb{R}^{n+1}$ tangent vector field on S^n
 so $v(x) \perp x$



Cor Hairy ball theorem

\exists nowhere zero v.f. on $S^n \iff n$ odd

(case $n=2$: "you cannot comb a ball of hair without creating a tuft")

Pf Suppose $v(x) \neq 0 \forall x$

$$\implies \text{hpy } F: S^n \times [0, 1] \rightarrow S^n$$

$$\implies F(x, t) = \cos(\pi t)x + \sin(\pi t) \frac{v(x)}{\|v(x)\|}$$

$$\implies F_0 = \text{id}, F_1 = -\text{id}$$

$$\implies 1 = \text{deg } F_0 = \text{deg } F_1 = (-1)^{n+1}$$

$$\implies n \text{ odd}$$

For n odd $\exists v(x) = (-x_2, x_1, \dots, -x_{2k}, x_{2k-1}) \in \mathbb{R}^{2k}$. \square

Cultural Remark Adams in 1962 proved using alg. topology:

(max # pointwise linearly independent vector fields on $S^n = 2^b + 8a - 1$ where $n+1 = 2^{4a+b}$ (odd number), $0 \leq b \leq 3$, $a, b \in \mathbb{N}$, $n \geq 1$. \leftarrow get 0 if n even $\implies \text{cor} \checkmark$)

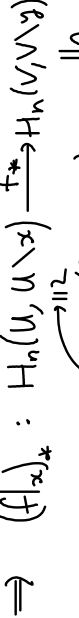
Local degree

$$f: S^n \rightarrow S^n$$

$$x \mapsto y = f(x)$$

\star Suppose points $x \neq y$ near x do not map to y : \exists nbhds $x \in U, y \in V$ s.t. $(U, U \setminus x) \xrightarrow{f} (V, V \setminus y)$

$\implies (f|_x)_* : H_n(U, U \setminus x) \xrightarrow{f_*} H_n(V, V \setminus y)$



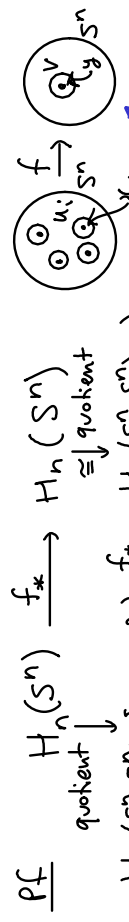
will use this again later: $H_n(S^n) \cong H_n(S^n, S^n \setminus \text{pt})$

deg f

RMK By left vertical arrows: $H_n(U, U \setminus x) \cong \mathbb{Z}$ again we made a choice of iso by picking a generator "1" by combining excision, quotient and choice of $H_n(S^n) \cong \mathbb{Z}$.

Lemma $f: S^n \rightarrow S^n, f^{-1}(y) = \{x_1, \dots, x_k\}$ ($n \geq 1$)

$$\Rightarrow \deg f = \sum \deg_{x_i} f$$



$\text{exc. } S^n \setminus \cup U_i \cong \text{exc. } S^n \setminus V \cong H_n(V, V \setminus Y)$

$\oplus H_n(U_i \setminus \{x_i\}) \xrightarrow{\oplus (f|_{x_i})_*} H_n(V, V \setminus Y)$

map to each summand is exc. of $S^n \setminus U_i$ so iso.

is: $1 \in \mathbb{Z} \xrightarrow{\deg f} \mathbb{Z} \xrightarrow{\text{id}} \mathbb{Z}$

$(1, \dots, 1) \in \oplus \mathbb{Z} \xrightarrow{\oplus \deg_{x_i} f} \mathbb{Z} \xrightarrow{\text{id}} \mathbb{Z}$ (see Rmk on previous page about choice of "1")

Example $p: \mathbb{C} \rightarrow \mathbb{C}$ polynomial $a_n z^n + a_{n-1} z^{n-1} + \dots + a_0, a_i \in \mathbb{C}$

$\Rightarrow f: S^2 = \mathbb{C}P^1 \rightarrow \mathbb{C}P^1 = S^2$ (where view $\mathbb{C}P^1 = \mathbb{C} \cup \infty \cong S^2$)

$\Rightarrow \text{hpy } F(z, t) = a_n z^n + t(a_{n-1} z^{n-1} + \dots + a_0)$

$F_0 = a_n z^n$ and $F_1 = f$

$\Rightarrow \deg f = \deg(a_n z^n) \stackrel{\text{Lemma}}{=} \sum_{k=0}^{n-1} \deg a_k z^k \leftarrow w_k = e_n$

$\stackrel{\text{Rmk}}{=} \deg$ of the poly p .

Cor (Fundamental Thm of Algebra) $n \geq 1 \Rightarrow p$ has a root

PF $p^{-1}(0) = \emptyset \Rightarrow f^{-1}(0) = \emptyset \Rightarrow \deg f = 0 \Rightarrow \mathbb{Z} \square$

Cultural Rmk For smooth $f: S^n \rightarrow S^n$

$\deg f =$ (the number of preimages of a generic point)

(i.e. almost any point works)

Example $S^2 \rightarrow S^2$

$S^2 \setminus \text{North pole} \cong \mathbb{C}$

map $\mathbb{C} \rightarrow \mathbb{C}, z \mapsto z^d$ and North \mapsto North

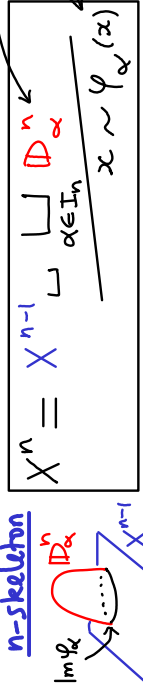
$\Rightarrow \deg = d = \#$ preimages of a point

except if pick North/South pole

$\left(\begin{matrix} z=0 \\ z=\infty \end{matrix} \right) \left(\begin{matrix} z=0 \\ z=\infty \end{matrix} \right)$

8. CELLULAR HOMOLOGY

Def CW complex X is sequence $\phi = X^{-1} \subseteq X^0 \subseteq X^1 \subseteq X^2 \subseteq \dots$ s.t. X^0 is any set with discrete topology. closed n -discs labelled by some index set I_n



$\Rightarrow X = \bigcup_{n \geq 0} X^n$ top space with weak topology:

$U \subseteq X$ open $\iff U \cap X^n \subseteq X^n$ open $\forall n$.

$(\iff U \cap D_\alpha^n \subseteq D_\alpha^n$ open $\forall n, \alpha$)

Call X n-dimensional if $X = X^n$ and this is the least such n .

Example $S^n = (D^0 \sqcup D^1) / (\partial^0 \sim \partial^1)$ \rightarrow attach $D^2 = S^2$

Example $X = \mathbb{R}P^2 = (D^0 \sqcup D^1) / (\partial^0 \sim \partial^1)$ \rightarrow attach $D^2 = S^2$

$X^0 = \bullet = D^0$ $X^1 = S^1 = (D^0 \sqcup D^1) / (\partial^0 \sim \partial^1)$ $\partial D^1 = S^0 = \{0, 1\} \xrightarrow{\varphi_1} X^0 = \bullet$

$X^2 = (D^1 \sqcup D^2) / (\text{wrap } \partial \text{ of } D^2 \text{ twice around } D^1)$

$= (X^1 \sqcup D^2) / (\partial D^2 = S^1 \xrightarrow{\varphi_2} X^1 = S^1 \xrightarrow{\varphi_1} X^0 = \bullet)$

Fact If we homotope φ_α , we get a homotopy equivalent space

Example If we use another degree 2 map φ_2 above, get $X \simeq \mathbb{R}P^2$.

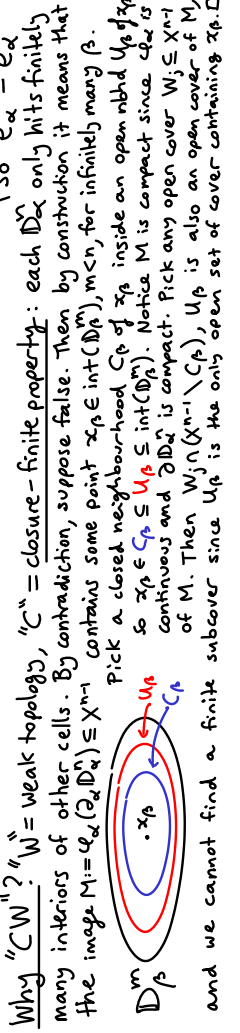
Cultural Rmk Every CW-complex X is homotopy equivalent to a simplicial complex Y (so in particular a Δ -complex). [If X finite/ n -dim then can ensure Y is finite/ n -dim]

X is partitioned as a set by interiors of n-cells

$X^n = X^{n-1} \cup \bigcup_{\alpha \in I_n} \text{int}(D_\alpha^n) \cup \bigcup_{\alpha \in I_{n-1}} \text{int}(D_\alpha^{n-1}) \cup \dots \cup \bigcup_{\alpha \in I_0} \text{int}(D_\alpha^0)$

$e_\alpha^n = \text{Image}(D_\alpha^n \rightarrow X)$

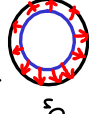
Why "CW"? "W" = weak topology, "C" = closure-finite property: each D_α^n only hits finitely many interiors of other cells. By contradiction, suppose false. Then by construction it means that the image $M_i = \varphi_\alpha(D_\alpha^i) \subseteq X^{i-1}$ contains some point $x_\beta \in \text{int}(D_\beta^{i-1})$, m.c.n, for infinitely many β .



Examples $\mathbb{R}P^k = S^k / (\mathbb{Z}/2\text{-action by } \pm \text{id})$ 
 $X^k = \mathbb{R}P^k$ inductively
 $X^n = X^{n-1} \cup e^n$ with $\varphi: S^{n-1} \rightarrow X^{n-1} = \mathbb{R}P^{n-1} = S^{n-1} / \pm \text{id}$
 $x \mapsto [x] = [-x]$

Complex projective space $\mathbb{C}P^n = (S^{2n+1} \subseteq \mathbb{C}^{n+1}) / (S^1\text{-action by } \lambda \cdot \text{Id})$
 $X^0 = X^1 = pt = \mathbb{C}P^0$
 $X^2 = X^3 = X^0 \cup e^2 = \mathbb{C}P^1$ $\varphi: S^1 \rightarrow pt$ $\mathbb{C}P^1 \cong S^2$
 $X^4 = X^5 = X^2 \cup e^4 = \mathbb{C}P^2$ $\varphi: S^3 \rightarrow \mathbb{C}P^1 = S^3 / (S^1\text{-action})$
 $x \mapsto [x] = [\lambda x], \forall \lambda \in S^1$
 $X^{2n} = X^{2n-2} \cup e^{2n} = \mathbb{C}P^n$ $\varphi: S^{2n-1} \rightarrow X^{2n-2} = \mathbb{C}P^{n-1}$
 $x \mapsto [x]$

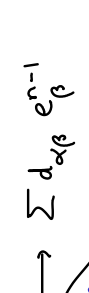
In coordinates: $\mathbb{C}P^n = \{ [z_0 : \dots : z_n] : \text{not all } z_i \in \mathbb{C} \text{ are } 0 \}$ and $[z] \sim [\lambda z], \forall \lambda \in \mathbb{C}^*$
 Can rescale so that $\sum |z_i|^2 = 1$ so $z \in S^{2n-1}$ and left with rescaling by $\lambda \in S^1 \subseteq \mathbb{C}^*$.
 $\mathbb{C}P^{n-1} \cong X^{2n-2} = \{ [z_0 : \dots : z_{n-1} : 0] \} \subseteq \mathbb{C}P^n = X^{2n}$ and \leftarrow notice this is 0 if $w_0 = \dots = w_{n-1} = \sqrt{1 - \sum |w_j|^2}$
 $e^{2n}: \mathbb{D}^{2n} = \{ (w_0, \dots, w_{n-1}) : \sum |w_j|^2 \leq 1 \} \rightarrow X^{2n}$ via $[w_0 : \dots : w_{n-1} : \sqrt{1 - \sum |w_j|^2}]$

Observe: For X CW complex, for $n \geq 1$:
 • (X^n, X^{n-1}) is a good pair \leftarrow (since \exists nbhd of $\partial \mathbb{D}^n$ that deformation retracts to $\partial \mathbb{D}^n$)
 • $X^n / X^{n-1} \cong \bigvee_{\alpha \in I_n} S^n$

 X^{n-1} identified to a point

Def Cellular complex for X a CW cx,
 $C_n^{CW}(X) = H_n(X^n, X^{n-1}) \cong H_n(X^n / X^{n-1})$
 $=$ free abelian gp gen. by the n -cells e_α^n
 since $\Delta^n \cong \mathbb{D}^n \rightarrow (e^n \subseteq X^n) \rightarrow \mathbb{D}_\alpha^n / \partial \mathbb{D}_\alpha^n = S_\alpha^n$ generate
 as usual we use the standard orientations of $\Delta^n, \mathbb{D}^n, S^n$.
 Will build cellular differential $d: C_*^{CW} \rightarrow C_*^{CW}$, prove $d \circ d = 0$
 \Rightarrow get $H_*^{CW}(X) = H_*(C_*^{CW}(X), d)$

Example $C_k^{CW}(\mathbb{C}P^n) = \begin{cases} \mathbb{Z} \cdot e^k & \text{for } k=0,2,4,\dots,2n \\ 0 & \text{else} \end{cases}$ hence $d=0$ so $H_*^{CW}(\mathbb{C}P^n) = C_*^{CW}(\mathbb{C}P^n) \cong \begin{cases} \mathbb{Z} & 0 \leq * \leq 2n \text{ even} \\ 0 & \text{else} \end{cases}$

Cellular differential: $d e_\alpha^n = \sum_{\beta \in I_{n-1}} d_{\alpha\beta}^n \cdot e_\beta^{n-1}$
 now describe the coefficients $d_{\alpha\beta}^n \in \mathbb{Z}$ and why that is a finite sum.

$d: C_n^{CW} \rightarrow C_{n-1}^{CW}$
 $H_n(X^n, X^{n-1}) \xrightarrow{\text{Connecting map of LES}} H_{n-1}(X^{n-1}) \xrightarrow{\text{quotient map}} H_{n-1}(X^{n-1}, X^{n-2}) \rightarrow \sum d_{\alpha\beta}^{n-1} e_\beta^{n-1}$
 $e_\alpha^n \mapsto [\partial \mathbb{D}_\alpha^n \xrightarrow{\varphi_\alpha} \varphi_\alpha(S^{n-1})] \rightarrow \sum d_{\alpha\beta}^{n-1} e_\beta^{n-1}$
 Recall LES
 $H_*(X^{n-1}) \rightarrow H_*(X^n) \xrightarrow{q_*} H_*(X^n, X^{n-1}) \xrightarrow{q_*} H_*(X^{n-1}, X^{n-2}) \rightarrow \dots$
 \leftarrow S^n \leftarrow S^n \leftarrow S^n
 \leftarrow $\mathbb{D}_\alpha^n \rightarrow e_\alpha^n \subseteq X^n$
 \leftarrow $\partial \mathbb{D}_\alpha^n \subseteq X^{n-1}$


So $(\varphi_\alpha)_* [S^{n-1}]$ is our chosen generator of $H_{n-1}(S^{n-1})$ from Chapter 5
 here it is important that we chose identifications $\Delta^n \cong \mathbb{D}^n, S^n \cong \mathbb{D}^n / \partial \mathbb{D}^n$ compatibly with orientations.
 Quotient by $\bigvee_{I_{n-1}} S^{n-1}$

Therefore: $d_{\alpha\beta}^n = \text{deg}(S^{n-1} \xrightarrow{\varphi_\alpha} X^{n-1} \xrightarrow{q} X^{n-1} / X^{n-2} \cong \bigvee_{I_{n-1}} S^{n-1} \rightarrow S^{n-1})$
 $\cong \mathbb{D}_\beta^n / \partial \mathbb{D}_\beta^n$

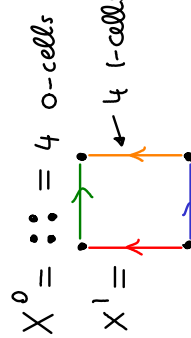
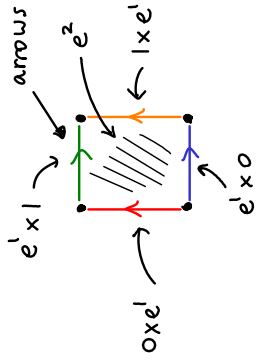
Rmk Only finitely many $d_{\alpha\beta}^n \neq 0$ (for fixed α) because φ_α, q are continuous and S^{n-1} compact, so get a compact image in $\bigvee_{\beta} S^{n-1}$, therefore cannot surject onto ∞ many S_β^{n-1} .
 Recall if don't surject then $\text{deg}=0$
 indeed see proof of "closure-finite": it shows φ_α only hits finitely many interiors of \mathbb{D}_β^{n-1}
 by LES $d_{n-1} \circ d_n = q_{n-2} \circ d_{n-1} \circ q_{n-1} \circ d_n = 0$
 \square

Lemma $d \circ d = 0$
PF $d_n = q_{n-1} \circ d_n = q_{n-2} \circ d_{n-1} \circ q_{n-1} \circ d_n = 0$
 \square

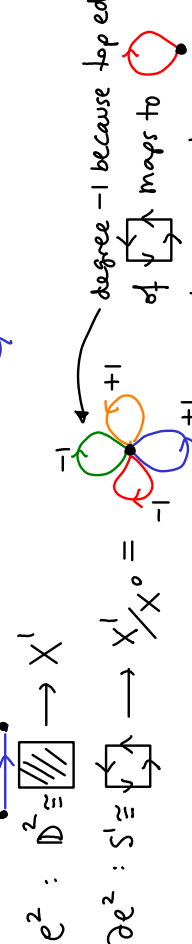
Cor $\text{rank } H_n^{CW}(X) \leq \# \text{ n-cells}$
PF $\# \text{ n-cells} = \text{rank } C_n^{CW}(X) \geq \text{rank } \ker d_n^{CW} \geq \text{rank } H_n^{CW}(X) \square$

Example $I \times I$ $I = [0,1]$ $\mathbb{D}^1 = [-1,1]$

arrows here tell us how we map $[-1,1] \rightarrow$ edge (so orientation)



orientations of cells tell us how to orient the circles



$\partial e^2 = +e^1 x_0 + 1 x e^1 - e^1 x_1 - o x e^1$
 (= $(\partial e^1) \times e^1 - e^1 \times (\partial e^1)$ ← we come back to this later)

Example $\mathbb{R}P^n$ recall: 1 cell in each dim, $\varphi: S^k \rightarrow X^k = \mathbb{R}P^k$
 $x \mapsto [x]$

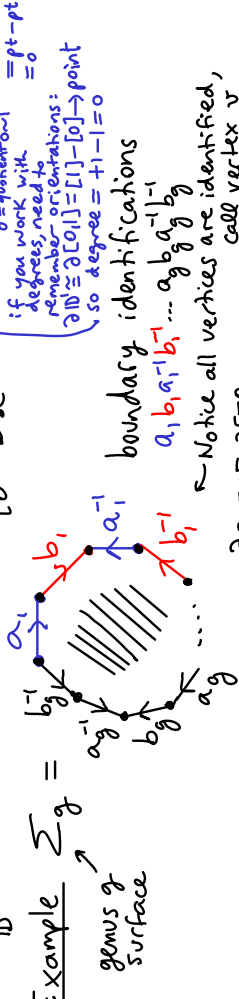
$S^{k-1} \xrightarrow{\varphi} X^{k-1} / X^{k-2} = \mathbb{R}P^{k-1} / \mathbb{R}P^{k-2} \cong S^{k-1}$
 $\xrightarrow{-id(\Delta)}$
 $\Rightarrow d\alpha_\beta = \begin{cases} 2 & k \text{ even} \\ 0 & k \text{ odd} \end{cases}$

$C_*^{CW}(\mathbb{R}P^n) \rightarrow \mathbb{Z} \xrightarrow{k=n} \mathbb{Z} \xrightarrow{k=n-1} \mathbb{Z} \xrightarrow{k=n-2} \mathbb{Z} \rightarrow \dots \rightarrow \mathbb{Z} \xrightarrow{k=1} \mathbb{Z} \rightarrow 0$
 (2 if n even, 0 if n odd resp.)

$H_*^{CW}(\mathbb{R}P^n) \cong \begin{cases} \mathbb{Z} & * = 0, \text{ also } * = n \text{ if odd} \\ \mathbb{Z}/2 & * = 1, 3, 5, \dots < n \\ 0 & \text{else} \end{cases}$

Example S^n : $C_*^{CW}(S^n): n > 2: 0 \rightarrow \mathbb{Z} \cdot \mathbb{D}^n \xrightarrow{0} 0 \rightarrow \dots \rightarrow 0 \rightarrow \mathbb{Z} \cdot \mathbb{D}^0 \rightarrow 0$

$n=1: 0 \rightarrow \mathbb{Z} \cdot \mathbb{D}^1 \xrightarrow{0} \mathbb{Z} \cdot \mathbb{D}^0 \rightarrow 0$
 $\Rightarrow H_*^{CW}(S^n) \cong \begin{cases} \mathbb{Z} & * = 0, n \\ 0 & \text{else} \end{cases}$

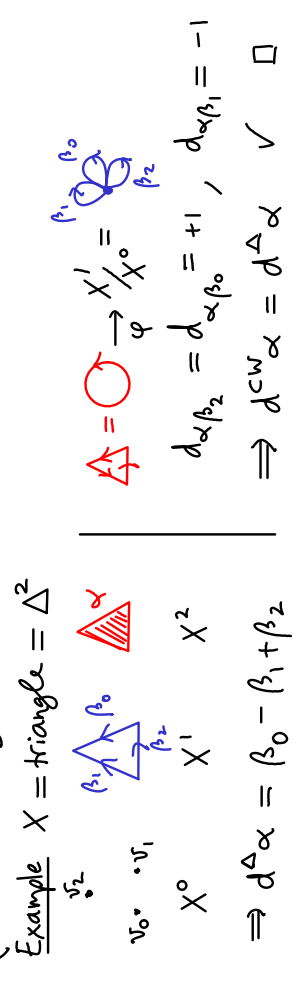


$0 \rightarrow \mathbb{Z} \xrightarrow{\partial} \mathbb{Z} \xrightarrow{\partial=0} \mathbb{Z} \rightarrow 0 \Rightarrow H_*(\Sigma_g) \cong \begin{cases} \mathbb{Z} & * = 0, 2 \\ \mathbb{Z}^{2g} & * = 1 \\ 0 & \text{else} \end{cases}$

Example Non-orientable surface $N_k: \mathbb{D} \rightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \rightarrow 0$
 $1 \mapsto (-2, \dots, -2)$
 (since $\mathbb{D} \mapsto -a_1 - a_1 - a_2 - a_2 - \dots - a_k - a_k$)

Lemma $X \Delta$ -cx structure \Rightarrow induces CW-cx structure on X and $(C_*^{CW}(X), d^{CW}) \cong (C_*^\Delta(X), d^\Delta)$
 $\Rightarrow H_*^{CW}(X) \cong H_*^\Delta(X)$

Pf $X^n = \cup n$ -simplices of X and degrees are ± 1 depending on orient.
 so can identify d^{CW} and d^Δ . \square



Theorem X CW cx (or Δ -cx) \implies $H_*^{CW}(X) \cong H_*(X)$

$\implies H_*^{\Delta}, H_*^{CW}$ independent of choice of CW-cx/ Δ -cx structure.

Pf ① $H_*(X^n, X^{n-1}) \cong \tilde{H}_*(X^n/X^{n-1}) \cong \tilde{H}_*(\bigvee_{\alpha} S^n) \cong \bigoplus_{\alpha} \tilde{H}_*(S^n)$
 lives in degree n
 $= 0 \iff * \neq n$

LES for $(X^n, X^{n-1}) \implies H_*(X^{n-1}) \rightarrow H_*(X^n) \rightarrow H_*(X^n/X^{n-1})$ iso for $* < n-1$
 $* > n$

② for $* < n$: $H_*(X^n) \cong H_*(X^{n+1}) \cong H_*(X^{n+2}) \cong \dots \cong H_*(X)$

by compactness each sing-chain lands in X^N , some N , by the same proof as in the "closure-finite" property (start of Chapter)

③ for $* > n$: $H_*(X^n) \cong H_*(X^{n-1}) \cong H_*(X^{n-2}) \cong \dots \cong H_*(X^{n-1}) = 0$

④ LES: $\dots \rightarrow H_n(X^{n-1}) \rightarrow H_n(X^n) \xrightarrow{q_n} H_n(X^n, X^{n-1}) \rightarrow \dots$
 $\stackrel{0}{=} \text{by } ③$

$\implies q_n$ injective $\forall n$

⑤ LES: $\dots \xrightarrow{q_{n+1}^{n+1}} H_{n+1}(X^{n+1}, X^n) \xrightarrow{\delta_{n+1}^{n+1}} H_n(X^n) \xrightarrow{q_n} H_n(X^{n+1}) \rightarrow 0$
 hence surjective with $\text{Ker} = \text{im } \delta_{n+1}^{n+1}$

UPSHOT

④ $\text{Ker } \delta_n^n = \text{im } q_n^n$

$H_n^{CW}(X) = \frac{\text{Ker } d_n^{CW}}{\text{Im } d_{n+1}^{CW}} = \frac{\text{Ker } q_{n-1}^{n-1} \circ \delta_n^n}{\text{im } q_n^n \circ \delta_{n+1}^{n+1}}$
 exactness of LES $\dots \rightarrow H_n(X^{n-1}) \rightarrow H_n(X^n) \xrightarrow{q_n^n} H_n(X^n, X^{n-1}) \xrightarrow{\delta_n^n} \dots$

$= q_n^n (H_n(X^n)) / q_n^n (\text{im } \delta_{n+1}^{n+1})$

$\cong H_n(X^n) / \text{im } \delta_{n+1}^{n+1} \cong H_n(X^{n+1}) \cong H_n(X)$ ②

⑤ 1^{st} iso thm

Rmk by ① H_k not affected if attach $(k+2)$ -cells or higher

by ② Inclusion $X^n \rightarrow X$ induces iso $H_*(X^n) \rightarrow H_*(X)$ for $* < n$

Cor X n -dimensional cell cx $\implies H_*(X) = 0$ for $* > n$

Axioms for homology: Eilenberg-Steenrod axioms

It was no accident that $H_*^{\Delta}, H_*^{CW}, H_*^*$ all agreed.

Def A generalised homology theory (GHT)

is a functor F : Top Pairs = (Category of pairs of spaces and maps of pairs) \rightarrow Graded Abelian Gps

with a natural transformation $\delta : F_*(X, A) \rightarrow F_{*-1}(X, \emptyset)$ satisfying:

1) homology invariance: $f \cong g \implies F(f) = F(g)$

2) exactness: \exists LES $\dots \rightarrow F_*(A) \rightarrow F_*(X) \rightarrow F_*(X, A) \xrightarrow{\delta} F_{*-1}(A) \rightarrow \dots$

3) additivity: $(X, A) = \sqcup (X_i, A_i), \text{incl}_i : (X_i, A_i) \rightarrow (X, A)$
 then $\Sigma F(\text{incl}_i) : \bigoplus F(X_i, A_i) \cong F(X, A)$

4) excision: $\bar{E} \subseteq A^{\circ} \subseteq X \implies F(X \setminus E, A \setminus E) \xrightarrow{\cong} F(X, A)$
 $\xrightarrow{\uparrow F(\text{incl})} F(X, A)$

Remark (4) $\iff X = A^{\circ} \cup B^{\circ}, \text{incl} : (B, A \cap B) \rightarrow (X, A)$
 then $F(\text{incl}) : F(B, A \cap B) \cong F(X, A)$

Pf $B = X \setminus E, E = X \setminus B$ noticing that $(X \setminus E)^{\circ} \cup A^{\circ} = X$
 $E = A \setminus B$ noticing that $\bar{E} \subseteq \bar{A} \setminus B^{\circ} \subseteq A^{\circ} \setminus B^{\circ} \subseteq A^{\circ}$. \square
 $X = A^{\circ} \cup B^{\circ}$
 so $\partial B \subseteq A^{\circ}$

Rmk In (3), the topology on the disjoint union $\sqcup (X_i, A_i)$ is defined by: $U \subseteq \sqcup (X_i, A_i)$ open $\iff U \cap X_i \subseteq X_i$ open $\forall i$

FACT Theorem

a) $(F, \delta_F), (G, \delta_G)$ GHTs, $\alpha : F \rightarrow G$ a natural transformation commuting with δ_F, δ_G

such that $\alpha_{\text{point}} : F(\text{point}) \rightarrow G(\text{point})$ is an iso, then α is an iso.

b) If (F, δ_F) GHT satisfies (5) dimension: $F_*(\text{point}) = \begin{cases} \mathbb{Z} & * = 0 \\ 0 & \text{else} \end{cases}$

Then \exists natural iso $F \cong H_*$ (such an F is called a homology theory)

Rmk In (b) if require $F_0(\text{point}) = \mathbb{G}$ an abelian group (instead of \mathbb{Z}) $\implies F(X, A) \cong H_*(X, A; \mathbb{G})$ (homology with coefficients in \mathbb{G}) \leftarrow later in course

9. COHOMOLOGY

(C_*, ∂_*) chain complex s.t. C_n free \mathbb{Z} -module $\leftarrow C_* \cong \bigoplus_{\mathbb{Z}}$

Def **n-cochains**

$$C^n = \text{Hom}(C_n, \mathbb{Z})$$

boundary map

$$\partial^n : C^n \rightarrow C^{n+1}$$

(this is the dual of ∂)

$$\partial^n(\phi) = \phi \circ \partial_{n+1}$$

Notice ∂^* is degree +1 map (not -1)

$$H^m(C_*, \partial_*) = \text{Ker} \frac{\partial^m}{\text{Im} \partial^{m-1}} \leftarrow \begin{matrix} \text{cocycles} \\ \text{coboundaries} \end{matrix}$$

Remark If we use negative grading, (C^{-*}, ∂^{-*}) is a chain complex with homology so many results from H_* carry over to H^* . So abusively write

$$H^*(C_*, \partial_*) = H_*(\text{Hom}(C_*, \mathbb{Z}), \partial^*)$$

Warning A cochain $\varphi \in C^*$ takes values $\varphi(c) \in \mathbb{Z}$ on chains $c \in C_*$. However the cohomology class $\alpha = [\varphi] \in H^*$ does not have a well-defined value on c : $[\varphi] = [\varphi + \partial^*(\psi)]$ and $(\varphi + \partial^*(\psi))(c) = \varphi(c) + \psi(\partial_* c)$. If c is a cycle, so $\partial_* c = 0$ then $\alpha(c) = \varphi(c)$ is well-defined, so \exists pairing $H^* \times H_* \rightarrow \mathbb{Z}$

Remarks about dualisation

$\text{Hom}(\mathbb{Z}^n, \mathbb{Z}) \cong \mathbb{Z}^n$ generated by projection maps $\pi_i(x_1, \dots, x_n) = x_i$

$$\alpha : \mathbb{Z}^n \rightarrow \mathbb{Z}^m \Rightarrow \text{Hom}(\mathbb{Z}^n, \mathbb{Z}) \xleftarrow{\alpha^*} \text{Hom}(\mathbb{Z}^m, \mathbb{Z}) \quad \alpha^* \phi = \phi \circ \alpha$$

$$\begin{matrix} \mathbb{Z}^n & \xleftarrow{\text{transpose}(A)} & \mathbb{Z}^m \\ \uparrow & & \uparrow \\ \text{m} \times \text{n matrix} & & \mathbb{Z}^m \end{matrix}$$

Def X space \Rightarrow **singular cohomology** $H^*(X) = H^*(C^*(X), \partial^*)$

similarly define H_{Δ}^* , H_{CW}^*

Example $\mathbb{RP}^3 : C_{CW}^*(\mathbb{RP}^3) : 0 \rightarrow \mathbb{Z} \xrightarrow{\partial} \mathbb{Z} \xrightarrow{\partial} \mathbb{Z} \xrightarrow{\partial} \mathbb{Z} \rightarrow 0$

dualise : $C_{CW}^*(\mathbb{RP}^3) : 0 \leftarrow \mathbb{Z} \xleftarrow{\partial} \mathbb{Z} \xleftarrow{\partial} \mathbb{Z} \xleftarrow{\partial} \mathbb{Z} \rightarrow 0$

$$H^*(\mathbb{RP}^3) \cong H_{CW}^*(\mathbb{RP}^3) = \begin{cases} \mathbb{Z} & * = 0, 3 \\ \mathbb{Z}/2 & * = 2 \\ 0 & \text{else} \end{cases} \leftarrow \text{notice } H_1(\mathbb{RP}^3) \cong \mathbb{Z}/2 \text{ has moved to grading 2.}$$

Functoriality

$$f : X \rightarrow Y \Rightarrow f_* : C_* X \rightarrow C_* Y \quad \leftarrow \text{called pull-back}$$

$$\Rightarrow C^* X \xleftarrow{f^*} C^* Y \text{ is dual : } \boxed{f^* \phi = \phi \circ f_*}$$

Lemma f^* is a **cochain map** (meaning $\partial^* \circ f^* = f^* \circ \partial^*$)

$$\Rightarrow \boxed{f^* : H^* Y \rightarrow H^* X}$$

Pf $(\partial^* \circ f^*)(\phi) = \partial^*(\phi \circ f_*)$

$$= (\phi \circ f_*) \circ \partial$$

$$= (\phi \circ \partial) \circ f_* \quad \text{as } f_* \text{ chain map}$$

$$= f_* \circ (\phi \circ \partial)$$

$$= f_* \circ (\partial^* \phi)$$

$$= (f_* \circ \partial^*)(\phi)$$

Properties $\cdot \text{id}^* = \text{id}$

$\cdot (f \circ g)^* = g^* \circ f^*$ notice order!

$$\Rightarrow \boxed{H^* : \text{Top} \rightarrow \text{Graded AbGps}}$$

Contravariant functor

Exercise $H^0(X) = \prod_{\text{Path } X} \mathbb{Z}$ where $\text{Path } X = \{\text{path-components of } X\}$

Homotopy invariance

Lemma $f_*, g_* : C_* \xrightarrow{\text{free}} \tilde{C}_*$ chain hpic $\Rightarrow f^* = g^* : H^* \tilde{C} \rightarrow H^* C$

Pf $f_* - g_* = \tilde{\partial} \circ h + h \circ \partial$ same $h : C_* \rightarrow \tilde{C}_*[1]$

$f^* - g^* = h^* \circ \tilde{\partial}^* + \partial^* \circ h^*$ for dual $h^* : \tilde{C}^* \rightarrow C^*[-1]$.
(notice degree -1, not +1) \square

Def h^* called **cochain homotopy**

Cor $f \simeq g : X \rightarrow Y \Rightarrow f^* = g^* : H^* Y \rightarrow H^* X \quad \square$

Algebra: dual of SES

Lemma $0 \rightarrow A \xrightarrow{i} B \xrightarrow{j} C \rightarrow 0$ exact, A, B, C free
 $\Rightarrow 0 \leftarrow A^* \xleftarrow{i^*} B^* \xleftarrow{j^*} C^* \leftarrow 0$ exact
Pf C free $\Rightarrow \exists$ splitting $B \xrightarrow{j} C \xleftarrow{s} B$ $j \circ s = \text{id}$
 Pick preimages b_i for basis e_i of C , then $s(e_i) = b_i$
 $\Rightarrow A \oplus C \xrightarrow{i \oplus s} B$

dual $\Rightarrow A^* \oplus C^* \xleftarrow{i^* \oplus s^*} B^* \xleftarrow{j^*} C^* \leftarrow 0$
 $\Rightarrow 0 \leftarrow A^* \xleftarrow{i^*} B^* \xleftarrow{j^*} C^* \leftarrow 0$
 where $0 = (j \circ i)^* = i^* \circ j^*$ so $\text{Im } j^* \subseteq \text{Ker } i^*$
 prove \supseteq : $i^* b = 0 \Rightarrow b - j^* s^* b \in \text{Ker } i^* \cap \text{Ker } s^* = \{0\}$
 $\Rightarrow b = j^* s^* b \in \text{Im } j^*$
 $\Rightarrow \text{Ker } i^* = \text{Im } j^* \quad \square$

Relative cohomology
 $H^*(X, A) = H^*(\text{Hom}(C_*(X, A), \mathbb{Z}))$
 recall $C_*(X, A) = C_*(X)/C_*(A)$
 and $\text{homs } C_*(X)/C_*(A) \rightarrow \mathbb{Z}$
 correspond precisely to $\text{homs } C_*(X) \rightarrow \mathbb{Z}$
 which vanish on $C_*(A)$.
 So relative cocycles are cocycles on X
 which vanish on chains in A .

Excision, LES, Mayer-Vietoris
 By previous Lemma get dual results:
Excision $\bar{E} \subseteq A^0 \subseteq X \Rightarrow H^*(X \setminus E, A) \xleftarrow{\cong} H^*(X, A)$
LES for pair $(X, A) \quad \dots \leftarrow H^{[+]}(X, A) \xleftarrow{\delta} H^*(A) \xleftarrow{i^*} H^*(X) \xleftarrow{j^*} H^*(A) \leftarrow \dots$
M.V. $X = A \cup B \Rightarrow \dots \leftarrow H^{*+1}(X) \leftarrow H^*(A \cap B) \leftarrow H^*(A) \oplus H^*(B) \leftarrow H^*(X) \leftarrow \dots$
 where $A \cap B \xrightarrow{i_A^*} A \xrightarrow{j_A^*} X$
 $\xrightarrow{i_B^*} B \xrightarrow{j_B^*} X$
 $i_A^* \oplus i_B^* \quad j_A^* \oplus j_B^*$ are the obvious maps

Axioms for cohomology These are analogous to the axioms for homology
 except we reverse all arrows, and we change axiom (3): \prod instead of \oplus
additivity: $(X, A) = \sqcup (X_i, A_i)$, $\text{incl}_i: (X_i, A_i) \rightarrow (X, A)$
 then $\prod F(\text{incl}_i): \prod F(X_i, A_i) \xleftarrow{\cong} F(X, A)$

10. CUP PRODUCT

Theorem $H^*(X)$ is unital graded-commutative ring via
 $\cup: H^k(X) \times H^l(X) \rightarrow H^{k+l}(X)$ determined by

$$\cup: C^k(X) \times C^l(X) \rightarrow C^{k+l}(X)$$

$$(\phi \cup \psi)(\sigma) = \phi(\sigma|_{[e_0, \dots, e_k]}) \cdot \psi(\sigma|_{[e_{k+1}, \dots, e_{k+l}]})$$

① $1 \in C^0(X)$ "constant function 1" $\Rightarrow 1 \cup \phi = \phi \cup 1 = \phi$
 ② $\phi \cup \psi = (-1)^{\text{deg } \phi \cdot \text{deg } \psi} \psi \cup \phi$ ($1(\text{point}) = 1$ so $1(\sum n_i \cdot \beta_i) = \sum n_i$)

Useful trick If X is Δ -CX, $C_*^\Delta(X)$ inclusion $\rightarrow C_*(X)$, so $C_*^\Delta(X) \xleftarrow{\text{restriction}} C^*(X)$.
 and can define cup product on $C_*^\Delta(X)$ so that:
 $H_*^\Delta(X) \times H_*^\Delta(X) \rightarrow H_*^\Delta(X) \leftarrow$ at chain level
 $(\phi \cup \psi)([v_0, \dots, v_n]) = \phi([v_0, \dots, v_k]) \cdot \psi([v_{k+1}, \dots, v_n])$
 $\cong \uparrow$
 $H^*(X) \times H^*(X) \rightarrow H^*(X)$
 $\cong \uparrow$
 $H^*(X) \times H^*(X) \rightarrow H^*(X)$

So you can compute cup products on $H^*(X)$ by picking simplicial cocycle representatives:
 so define values on the simplicial chains defining the Δ -CX structure, and use

Proof of Theorem
 $(\phi \cup \psi)(\sigma) = (\phi \cup \psi)(\partial \sigma)$
 $= (\phi \cup \psi) \sum (-1)^i \sigma|_{[e_0, \dots, \hat{e}_i, \dots, e_n]}$
 $= \sum_{i \leq k} (-1)^i \phi(\sigma|_{[e_0, \dots, \hat{e}_i, \dots, e_{k+1}]}) \cdot \psi(\sigma|_{[e_{k+1}, \dots, e_n]})$
 $+ \sum_{i > k} (-1)^i \phi(\sigma|_{[e_0, \dots, e_k]}) \cdot \psi(\sigma|_{[e_k, \dots, \hat{e}_i, \dots, e_n]}) \cdot (-1)^{i-k} \underbrace{(-1)^{k-i}}_1$
 $= ((\partial^* \phi) \cup \psi)(\sigma) + (-1)^k \phi \cup \partial^* \psi$
 indices $[\phi] \cup [\psi] = [\phi \cup \psi]$: \cong
 well-defined: \bullet cycles \rightarrow cycle: $\partial(\phi \cup \psi) = (\partial \phi) \cup \psi \pm \phi \cup (\partial \psi) = 0$
 $\bullet [\phi] = [\phi + \partial \alpha] \cup \psi = \partial(\alpha \cup \psi) \pm (\partial \phi) \cup \psi = 0$ (using $\partial \psi = 0$)
 \bullet Similarly $[\phi] \cup [\partial \beta] = 0$

cannot do this for $C_*^{\text{CW}}(X)$ because there is no meaningful analogue for \mathbb{D}^n of the "bottom face" $[e_0, \dots, e_k]$ and "top face" $[e_k, \dots, e_n]$

indices $[\phi] \cup [\psi] = [\phi \cup \psi]$: \cong
 well-defined: \bullet cycles \rightarrow cycle: $\partial(\phi \cup \psi) = (\partial \phi) \cup \psi \pm \phi \cup (\partial \psi) = 0$
 $\bullet [\phi] = [\phi + \partial \alpha] \cup \psi = \partial(\alpha \cup \psi) \pm (\partial \phi) \cup \psi = 0$ (using $\partial \psi = 0$)
 \bullet Similarly $[\phi] \cup [\partial \beta] = 0$

bilinear, associative, distributive: true at chain level

unital: $(\partial 1)(\sigma) = 1(\sigma|_{[e_0]}) - 1(\sigma|_{[e_1]}) = 1 - 1 = 0$
 $(1 \cup \psi)(\sigma) = 1(\sigma|_{[e_0]}) \cdot \psi(\sigma|_{[e_0 \rightarrow e_n]}) = \psi(\sigma)$ ($\psi|_{1} = \phi$ similar)

graded-comm. sketch proof: \leftarrow **non-examinable**
 Let $r: C_n(X) \rightarrow C_n(X)$, $r(\sigma) = \epsilon_n \bar{\sigma}$ where: $\epsilon_n = (-1)^{\frac{n(n+1)}{2}}$

and $\bar{\sigma} [v_0, \dots, v_n] = \sigma [v_n, \dots, v_0]$ \leftarrow reverse order of vertices
 is product of $n+(n-1)+\dots+1$ transpositions $\frac{n(n+1)}{2}$
 (idea: each transposition is a reflection in hyperplane, so reverses orientation of simplex, so insert ϵ_n to compensate)

one checks: \bullet r chain map
 \bullet $r^* \psi \cup r^* \psi = r^*(\psi \cup \psi)$
 $\epsilon_{kl} \epsilon_l \leftarrow$ differ by $(-1)^{kl}$
 \bullet $r \simeq id$ so can drop $r^* = id$ on cohomology
 $(r-id = \partial \theta + \theta \rho$ with v_i, w_i like for prism operator)
 $(P\sigma = \sum (-1)^i \epsilon_{n-i} (\sigma \circ \pi_i) [v_0, \dots, v_i, w_n, \dots, w_i])$
 projection $\Delta^n \times I \xrightarrow{\pi} \Delta^n$

Naturality of cup product

Lemma $f: X \rightarrow Y \implies f^*: H^* Y \rightarrow H^* X$ hom of unital rings
Pf $f^*(\psi \cup \psi)(\sigma) = (f_* \psi)(f_* \psi)(\sigma)$
 $= \psi(f_* \sigma|_{[e_0, \dots, e_k]}) \cdot \psi(f_* \sigma|_{[e_{k+1}, \dots, e_n]})$
 $= ((\psi \circ f_*) \cup (\psi \circ f_*))(\sigma)$
 $= (f^* \psi \cup f^* \psi)(\sigma)$
 unital: $f^*(1) = 1 \circ f_* = 1 \quad \square$

UPSHOT
 $H^*: \text{Top} \rightarrow \left\{ \begin{array}{l} \text{Graded-commutative unital rings} \\ \text{with graded unital ring homs} \end{array} \right\}$
 contravariant functor.

Warning An (iso)morphism $H^*(Y) \rightarrow H^*(X)$ of groups will also preserve the ring structure if f^* is induced by a map of spaces $X \rightarrow Y$ (by above Lemma).
 \implies Cor The excision theorem iso on cohomology is an iso of rings. However the connecting hom in MV. or LES cannot possibly be a ring hom since it raises gradings by 1 ($\implies \delta(a \cup b)$ and $\delta(a) \cup \delta(b)$ have different gradings!)

Example $H^1(T^2) \times H^1(T^2) \xrightarrow{\cong} H^2(T^2)$ is bilinear form $\mathbb{Z}^2 \times \mathbb{Z}^2 \rightarrow \mathbb{Z}$ with matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
Pf By the Useful Trick, it is enough to work with H_Δ^* instead of H^* .



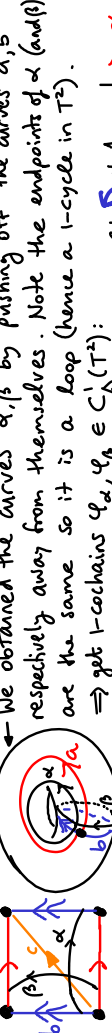
*	$H_\Delta^1(T^2)$	$H_\Delta^2(T^2)$
0	\mathbb{Z}	\mathbb{Z}
1	$\mathbb{Z} \oplus \mathbb{Z}$	\mathbb{Z}
2	\mathbb{Z}	\mathbb{Z}

$\mathbb{Z} \xrightarrow{\alpha^*} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\beta^*} \mathbb{Z} \xrightarrow{\gamma^*} \mathbb{Z}$
 \leftarrow abbreviate $\begin{cases} A = \alpha^* + c^* \\ B = \beta^* + c^* \end{cases}$
 \leftarrow (Remark $[D_1^*] = -[D_2^*]$ in $H_\Delta^2(T^2)$)

Claim $A \cup B = D_1^*$

Pf $\Delta^2 \xrightarrow{D_1} (A \cup B)(D_1) = A(D_1|_{[e_0, e_1]}) \cdot B(D_1|_{[e_1, e_2]}) = A(\alpha)B(b) = 1 \cdot 1 = 1$
 $\Delta^2 \xrightarrow{D_2} (A \cup B)(D_2) = A(b)B(\alpha) = 0 \quad \square$
 (can also check these by hand)

Graded-comm. $\implies B \cup A = -D_1^*, A \cup A = (-1)^{|A|} A \cup A$ so $= 0$, similarly $B \cup B = 0 \quad \square$
Remark Recall that to specify a cochain in $C_k^*(X)$ one needs to specify values on all generators of $C_k^*(X)$ so not just on generators of $H_k^*(X)$ (e.g. above A and α agree on gens a, b of $H_1^*(T^2)$ but disagree on $c \in C_1^*(T^2)$, note α is 1-cochain $\in C_1^*(T^2)$ but is not a 1-cycle. Some (but not all) k -cochains ψ can be specified by drawing a "nice" $(n-k)$ -dimensional subspace $\Sigma \subseteq X$ and defining $\psi(c) = \#(\text{times } \Sigma \text{ intersects } c)$ for all $c \in C_k^*(X)$ where one must explain with what sign \pm an intersection point is counted and one has ensured that Σ intersects the generators of $C_k^*(X)$ in a finite # points.

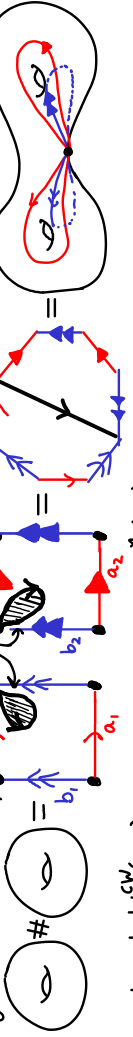


Fact Since T^2 is an orientable manifold, $\psi_\alpha \cup \psi_\beta = (\alpha \cdot \beta)$ vol where vol is a generator of $H^2(T^2)$. Later in the course: vol is the "Poincaré dual" of the point class, and corresponds to the dual of the oriented sum of the top simplices. Above: $\text{vol} = D_1^*$ and $\psi_\alpha \cup \psi_\beta = B \cup (-A) = A \cup B = (\alpha \cdot \beta) \text{ vol} = \text{vol} = D_1^*$.

Non-examinable comments about intersection numbers
 Defining intersection numbers rigorously is tricky, even when using smooth chains. One can calculate $\psi_\Sigma(c)$ on a cycle c by first deforming c to a smooth homologous cycle \tilde{c} which is "transverse" to Σ , and then we count intersection points $\Sigma \cap \tilde{c}$ (with "orientation signs").
 The fact that we consider the intersection number $a \cdot a = 0$ is because we can push a off itself.



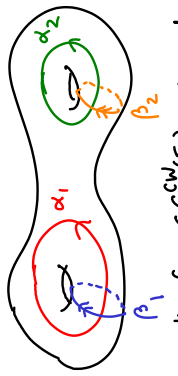
Exercise Σ_2 (genus 2 surface)



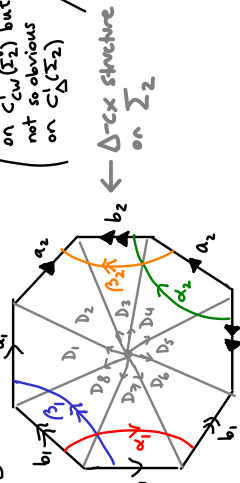
remove balls & glue handles

$H_0^*(\Sigma_2)$	\mathbb{Z}
$H_1^*(\Sigma_2)$	$\mathbb{Z} \oplus \mathbb{Z}$
$H_2^*(\Sigma_2)$	\mathbb{Z}

Deform curves a_1, b_1, a_2, b_2 to get $\alpha_1, \beta_1, \alpha_2, \beta_2$



$\mathbb{Z} \langle \alpha_1^*, \beta_1^*, \alpha_2^*, \beta_2^* \rangle$ ← dual basis in C_{CW}^* (easy to define on $C_{CW}^*(\Sigma_2)$ but not so obvious on $C_{\Delta}^*(\Sigma_2)$)



Then notice for $c \in C_{CW}^*(\Sigma_2)$ signed count

$\alpha_i^*(c) = -\#(\beta_i \text{ intersects } c)$ so can extend this to a definition of $\alpha_i^*, \beta_i^* \in C_{\Delta}^*(\Sigma_2)$ by allowing $c \in C_{\Delta}^*(\Sigma_2)$.
 $\beta_i^*(c) = \#(\alpha_i \text{ intersects } c)$ Check that α_i^*, β_i^* are 1-cocycles in $C^1(\Sigma_2)$.

Exercise: $\alpha_i^* \cup \beta_j^* = \delta_{ij}^* = \delta_{ij}^* = -\beta_j^* \cup \alpha_i^*$ (key idea: a loop has total signed intersection $\neq 0$ with the boundary of a triangle)

Hint: represent D as a sum of triangles in (least picture)

orientation signs: $\alpha_i^* \cup \alpha_j^* = \beta_i^* \cup \beta_j^* = 0$ so same as geometric intersection numbers of corresponding curves

$D = -D_1 - D_2 + D_3 + D_4 + D_5 - D_6$ using + if outer edge is oriented anticlockwise

Cultural Rmk on general theory (Intersection Theory/Differential Topology)

M^m oriented m -manifold $\Rightarrow H_n(N) \xrightarrow{\text{incl}} H_n(M) \xleftarrow{\text{see later in course}}$
 $N^n \subseteq M^m$ oriented n -dim submfld \exists generator $[N] \xrightarrow{\text{with signs}}$
 N, M also smooth (see Differential Geometry course) $\Rightarrow \omega_N \in H^{m-n}(M)$ counts # intersections with N

$n_1, n_2 \in \mathbb{Z}$ (can always "homotope" N_1, N_2 to achieve transversality, and class ω_N does not change if homotope)
 $N_1, N_2 \subseteq M^m$ compact oriented smooth submflds $\Rightarrow \omega_{N_1} \cup \omega_{N_2} = \omega_{N_1 \cup N_2}$ (homotope the inclusion maps)

$N_1, N_2 \subseteq M^m$ compact oriented smooth submflds \Rightarrow the tangent spaces to N_1, N_2 at p span the tangent space to M at p (in particular if $n_1 + n_2 = m$, and M connected, then $H^m(M) \cong \mathbb{Z}$ s.t. $\omega_{N_1} \cup \omega_{N_2} \mapsto \#(N_1 \cap N_2) \in \mathbb{Z}$)

N_1, N_2 means the best vector space approximation at p (determined by the local smooth coordinates) (if $n_1, n_2 = m$, get $N_1 \cap N_2$ is dim $\Sigma = 1$ points. Sign compares oriented bases of tang. spaces: $T_M = T_{N_1} \oplus T_{N_2}$)

Fact (Thom 1954) Not all $a \in H^j(M)$ arise as ω_N for connected compact oriented codim $= j$ smooth submfld N . But $\exists N \in N$ s.t. N does arise. They do arise for $H^*(M; \mathbb{Q}), H^*(M; \mathbb{R}), H^*(M; \mathbb{Z}/2)$

11. KÜNNETH FORMULA AND PRODUCT SPACES

Algebra: tensor products

R ring (comm. with 1) e.g. abelian groups = \mathbb{Z} -mods
 Def A, B R -modules \Rightarrow **Tensor product** is R -module
 vector spaces/ \mathbb{F} = \mathbb{F} -mods

$A \otimes_R B = \langle (a, b) : a \in A, b \in B \rangle / \langle \text{relations of bilinearity \& rescaling} \rangle$

(or $A \otimes B$) R -mod generated write $a \otimes b$ for its class

bilinearity: $(a_1 + a_2) \otimes b = a_1 \otimes b + a_2 \otimes b$

$a \otimes (b_1 + b_2) = a \otimes b_1 + a \otimes b_2$

rescaling: $r \cdot (a \otimes b) = (ra) \otimes b = a \otimes (rb) \quad \forall r \in R$

So general element looks like $\sum a_k \otimes b_k$ (finite sum) ← **NOT UNIQUE!**

Don't confuse with $A \times B$: e.g. $0 \otimes b = 0 \quad \forall b$

Rmk Can define $A \otimes_R B$ also by a universal property: for all R -mods C ,

$$\text{Hom}_R(A \otimes_R B; C) \xrightarrow{\text{natural}} \{R\text{-bilinear maps } A \times B \rightarrow C\}$$

Using above description of $A \otimes B: \varphi \mapsto (A \times B \rightarrow C, (a, b) \mapsto \varphi(a \otimes b))$

Example $(R = \mathbb{F}) \quad V, W$ v.s. \mathbb{F} $\Rightarrow V \otimes W$ v.s. \mathbb{F} basis $v_i \otimes w_j$
 basis v_i \leftarrow basis w_j \leftarrow basis $v_i \otimes w_j$
 $\dim_{\mathbb{F}} V \otimes W = \dim V \cdot \dim W$

Exercise V, W finite dim \mathbb{F} $\Rightarrow V^* \otimes W \cong \text{Hom}_{\mathbb{F}}(V, W)$

Hint $f: V \rightarrow \mathbb{F}, u \in W, f \otimes w \mapsto (V \rightarrow W, v \mapsto f(v) \cdot w)$

Examples $(R = \mathbb{Z}) \cdot \mathbb{Z}^n \otimes \mathbb{Z}^m \cong \mathbb{Z}^{nm}$ (e.g. $(\mathbb{R}^n)^* \otimes \mathbb{R}^m \cong \text{Mat}_{n \times m}(\mathbb{R}) \cong \text{Hom}(\mathbb{R}^n, \mathbb{R}^m)$)
 $\cdot \mathbb{Z}/n \otimes \mathbb{Z} \cong \mathbb{Z}/n$ $\leftarrow \sum a_i \otimes b_i = (\sum a_i b_i) \otimes 1$
 $\cdot \mathbb{Z}/2 \otimes \mathbb{Z}/3 = 0$ $\leftarrow 1 \otimes x = 3 \otimes x = 1 \otimes 3x = 0$
 $\cdot \mathbb{Z}/2 \otimes \mathbb{Z}/4 \cong \mathbb{Z}/2$ $\leftarrow \{1 \otimes 1 = 3 \otimes 1 = 1 \otimes 3 = 0\}$
 $\cdot A \otimes B \cong B \otimes A$

Examples $A \otimes B \cong B \otimes A$
 $\cdot (\bigoplus_i A_i) \otimes (\bigoplus_j B_j) \cong \bigoplus_{i,j} (A_i \otimes B_j)$ hence now know $A \otimes B$ for any $f.g.$ R -mods A, B .
 $\cdot A \otimes R \cong A$ (so " \otimes_R does nothing")
 $\cdot A \otimes R/d \cong A/d \cdot A$

for example $\mathbb{Z}/2 \otimes \mathbb{Z}/4 \cong (\mathbb{Z}/4)/2 \cdot \mathbb{Z}/4 \cong \mathbb{Z}/2$ $\leftarrow \left(\frac{\text{Rmk } (\mathbb{Z}/m)/m \cdot \mathbb{Z}/n}{\cong \mathbb{Z}/\text{gcd}(m,n)} \right)$

More generally: $\left\{ \begin{array}{l} R/I \otimes R/J \cong R/(I+J) \text{ for ideals } I, J \subseteq R. \\ A \otimes_R R/J \cong A/J \cdot A \end{array} \right.$

Warning $\otimes A$ often not an exact functor, i.e. does not preserve exact sequences
indeed it can ruin injectivity: $0 \rightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \rightarrow \mathbb{Z}/2 \rightarrow 0$

Fact $\cdot \otimes \mathbb{Z}$ and $\otimes \mathbb{R}$ are exact functors on \mathbb{Z} -mods

More generally $\otimes \text{Frac}(R)$ is exact on R -mods where $\text{Frac}(R)$ is a fraction field, and R is an integral domain

example A f.g. \mathbb{Z} -mod $\Rightarrow A \cong \mathbb{Z} \oplus \mathbb{Z}/d_1 \oplus \dots \oplus \mathbb{Z}/d_k$ some $d_i \neq 0$

$\Rightarrow A \otimes \mathbb{Q} \cong \mathbb{Q}^r \Rightarrow r = \text{rank } A = \dim_{\mathbb{Q}}(A \otimes \mathbb{Q})$

Corollary Rank-nullity thm holds for \mathbb{Z} -modules if use rank instead of dim.

Pf $0 \rightarrow A \rightarrow B \xrightarrow{f} C \rightarrow 0$ exact $\Rightarrow 0 \rightarrow A \otimes \mathbb{Q} \rightarrow B \otimes \mathbb{Q} \rightarrow C \otimes \mathbb{Q} \rightarrow 0$ exact
ker $e \xrightarrow{\text{im } f} \Rightarrow \dim(\text{ker } e) + \dim(\text{im } f) = \dim(B \otimes \mathbb{Q})$
rank-nullity for \mathbb{Q} -vector spaces. \square

Tensor product of chain cxes

C_*, \tilde{C}_* chain cxes $\Rightarrow (C_* \otimes \tilde{C}_*)_n = \bigoplus_{i+j=n} C_i \otimes \tilde{C}_j$ of R -mods

$\partial(x \otimes y) = (\partial x) \otimes y + (-1)^{\deg x} x \otimes \partial y$

Think of ∂ as an operator of $\deg = -1$ acting from left since ∂ "jumps over x " get $(-1)^{\deg x} \cdot \deg x$

Exercise $\partial \circ \partial = 0$ \leftarrow would fail without sign recall $\mathbb{Z}_k = \text{ker } \partial = \text{im } \partial = \text{boundaries}$

$\mathbb{Z}_i \otimes \tilde{\mathbb{Z}}_j \subseteq \mathbb{Z}_{i+j}(C_* \otimes \tilde{C}_*)$ and $\mathbb{Z}_i \otimes \tilde{\mathbb{Z}}_j \subseteq \mathbb{B}_{i+j}(C_* \otimes \tilde{C}_*)$

Cor \exists natural maps

$$\begin{aligned} H_i(C_*) \otimes H_j(\tilde{C}_*) &\rightarrow H_{i+j}(C_* \otimes \tilde{C}_*) \\ \sum [c_k] \otimes [\tilde{c}_k] &\mapsto \sum [c \otimes \tilde{c}_k] \end{aligned}$$

FACT:

Algebraic Künneth Thm

$C_*, H_*(C_*)$ f.g. free R -mods $\xleftarrow{\text{PID}}$ (principal ideal domain) $\Rightarrow \bigoplus_{i+j=n} H_i(C_*) \otimes H_j(\tilde{C}_*) \cong H_n(C_* \otimes \tilde{C}_*)$ via \leftarrow

Algebra: Euler characteristic

C finitely generated graded abelian gr (so \mathbb{Z} -mod) | more generally: R -mod for PID R

Def Euler characteristic $\chi(C) = \sum (-1)^i \text{rank } C_i$

Example/Motivation X finite CW-cx then take $C = C_*^{CW}(X)$ to get

$\chi(X) = \#(0\text{-cells}) - \#(1\text{-cells}) + \#(2\text{-cells}) - \dots$

Lemma If C_* f.g. chain cx $\Rightarrow \chi(C_*) = \chi(H_*(C_*)) = \sum (-1)^i \text{rank } H_i(C_*)$

Pf Abbreviate $|C_i| = \text{rank } C_i = (\dim_{\mathbb{Q}}(C_i \otimes \mathbb{Q}))$
By previous Corollary about rank-nullity:

$0 \rightarrow \mathbb{Z}_i \rightarrow C_i \xrightarrow{\partial} B_{i-1} \rightarrow 0 \Rightarrow |C_i| = |\mathbb{Z}_i| + |B_{i-1}|$
 $0 \rightarrow B_i \rightarrow \mathbb{Z}_i \rightarrow H_i \rightarrow 0 \Rightarrow |H_i| = |\mathbb{Z}_i| - |B_i|$
 $\Rightarrow \chi(C_*) - \chi(H_*) = \sum (-1)^i |B_{i-1}| + \sum (-1)^i |B_i| = \sum (-1)^i (|B_{i-1}| + |B_i|) = 0. \square$

Cor X space $\Rightarrow \chi(X) := \sum (-1)^i \text{rank } H_i(X) = \sum (-1)^i \text{rank } C_i(X)$
 \leftarrow if finite rank $H_*(X)$
 \leftarrow if finite rank $C_*(X)$

So $\chi(X)$ is invariant up to hty equivalence! Example $\chi(\text{platonic solid}) = \chi(S^2) = 2$

Product spaces

X, Y CW-cxes $\Rightarrow X \times Y$ CW-cx with cells $e_\alpha \times e_\beta$ attaching maps $\partial(D_\alpha^i \times D_\beta^j) = (\partial D_\alpha^i) \times D_\beta^j \cup D_\alpha^i \times (\partial D_\beta^j)$
 $\downarrow \text{id} \times \text{id}$
 $X^{i-1} \times Y^j \cup X^i \times Y^{j-1}$

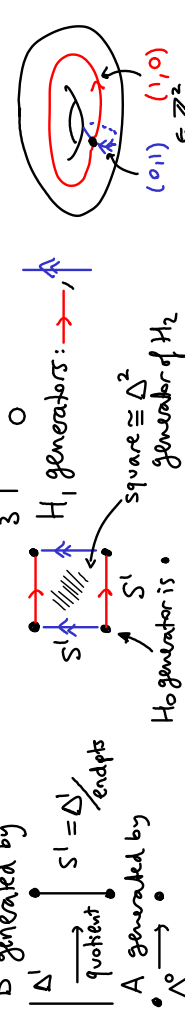
Cor $\chi(X \times Y) = \chi(X) \cdot \chi(Y)$
 \forall finite CW-cxes X, Y

Pf $\sum (-1)^k \text{rank } H_k^{CW}(X \times Y) = \sum_k \sum_{i+j=k} (-1)^{i+j} \text{rank } C_i^{CW}(X) \cdot \text{rank } C_j^{CW}(Y)$
 $= \sum (-1)^k \text{rank } C_k^{CW}(X \times Y) = (d e_\alpha^i \times e_\beta^j + (-1)^i e_\alpha^i \times (d e_\beta^j))$

Lemma $d(e_\alpha^i \times e_\beta^j) = (d e_\alpha^i) \times e_\beta^j + (-1)^i e_\alpha^i \times (d e_\beta^j)$
hence $C_*^{CW}(X \times Y) \cong C_*^{CW}(X) \otimes C_*^{CW}(Y)$
(proof later)

Hence if $H_*(Y)$ free then by Künneth $H_*(X \times Y) \cong H_*(X) \otimes H_*(Y)$.
Example $H_*(S^1) \otimes H_*(S^1) \cong H_*(S^1 \times S^1) \leftarrow$ torus

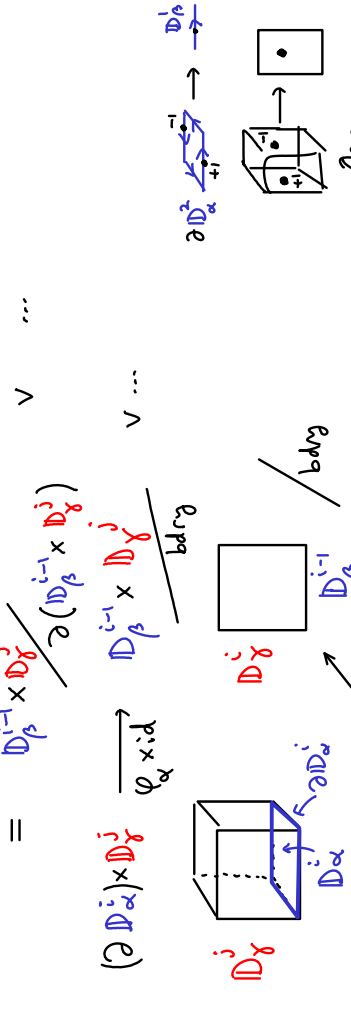
$A \otimes A$	0	1	2	3
$A \otimes B$	0	1	2	3
$B \otimes A$	0	1	2	3
$B \otimes B$	0	1	2	3



for R -mods, do $\dim_{\mathbb{F}}(C_i \otimes \mathbb{F})$ with $\mathbb{F} = \text{Frac}(R)$ (R integral domain) [Corollary still holds, same proof]

Pf $(\partial D_\alpha^i) \times D_\beta^j \xrightarrow{\varphi_\alpha \times \text{id}} X^{i-1} \times Y^j \rightarrow X^{i-1} \times Y^j$
 This proof is Non-examinable

$X^{i-1} = X^{i-2} \cup (D_\beta^{i-1} \cup \dots)$
 $Y^j = Y^{j-1} \cup (D_\alpha^{j-1} \cup \dots)$
 $X^{i-1} \times Y^j = X^{i-2} \times Y^j \cup X^{i-1} \times Y^{j-1} \cup (D_\beta^{i-1} \times D_\alpha^j \cup \dots)$
 get \sim from attaching maps

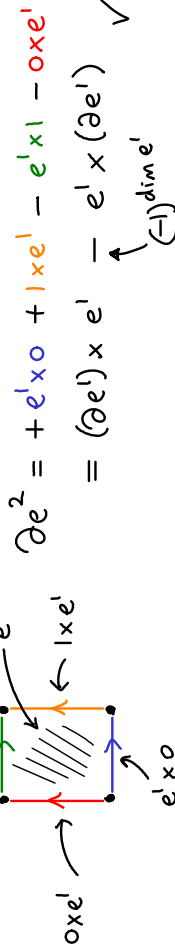


By considering local degrees now we see we get degree = $d_\alpha d_\beta$ for this.
 \Rightarrow get contribution $(d_\alpha^i) \times e_\beta^j$ ✓

similarly $D_\alpha^i \times \partial D_\beta^j \xrightarrow{\text{id} \times \varphi_\beta} D_\alpha^i \times D_\beta^{j-1} / \text{bdry}$
 \Rightarrow degree $(-1)^i d_\alpha d_\beta$
 so get $(-1)^i e_\alpha^i \times d_\beta^j$

$(-1)^i$ caused by orientations.
 could reorder factors: $D_\alpha^i \times D_\beta^j \cong D_\beta^j \times D_\alpha^i$ by $(\circ \text{Id}_i \circ)$
 whose det = $(-1)^{ij}$. Then $\partial D_\beta^j \times D_\alpha^i \rightarrow D_\beta^{j-1} \times D_\alpha^i / \text{bdry}$ gives degree $d_\alpha d_\beta$.
 Swap factors $D_\beta^{j-1} \times D_\alpha^i / \text{bdry}$ by $(\circ \text{Id}_{j-1} \circ)$, det = $(-1)^{i(j-1)}$. Total sign = $(-1)^i$.

Example Recall after definition of H_*^{CW} we had example IX I:
 arrows here tell us how we map $[-1, 1] \rightarrow \text{edge}$ (so orientation)

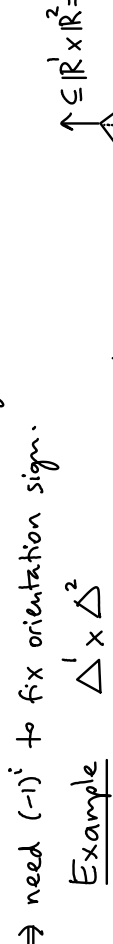


$\partial e^2 = +e^1 \times 0 + 1 \times e^1 - e^1 \times 1 - 0 \times e^1$
 $= (\partial e^1) \times e^1 - e^1 \times (\partial e^1)$ ✓
 $(-1)^{\dim e^1}$

A further comment on orientation sign $(-1)^i$
 $D^i \times D^j \cong \Delta^i \times \Delta^j$
 $[v_0, \dots, v_i] \times [w_0, \dots, w_j]$
 viewed in $\mathbb{R}^i, \mathbb{R}^j$
 project $\mathbb{R}^{i+j} \rightarrow \mathbb{R}^i$
 $(t_0, \dots, t_i) \mapsto (t_0, \dots, t_i)$

$\partial(D^i \times D^j) \cong \partial \Delta^i \times \Delta^j \cup \Delta^i \times \partial \Delta^j$
 $\cong \sum_k (-1)^k [v_0, \dots, \hat{v}_k, \dots, v_i] \times [w_0, \dots, w_j]$
 $\cong \sum_k (-1)^k [w_0, \dots, \hat{w}_k, \dots, w_j]$

would be correct orientation sign for basis $w_1, \dots, w_k, \dots, w_j, \dots, w_0$ but actually we have $[w_0, \dots, w_k, \dots, w_j] \times [v_0, \dots, v_i] \subseteq \mathbb{R}^i \times \mathbb{R}^j$
 and $(-1)^{ik}$ is the orientation sign for the basis $v_1, \dots, v_i, \dots, v_0, \dots, w_0, \dots, w_j$
 for the hyperplane in \mathbb{R}^{i+j+1} containing \Rightarrow need $(-1)^i$ to fix orientation sign.



Example $\Delta^1 \times \Delta^2 \subseteq \mathbb{R}^3$
 $e_3 \times [v_0, v_1] \times [w_0, w_1, w_2] \subseteq \mathbb{R}^2$
 $[v_0, v_1] \times [w_0, w_1, w_2]$
 out $w_2 - w_1$ is negative \mathbb{R}^3 -basis
 out $w_2 - w_1$ is positive \mathbb{R}^2 -basis
 differ due to $(-1)^i, i=1$.

Projections $X \times Y \begin{matrix} \xrightarrow{p_X} X \\ \xrightarrow{p_Y} Y \end{matrix}$

FACT: \leftarrow no conditions on X \leftarrow automatic if use field coefficients $\forall n$

Künneth Theorem If $H_n(Y)$ finitely generated, free $\forall n$

$$H_n(X \times Y) \cong \bigoplus_{i+j=n} H_i(X) \otimes H_j(Y)$$

$$P_X^* a \cup P_Y^* b \longleftarrow a \otimes b$$

and extend linearly

Recall for cellular homology this on generators is: (chain level) $e_i \times e_j \longleftarrow e_i \otimes e_j$

Rmk An indirect proof the Thm is to write down two generalised cohomology theories $F(X,A) = H^*(X,A) \otimes H^*(Y)$ and $G(X,A) = H^*(X \times Y, A \times Y)$, and consider the natural transformation $\alpha: F \rightarrow G$ given by \otimes , notice for $X = pt$ both F, G give $H^*(Y)$.

Example $X = S^n, Y = S^m, n \neq m$

$$H_*(S^n \times S^m) \cong \begin{cases} \mathbb{Z} & * = 0, n, m, n+m \\ 0 & \text{else} \end{cases}$$

$$H_*(S^n \times S^n) \cong \begin{cases} \mathbb{Z} & * = 0, 2n \\ \mathbb{Z}^2 & * = n \leftarrow \text{gens: } a_n^{(1)}, a_n^{(2)} \\ 0 & \text{else} \end{cases}$$

where $a_i = \text{dual}(e^i)$

Cor $H^k(T^n) \cong \wedge^k [x_1, \dots, x_n]$ = free abelian gp. on gens.

where $x_i = p_i^*(\text{gen. of } H^1(S^1))$ \leftarrow deg $x_i = 1$

Pf Künneth & induction ($T^n = T^{n-1} \times S^1$) \square

FACT cup product equals composition

$$\cup : H^i(X) \otimes H^j(X) \rightarrow H^{i+j}(X \times X)$$

$$\left(\Delta_{\sigma_1}^i \rightarrow X \right) \otimes \left(\Delta_{\sigma_2}^j \rightarrow X \right) \xrightarrow{\cup} \left(\Delta_{\sigma_1 \times \sigma_2}^{i+j} \rightarrow X \times X \right)$$

exterior product Δ^{i+j}

12. UNIVERSAL COEFFICIENTS THEOREM

Proof is non-examinable. for (C_*, ∂_*) chain cx:

$$\Rightarrow 0 \rightarrow Z_* = \ker \partial_* \xrightarrow{\text{incl}} C_* \xrightarrow{\partial_*} B_{*-1} \rightarrow 0 \text{ is SES}$$

$\cup_{\partial=0}$

FACT: Submodules of a free \mathbb{Z} -module are free

Rmk The same holds for R -mods if R is PID

Assume C_* free \mathbb{Z} -mod

FACT $\Rightarrow Z_*, B_*$ free (as $\ker \partial, \text{Im } \partial$ are submods of C_*)

\Rightarrow SES splits, choose splitting $C_* \xrightarrow{\partial_*} B_{*-1} \xrightarrow{\text{incl}} C_*$ so $\partial_* \circ s = \text{id}$

dual SES $0 \leftarrow Z^* \xleftarrow{\text{incl}^*} C^* \xleftarrow{\partial^*} B^{*-1} \leftarrow 0$

$0 \leftarrow Z^n \xleftarrow{\partial} C^n \xleftarrow{\partial} B^{n-1} \leftarrow 0$

$0 \leftarrow Z^{n-1} \xleftarrow{\partial} C^{n-1} \xleftarrow{\partial} B^{n-2} \leftarrow 0$

Connecting map $0 \leftarrow \partial^* \psi \xleftarrow{\partial^*} \varphi \xleftarrow{\partial^*} \phi \xleftarrow{\partial^*} \psi \xleftarrow{\partial^*} 0$

LES: $\varphi|_{Z^*} = \phi \xleftarrow{\partial^*} \psi \xleftarrow{\partial^*} \varphi \xleftarrow{\partial^*} \psi \xleftarrow{\partial^*} \varphi$

LES $\Rightarrow 0 \leftarrow \ker \delta^n \xleftarrow{\delta^n} Z^n \xleftarrow{\partial^*} H^n C \xleftarrow{\partial^*} H^{n-1} C \xleftarrow{\partial^*} \ker \delta^{n-1} \leftarrow 0$

$(H^* B = B^*, H^* Z = Z^* \text{ since } \partial^* = 0)$

$\ker \delta^n = \{ \phi \in Z^n : \phi(B_n) = 0 \} \Rightarrow$ so: $\phi : Z_n \rightarrow Z$

$= \text{Hom}(H_n(C_*), \mathbb{Z})$

Universal Coefficients Thm:

$$0 \rightarrow B^{n-1}/\text{Im } \delta^{n-1} \rightarrow H^n(C) \rightarrow \text{Hom}(H_n(C_*), \mathbb{Z}) \rightarrow 0 \text{ is SES}$$

$$\text{Ext}^1(H_{n-1}(C_*), \mathbb{Z}) \rightarrow [\varphi] \rightarrow \text{Hom}(H_n(C_*), \mathbb{Z}) \rightarrow 0 \text{ and natural}$$

and SES splits (but not naturally): $B^{n-1}/\text{Im } \delta^{n-1} \xrightarrow{\partial^*} H^n(C)$

$\Rightarrow H^n(C) \cong \text{Hom}(H_n(C), \mathbb{Z}) \oplus \text{Ext}^1(H_{n-1}(C), \mathbb{Z})$

MOTIVATION: What is difference between $H^*(\text{Hom}(C_*, \mathbb{Z}))$ and $\text{Hom}(H_n(C_*), \mathbb{Z})$ Similarly: $H_n(C_* \otimes \mathbb{Q})$ vs. $H_n(C_*) \otimes \mathbb{Q}$.

\mathbb{Z} -module \equiv abelian gp free means: $\bigoplus_{\text{indexing set}} \mathbb{Z}$

PID = principal ideal domain = integral domain R s.t. every ideal = $R \cdot a$ some a

\leftarrow recall just pick preimages under ∂_* of a basis for B_*

note: $\text{incl}^* = \text{restrict to } Z_*$ since $\text{incl}^* \phi : Z_* \rightarrow B_* \rightarrow Z_*$

Rmk Although $\partial^* = 0 : B^* \rightarrow B^{*+1}$ the map $\partial^* : B^{n-1} \rightarrow C^n$ need not = 0: $\psi : B_{n-1} \rightarrow Z \Rightarrow \partial^* \psi = \psi \circ \partial : C_n \rightarrow B_{n-1} \rightarrow Z$

$$\delta(\phi) = \phi|_{B_*}$$

see next Lemma

$$s^* \circ \partial^* = \text{id}$$

(since $\partial \circ s = \text{id} \Rightarrow \text{id} = (\partial \circ s)^* = s^* \circ \partial^*$)

Lemma $\text{Ext}^1(H_{n-1}(C); \mathbb{Z}) \cong B^{n-1}/\text{Im } S^{n-1}$ canonically

Algebra background on Extension groups $\text{Ext}_R^i(M, R)$ ($= \text{Ext}_R^i(M, R)$)

our case

M R -module, R ring (comm. with 1)

$\Rightarrow \exists$ free resolution:

$$\dots \rightarrow P_2 \xrightarrow{\psi_2} P_1 \xrightarrow{\psi_1} P_0 \xrightarrow{\psi_0} M \rightarrow 0 \quad \text{exact, } P_i \text{ free } R\text{-mods}$$

(pick gens x_α for $M \Rightarrow P_0 = \bigoplus_\alpha R \xrightarrow{\psi_0} M, e_\alpha \mapsto x_\alpha$
 " " y_β for $\text{Ker } \psi_0 \Rightarrow P_1 = \bigoplus_\beta R \xrightarrow{\psi_1} \text{Ker } \psi_0, e_\beta \mapsto y_\beta$
 continue inductively)

Take $\text{Hom}(\cdot; R)$ and drop $\text{Hom}(M; R)$

$$0 \rightarrow \text{Hom}(P_0; R) \xrightarrow{\psi_1^*} \text{Hom}(P_1; R) \xrightarrow{\psi_2^*} \dots$$

Is cochain complex but not exact

\Rightarrow take cohomology groups:

Def $\text{Ext}^0(M; R) = \text{Ker } \psi_1^*$

$\text{Ext}^1(M; R) = \text{Ker } \psi_2^* / \text{Im } \psi_1^*$

Fact independent of choices P_i, ψ_i

Example 1 $\text{Ext}^0(M; R) \cong \text{Hom}(M, R)$

$$P_1 \xrightarrow{\psi_1} P_0 \xrightarrow{\psi_0} M$$

$\downarrow \phi$ descends: $m \mapsto \phi(\psi_0^{-1}m)$
 will be defined since $\phi(\text{Ker } \psi_0) = 0$

Example 2 $\text{Ext}^1(M; R) = \left\{ \phi : P_2 \xrightarrow{\psi_2} P_1 \rightarrow P_0 \right\} / \left\{ \phi = \psi_0 \psi_1 : P_1 \xrightarrow{\psi_1} P_0 \right\}$

Rmk If R PID, then $\text{Ker}(P_0 \rightarrow M)$ is free (since submod of free mod P_0)
 \Rightarrow can pick $P_1 = \text{Ker}(P_0 \rightarrow M)$, $P_k = 0$ for $k \geq 2 \Rightarrow \text{Ext}_R^k(M; R) = 0 \quad k \geq 2$

(Co)homology with coefficients in a ring/field/module

Motivation

So far we had (C_*, ∂_*) chain cx of abelian groups } in graded sense
 $\Rightarrow H_*(C_*) = \text{Ker } \partial_* / \text{Im } \partial_*$ abelian group (since $\text{Ker } \partial, \text{Im } \partial$ are)
 We cannot use a chain cx of (non-abelian) groups, because
 $\text{Im } \partial_*$ need not be a normal subgroup of $\text{Ker } \partial_*$.

However, abelian groups can be thought of as \mathbb{Z} -modules,
 then given any abelian group G , define homology with coeffs in G
 $H_*(C_*; G) = H_*(C_* \otimes_{\mathbb{Z}} G)$ with differential $\partial_* \otimes \text{id}$

Def X space $\Rightarrow H_*(X; G) = H_*(C_*(X) \otimes_{\mathbb{Z}} G)$

Explanation:

$C_k(X)$ free \mathbb{Z} -mod $\cong \bigoplus_{I_k} \mathbb{Z} \Rightarrow C_k(X) \otimes_{\mathbb{Z}} G \cong \bigoplus_{I_k} G$: just replace \mathbb{Z} by G (as $\mathbb{Z} \otimes \cong \cdot$)

Why care? We hope to get more/new invariants of spaces

Example $X = \mathbb{R}P^2 =$ 

$C_*(\mathbb{R}P^2; G)$

*	$C_*(\mathbb{R}P^2; G)$
0	$G \oplus G \oplus G \oplus G$
1	$G \oplus G \oplus G \oplus G$
2	$G \oplus G \oplus G \oplus G$

for $G = \mathbb{Z}/2$: $0 \rightarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2 \xrightarrow{\partial_2} \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \xrightarrow{\partial_1} \mathbb{Z}/2 \oplus \mathbb{Z}/2 \rightarrow 0$
 $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix} \right)$

$\Rightarrow H_*(\mathbb{R}P^2; \mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2 & * = 0 \\ \mathbb{Z}/2 & * = 1 \\ \mathbb{Z}/2 & * = 2 \\ 0 & \text{else} \end{cases}$ compare: $H_*(\mathbb{R}P^2) \cong \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z}/2 & * = 1 \\ 0 & \text{else} \end{cases}$ ($G = \mathbb{Z}$ case)

Form cochain complex using $\text{Hom}_{\mathbb{Z}}(\cdot, G)$ (= group homs in place of $\text{Hom}(\cdot, \mathbb{Z})$)

$H^*(C_*; G) = H_*(\text{Hom}_{\mathbb{Z}}(C_*, G))$ with differential ∂^* : $\partial^* \phi = \phi \circ \partial_*$
 $H^*(X; G) = H_*(\text{Hom}_{\mathbb{Z}}(C_*(X), G))$ so: $H^*(C_*(X); G)$

Universal coefficients thm (Same proof using $\text{Hom}_{\mathbb{Z}}(\cdot, G)$)

$0 \rightarrow \text{Ext}_{\mathbb{Z}}^1(H_{n-1}(C_*; G) \rightarrow H^n(C_*; G) \rightarrow \text{Hom}_{\mathbb{Z}}(H_n(C_*), G) \rightarrow 0$
 $[\varphi] \mapsto (\varphi : H_n(C_*) \rightarrow G)$

Example $X = \mathbb{R}P^2$, $G = \mathbb{Z}/2$, apply $\text{Hom}_{\mathbb{Z}}(\cdot, G)$ to the complex in previous example

$$0 \leftarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2 \xrightarrow{\partial^1} \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \xrightarrow{\partial^0} \mathbb{Z}/2 \oplus \mathbb{Z}/2 \leftarrow 0$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow H^*(\mathbb{R}P^2; \mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2 & * = 0 \\ \mathbb{Z}/2 & * = 1 \\ \mathbb{Z}/2 & * = 2 \\ 0 & \text{else} \end{cases}$$

compare: $H^*(\mathbb{R}P^2) \cong \begin{cases} \mathbb{Z} & * = 0 \\ 0 & * = 1 \\ \mathbb{Z}/2 & * = 2 \\ 0 & \text{else} \end{cases}$

($G = \mathbb{Z}$ case)

Compare $\text{Hom}(H_*(\mathbb{R}P^2), \mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2 & * = 0 \\ \mathbb{Z}/2 & * = 1 \\ 0 & * = 2 \end{cases}$

caused by $\text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}/2, \mathbb{Z}/2) \cong \mathbb{Z}/2$

Can generalise further:

$C_* =$ chain cx of ...	coefficients in:
abelian gps (\mathbb{Z} -mods)	$H_*(C_*; G) = H_*(C_* \otimes_{\mathbb{Z}} G)$
R-modules (comm. with 1)	$H_*(C_*; M) = H_*(C_* \otimes_{\mathbb{Z}} M)$

Rmk $H_*(C; M)$ will be an R-module since $\ker \partial, \text{Im } \partial$ are (∂_* is R-linear hom by assumption)

X space $\Rightarrow C_k(X; R) = C_k(X) \otimes_{\mathbb{Z}} R \cong \bigoplus_{\mathbb{Z}} R$: just replace \mathbb{Z} by R (as $\mathbb{Z} \otimes R \cong R$)

$$\Rightarrow H_*(X; M) = H_*(C_*(X; R) \otimes_{\mathbb{Z}} M)$$

so just replace each \mathbb{Z} by M in $C_*(X)$

Form cochain complex using $\text{Hom}_R(\cdot, M) (= R\text{-linear homs to } M)$ in place of $\text{Hom}(\cdot, \mathbb{Z})$

$$\begin{aligned} H^*(C_*; M) &= H_*(\text{Hom}_R(C_*, M)) \\ H^*(X; M) &= H_*(\text{Hom}_R(C_*(X; R), M)) \end{aligned}$$

with differential ∂^* : $\partial^* \phi = \phi \circ \partial_*$

so: $H^*(C_*(X; R), M) \leftarrow H^*(C_*(X; R), M)$

Rmk These are R-mods. If we use $M=R$, then they are also rings via cup product

Universal Coefficients Thm For R any PID, C_* chain cx of R-mods,

$$0 \rightarrow \text{Ext}_{\mathbb{Z}}^1(H_{n-1}(C_*), M) \rightarrow H^n(C_*; M) \rightarrow \text{Hom}_{\mathbb{Z}}(H_n(C_*), M) \rightarrow 0$$

is SES and natural.

$B^{n-1}/\text{im } \delta^{n-1}$ working over R using homs to M

$[\varphi] \mapsto (\varphi: H_n(C_*) \rightarrow M)$

and the SES splits but the splitting is not natural.

Example $R = \mathbb{F}$ field $\Rightarrow C_*, H_*, H^*$ are vector spaces/ \mathbb{F} .

Rmk all \mathbb{F} -mods (i.e. vector spaces/ \mathbb{F}) are free \mathbb{F} -mods $\cong \bigoplus \mathbb{F} b_i$ up to iso they are determined by $\dim_{\mathbb{F}} =$ cardinality of basis.

Cor $C_* =$ chain cx of \mathbb{F} -vector spaces $\Rightarrow H^n(C_*; \mathbb{F}) \cong (H_n(C_*))^*$ dual v.s.: $\text{Hom}_{\mathbb{F}}(H_n(C_*), \mathbb{F})$

Pf Pick any basis v_i for \mathbb{F} -v.s. B_{n-1} , extend it to a basis v_i, w_j of \mathbb{Z}_{n-1} (also works in ∞ dim case).

\Rightarrow can extend any \mathbb{F} -linear map $\psi: B_{n-1} \rightarrow \mathbb{F}$ to $\phi: \mathbb{Z}_{n-1} \rightarrow \mathbb{F}$ just pick any values $\phi(w_j) \in \mathbb{F}$ e.g. $\phi(w_j) = 0$.

$\Rightarrow B^{n-1}/\text{im } \delta^{n-1} = 0$ so $H^n(C_*; \mathbb{F}) \rightarrow \text{Hom}(H_n(C_*), \mathbb{F})$ iso \square

Cor $H^n(X; \mathbb{F}) \cong H_n(X; \mathbb{F})^*$ for any field \mathbb{F} .

$$H^n(X; M) \cong H_{CW}^n(X; M) \cong H_{\Delta}^n(X; M)$$

if X is CW-cx \uparrow if X is Δ -cx

Pf Cor holds for homology and the isos are natural. \leftarrow i.e. functorial w.r.t. maps

The universal coeff. thm SES is natural. So result holds by 5-Lemma. \square

Algebra: structure thm for f.g. abelian groups and R-mods

Fact 1 A f.g. abelian gp $\Rightarrow A \cong \mathbb{Z}^r \oplus \mathbb{Z}/p_1 \oplus \dots \oplus \mathbb{Z}/p_a$

where $p_i \in \mathbb{Z}$ prime (need not be distinct) \leftarrow free part \mathbb{F}

Also r, a, p_i, n_i are unique (up to reordering) \leftarrow torsion part T

Example $\mathbb{Z}/4 = \mathbb{Z}/2 \neq \mathbb{Z}/2 \oplus \mathbb{Z}/2$ (Chinese Remainder Thm)

$\mathbb{Z}/6 \cong \mathbb{Z}/2 \oplus \mathbb{Z}/3$

Fact 2 $T \cong \mathbb{Z}/d_1 \oplus \dots \oplus \mathbb{Z}/d_k$ with $d_1 | d_2 | \dots | d_k$ ($d_i \in \mathbb{N}$ unique)

Example $\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/3 \cong \mathbb{Z}/2 \oplus \mathbb{Z}/3 \cdot 2^2$ $d_1=2, d_2=12$

Fact 3 M f.g. R-mod, R PID, then:

$$\begin{aligned} M &\cong \mathbb{F} \oplus T \\ \mathbb{F} &\cong R^r \\ T &\cong R/p_1 \oplus \dots \oplus R/p_a \\ &\cong R/d_1 \oplus R/d_2 \oplus \dots \oplus R/d_k \end{aligned}$$

$r \in \mathbb{N}$ unique, called rank of M

$p_i \in \mathbb{R}$ primes, p_i unique up to ordering & mult by $d_i | d_{i+1}$

d_i called invariant factors

unique up to multⁿ by invertible elements \leftarrow e.g. ± 1 if $R = \mathbb{Z}$

Rmk $T = \{m \in M : rm = 0 \text{ some } r \neq 0 \in R\} =$ torsion elements

$\mathbb{F} \cong M/T$

Torsion shift

Easy Exercise $\text{Ext}_R^*(\bigoplus_i M_i, \bigoplus_j N_j) \cong \prod_i \text{Ext}_R^*(M_i, N_j)$ ← any R-mods M_i, N_j

Upshot To compute $\text{Ext}_R^i(M, R)$ for $M = R \oplus R/d \oplus \dots$ just need:

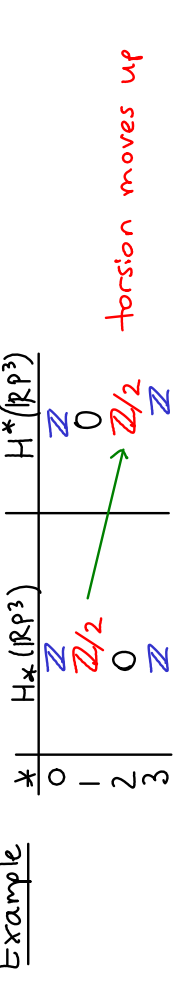
$\text{Ext}_R^1(R, R) = 0$ ← since $0 \rightarrow R \xrightarrow{1} R \rightarrow 0$
 $\text{Ext}_R^1(R/d, R) \cong R/d$ ← since $0 \rightarrow R \xrightarrow{d} R \rightarrow R/d \rightarrow 0$
 $\Rightarrow \text{Ext}_R^1(M, R) \cong \text{Torsion}(M)$

- Exercises
- $\text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}_m, \mathbb{Z}/m) \cong \mathbb{Z}/\text{gcd}(m, m)$
 - Gabelian gp $\Rightarrow \text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}, G) = 0$ and $\text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}/d, G) \cong G/d \cdot G$
 - R any ring (comm. with 1)
 $x \in R$ not zero divisor $\Rightarrow \text{Ext}_R^*(R/(x), N) \cong \begin{cases} N/x \cdot N & * = 1 \\ 0 & \text{else} \end{cases}$

Cor If $H_n(X; R)$ f.g. R-mod V_n , R PID,
 $\Rightarrow H_n(X; R) = R^n \oplus T_n$ (free & torsion parts)
 $\Rightarrow H^n(X; R) \cong R^n \oplus T_{n-1}$ ← torsion moves up!

Pf $0 \rightarrow T_{n-1} \rightarrow H^n(X; R) \rightarrow \text{Hom}(R^n \oplus T_{n-1}, R) \rightarrow 0$
 $\text{Hom}(R^n \oplus T_{n-1}, R) \cong (\text{Hom}(R, R))^n \oplus \text{Hom}(T_{n-1}, R)$
 $R \rightarrow R \xrightarrow{1} x \xrightarrow{1} x$
 x determines the hom
 \circ since $T_{n-1} \rightarrow R, 1 \mapsto 0$
 $(R$ is integral domain, so no torsion elts $\neq 0)$

$\Rightarrow 0 \rightarrow T_{n-1} \rightarrow H^n(X; R) \rightarrow R^n \rightarrow 0$
 free, so can split the SES (pick lifts of basis). \square



Universal coefficients Theorem in homology

FACT Theorem C_* chain cx of free R -mods, M R -module
 \Rightarrow natural SES $0 \rightarrow H_*(C_*) \otimes_R M \rightarrow H_*(C_* \otimes_R M) \rightarrow \text{Tor}_1^R(H_{*+1}(C_*), M) \rightarrow 0$
 $[C] \otimes m \mapsto [C \otimes m]$
 The SES splits, but the splitting is not natural.

Torsion groups: A, B R -mods (R comm. ring with 1) exact seqs
 $P_2 \xrightarrow{\psi_2} P_1 \xrightarrow{\psi_1} P_0 \xrightarrow{\psi_0} A \rightarrow 0$ free resolution P_i free R -mods
 $\Rightarrow \dots \rightarrow P_2 \otimes B \xrightarrow{\psi_2 \otimes \text{id}} P_1 \otimes B \xrightarrow{\psi_1 \otimes \text{id}} P_0 \otimes B \rightarrow 0$ not exact but is chain cx
 omit $A \otimes B$

$\text{Tor}_k^R(A, B) = H_k$ (this complex) ← fact independent of choices of P_i, ψ_i
 Rmk R PID $\Rightarrow \ker \psi_0$ free \Rightarrow pick $\{P_i = \ker \psi_i\}$ only $\text{Tor}_0^R, \text{Tor}_1^R$ can be non-zero
 Example $\text{Tor}_*^{\mathbb{Z}}(\mathbb{Z}/a, \mathbb{Z}/b) = ?$

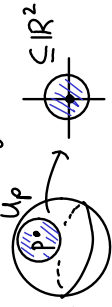
take $\cdot \otimes \mathbb{Z}/b \Rightarrow 0 \rightarrow \mathbb{Z} \xrightarrow{a} \mathbb{Z} \xrightarrow{\psi_0} \mathbb{Z}/a \rightarrow 0$ free resolution
 are $\mathbb{Z}/a \otimes \mathbb{Z}/b \Rightarrow 0 \rightarrow \mathbb{Z}/b \xrightarrow{a} \mathbb{Z}/b \rightarrow 0$ (since $\mathbb{Z} \otimes \mathbb{Z} \cong \mathbb{Z}$ any G)
 $\text{Tor}_0^{\mathbb{Z}}(\mathbb{Z}/a, \mathbb{Z}/b) = (\mathbb{Z}/b)/a \cdot \mathbb{Z}/b \cong \mathbb{Z}/\langle a, b \rangle \cong \mathbb{Z}/\text{gcd}(a, b) \cong \mathbb{Z}/a \otimes \mathbb{Z}/b$
 $\text{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/a, \mathbb{Z}/b) = \{x \in \mathbb{Z}/b : a \cdot x = 0\} \cong \mathbb{Z}/\text{gcd}(a, b)$
 Facts $\text{Tor}_0^R(A, B) = P_0 \otimes B / \text{Im}(\psi_0 \otimes \text{id}) \cong A \otimes B$ via: $b \xleftarrow{\text{gcd}(a,b)}$
 $\text{Tor}_*^R(A, B) \cong \text{Tor}_*^R(B, A)$

Exercise $\text{Tor}_*^R(\bigoplus_i A_i, \bigoplus_j B_j) \cong \bigoplus_i \text{Tor}_*^R(A_i, B_j)$
 $\text{Tor}_*^R(A, B) = 0$ for $* \geq 1$ if A or B is free (use $M \otimes_R R \cong M$)
 $\text{Tor}_*^R(R/u, M) \cong \begin{cases} M/u \cdot M & * = 0 \\ \text{u-torsion}(M) = \{x \in M : u \cdot x = 0\} & * = 1 \\ 0 & \text{else} \end{cases}$
 deduce $\text{Tor}_*^R(A, M)$ for f.g. R -mods A
 Example $H_*(\mathbb{R}P^2; \mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2 & * = 0 \\ \mathbb{Z}_2 & * = 1 \\ \mathbb{Z}_2 & * = 2 \end{cases}$
 $H_*(\mathbb{R}P^2) \otimes \mathbb{Z}_2 \cong \begin{cases} \mathbb{Z} \otimes \mathbb{Z}_2 & * = 0 \\ \mathbb{Z}_2 \otimes \mathbb{Z}_2 & * = 1 \\ 0 & * = 2 \end{cases}$
 caused by $\text{Tor}_1^{\mathbb{Z}}(H_1(\mathbb{R}P^2), \mathbb{Z}_2) = \text{Tor}_1(\mathbb{Z}_2, \mathbb{Z}_2) \cong \mathbb{Z}_2$

Künneth Thm
 R PID \Rightarrow natural SES: $0 \rightarrow \bigoplus_{i+j=n} H_i(C_*) \otimes H_j(D_*) \rightarrow H_n(C_* \otimes D_*) \rightarrow \bigoplus_{i+j=n-1} \text{Tor}_1^R(H_i(C_*), H_j(D_*)) \rightarrow 0$
 $(C_*$ free ch. cx. R -mods $\rightarrow D_*$ any ch. cx. R -mods)
 and the SES splits but the splitting is not natural. Example $R = \text{field} \Rightarrow$ that $\text{Tor}_1 = 0$
 Example (take $C_* = C^{-*}(X), D_* = C^{-*}(Y)$): $0 \rightarrow H^*(X) \otimes H^*(Y) \rightarrow \text{Tor}_1(H^*(X), H^*(Y)) \rightarrow 0$ exact (if $C_*(X)$ has ∞ rank then $C^{-*}(X)$ may not be free but it will be "flat" and Thm holds if C_* is flat R -mod)

13. MANIFOLDS: POINCARÉ-LÉFSCHETZ DUALITY

- M n -mfd is Hausdorff topological space s.t. $\forall p \in M$ \exists open neighbourhood $U_p \subseteq M$ homeomorphic to \mathbb{R}^n



(equivalently: to an open ball, or any open set in \mathbb{R}^n)

One also requires M second countable i.e. \exists countable basis of open sets

$\Leftrightarrow M$ is covered by countably many such U_p :
 \leftarrow exercise

A submanifold $N \subseteq M$ is a mfd s.t. inclusion $N \rightarrow M$ is an embedding (i.e. a homeomorphism onto its image)

- M n -mfd with boundary if also allow $U_p \cong$ upper half space \mathbb{H}^n which they form the boundary ∂M which is an $(n-1)$ -mfd without boundary.



equivalently: any open nbhd of $o \in \mathbb{H}^n$

FACT (Collar nbhd thm) $\partial M \subseteq M$ has an open neighbourhood $\cong \partial M \times (0,1]$



M is closed if compact without boundary.

Rmk For manifolds, connected components = path components. (since locally \cong disc, so locally path-connected, so conn. \Leftrightarrow path-con.)

Examples

closed mfds: $S^n, \mathbb{R}P^n, T^n = S^1 \times \dots \times S^1, \mathbb{C}P^n, O(n), SU(n)$

non-compact mfds: $\mathbb{R}^n, \text{Mat}_{m \times n} \cong \mathbb{R}^{mn}, GL(n, \mathbb{R})$

mfds with bdr: $D^n, D^1 \times S^1 =$ \cup $\setminus \text{disc} =$

FACT (Milnor 1959) Any mfd is homotopy equivalent to a CW-complex

fact If M is a compact manifold then $H_k(M)$ are finitely generated

Rmk M triangulable if $M \cong$ simplicial cx.

Not all mfds are triangulable, but most of those we encounter are.

Compact manifolds have f.g. homology \leftarrow Non-examinable proof

① X space is a Euclidean neighbourhood retract if

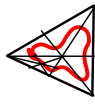
\exists embedding $j: X \rightarrow \mathbb{R}^m$ some N , s.t. $i(X)$ is a retract of a nbhd $V \subseteq \mathbb{R}^m$ \leftarrow (homeo onto image)

② X is weakly locally contractible if \forall nbhd $x \in U \subseteq X, \exists$ nbhd $x \in V \subseteq U$ s.t. V is contractible inside U .

FACT Compact $X \subseteq \mathbb{R}^n$ is ① \Leftrightarrow ②

Rmk If we find nbhd V as in ① with retraction $V \xrightarrow{f} X$ then any smaller nbhd V' also retracts using $f|_{V'}: V' \rightarrow X$. Similarly in ② $V' \subseteq V$ is contractible: restrict the hcy.

Lemma A X compact & ① $\Rightarrow X$ is the retract of a finite simplicial cx
pf $i(X) \subseteq \mathbb{R}^n$ compact \Rightarrow lies inside some large n -simplex $\Delta^n \rightarrow \mathbb{R}^n$
 Apply barycentric subdivision until simplices have diameter $< \text{dist}(X, \partial V)$.



Simpl. cx. = \cup {simplices which intersect X } using the restriction of retraction $V \rightarrow X$.

Rmk Also deduce X has f.g. homology since retractions are surjective on H_k .

$(\oplus \mathbb{Z} \rightarrow H_k(\text{finite simpl. cx.}) \xrightarrow{\text{retract}} H_k(X))$ so get surjection from free \mathbb{Z} -mod, so f.g.)

Lemma B M compact mfd $\Rightarrow M$ embeds into \mathbb{R}^N , some N .

pf "Just do it proof":

$\forall p \in M, \exists$ homeo $D^n \xrightarrow{\psi_p} \text{nbhd}(p \in M)$

Pick finite subcover of ψ_p : of $M = \cup_{p \in M} \psi_p(D^n)$. Say $i=1, \dots, k$

$\psi_i: M \xrightarrow{\psi_i} D^n \xrightarrow{\text{embed}} S^n \subseteq \mathbb{R}^{n+1}$ define embedding $(\psi_{p_1}, \dots, \psi_{p_k}): M \rightarrow \mathbb{R}^k \cdot (n+1)$
 \leftarrow (send $M \times \text{im}(\psi_i)$ to the point corresponding to $\partial D^n \in \mathbb{R}^n / \partial D^n$)

Finally use: a continuous bijection from a compact space to a Hausdorff space is \cong

Rmk Same works if M has boundary, just consider its double $M \cup M$ $\xrightarrow{\text{identify along } \partial M}$ and apply the Lemma to the double.

Cor M compact mfd (possibly with bdr) $\Rightarrow M$ has f.g. homology

Pf Mfds satisfy ② since locally ball \cong pt. M embeds in \mathbb{R}^N by Lemma B.

① holds by FACT. Done by Lemma A. \square

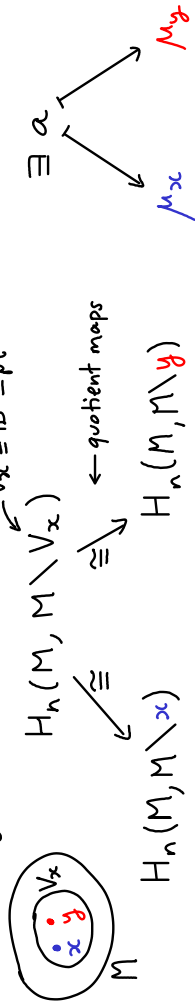
Local orientations and orientability

Def A local orientation of M at $x \in M$ is a choice of generator

$$\mu_x \in H_n(M, M \setminus x) \cong H_n(D^n, D^n \setminus \{0\}) \cong \tilde{H}_n(S^n) \cong \mathbb{Z} \quad \begin{matrix} \text{excise complement of nbhd } V_x \cong \mathbb{D}^n \\ \text{choice of } \mu_x \text{ is not canonical!} \\ \text{choice of } \mu_x \text{ is not canonical!} \end{matrix}$$

Def An orientation of M is a locally consistent choice $x \mapsto \mu_x$

meaning:

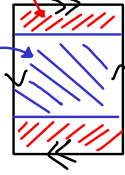


Def M orientable if \exists orientation on M

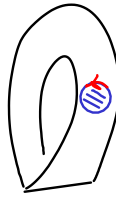
oriented if we chose an orientation

Examples $S^n, \mathbb{R}^n, \mathbb{C}P^n$, orientable surfaces Σ_g , $\mathbb{R}P^n \leftarrow \text{odd } n$

Non-example $\mathbb{R}P^2 = \text{Möbius band} \cup D^2$

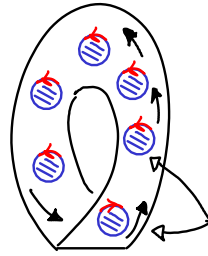


by local consistency can move disc continuously and preserves orientation



choice of μ_x is choice of orientation of boundary circle of small disc containing x

$\Rightarrow \mathbb{R}P^2$ not orientable



discs are differently oriented \Rightarrow contradicts local consistency.

The fundamental class $[M]$

FACT Theorem For M closed n -mfd: natural map from the LES

$$1) M \text{ orientable connected} \Rightarrow H_n(M) \cong H_n(M, M \setminus x) = \mathbb{Z} \cdot \mu_x \Rightarrow \exists [M] \xleftarrow{\mu_x} \mu_x$$

once we choose an orientation $(\mu_x)_{x \in M}$ called fundamental class

(if swap orientation: for $-\mu_x$ get $-[M]$)

$$2) M \text{ not orientable connected} \Rightarrow H_n(M) = 0 \Rightarrow H_n(M; \mathbb{F}_2) = \mathbb{F}_2 \cdot [M] \cong \mathbb{F}_2$$

(or any field of characteristic 2)

Construction of $[M]$ if M has Δ -complex structure

M compact \Rightarrow finite # n -simplices $\delta_1, \dots, \delta_N$

M oriented \Rightarrow pick orientations of $\delta_1, \dots, \delta_N$ to agree with given orientation of $M: \nu$ for $x \in \text{Int}(\delta_i)$



$$\mathbb{Z} \cdot \mu_x = H_n(M, M \setminus x) \xrightarrow{\text{exc}} H_n(\delta_i, \delta_i \setminus x) = \mathbb{Z} \cdot \delta_i$$

$$\mu_x \mapsto \delta_i$$

$$\Rightarrow [M] := \sum \delta_i \text{ satisfies } \partial [M] = 0 \checkmark \text{ (each facet arises twice with opposite signs)}$$

$$H_n(M) \rightarrow H_n(M, M \setminus x) \cong H_n(\delta_i, \delta_i \setminus x)$$

$$[M] \xrightarrow{\mu_x} \delta_i$$

Not difficult to see that $H_n^\Delta(M) = \mathbb{Z} \cdot [M]$, so $\Rightarrow H_n(M) \cong H_n(M, M \setminus x)$

Also $\mathbb{Z} \cong \mathbb{Z} \cdot [M]$ since $C_{n+1}(M) = 0 \Rightarrow \tilde{H}^{n+1}(M) = 0$ (simplices since $\dim M = n$)

M non-orientable \Rightarrow each facet of δ_i appears twice in $\partial \sum \delta_i$

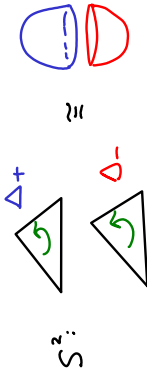
$\Rightarrow \partial \sum \delta_i = 0$ over \mathbb{F}_2 independently of choices of orientations of δ_i .

More generally:
 $[M] := \sum \pm \delta_i$
 where signs come from $H_n(M, M \setminus x) \cong H_n(\delta_i, \delta_i \setminus x)$
 $\mu_x \mapsto \pm \delta_i$
 (so compare orientation of μ_x with orientation of δ_i)

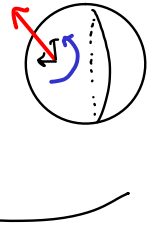


Examples

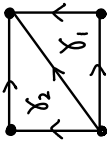
1) $S^n = \Delta^n \cup \Delta^n$
 glue bodies



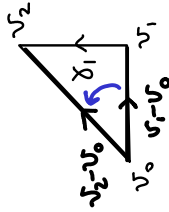
$[S^n] = \Delta_+ - \Delta_-$ if use canonical orientation we discussed
 hence $\partial[S^n] = \partial\Delta - \partial\Delta = 0$
 $D^n \subseteq \mathbb{R}^n$ canonical orientation
 $\Rightarrow S^{n-1} = \partial D^n$ using outward normal first rule



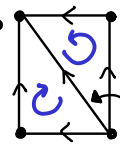
Δ -complex structure (compatibly with side identifications!)



Want orientation induced by square $\subseteq \mathbb{R}^2$



$v_1 - v_0, v_2 - v_0$ positive \mathbb{R}^2 -basis
 $\Rightarrow \delta_1$ agrees with orientation



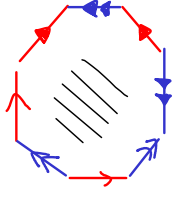
$[T^2] = +\delta_1 - \delta_2$
 \uparrow δ_2 orientation disagrees

RMK general feature: if two simplices are identified along a facet then the 2 respective outward normals are related by reflection.

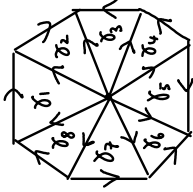


So consistency \Rightarrow either simplices are compatibly oriented and the two induced orientations on facet are opposite or not compatibly oriented but facet orientⁿ is same, then need sign like in example when build $[T^2]$

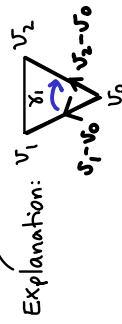
3) Recall $\Sigma_2 =$



Δ -cx structure (compatible with side identifications!):

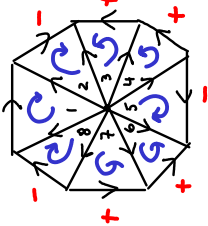


Use the orientation induced by polygon $\subseteq \mathbb{R}^2$
 $\Rightarrow [\Sigma_2] = -\delta_1 - \delta_2 + \delta_3 + \delta_4 - \delta_5 - \delta_6 + \delta_7 + \delta_8$

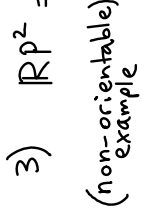


$v_1 - v_0, v_2 - v_0$ is negative \mathbb{R}^2 -basis

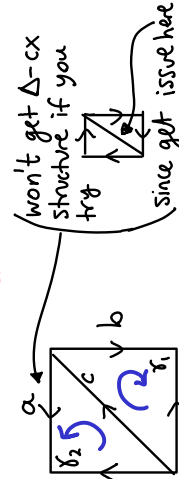
All simplices δ_i have $v_0 =$ centre of polygon



\Rightarrow sign \leftarrow $\begin{matrix} + \\ - \end{matrix}$ if overedge clockwise \leftarrow anti



(non-orientable example)



won't get Δ -cx structure if you try (since get issue here)

Use the orientation induced by square $\subseteq \mathbb{R}^2$

$\Rightarrow [\mathbb{RP}^2] = -\delta_1 + \delta_2$
 $\partial[\mathbb{RP}^2] = -(b - a + c) + (a - b + c)$
 $= -2b + 2a$
 $\neq 0$ so not cycle in $C_*^{CW}(\mathbb{RP}^2)$

However, working modulo 2:

$\partial[\mathbb{RP}^2] = 0 \in C_*^{CW}(\mathbb{RP}^2; \mathbb{F}_2)$ since $2=0$ in \mathbb{F}_2
 $\Rightarrow [\mathbb{RP}^2] \in H^2(\mathbb{RP}^2; \mathbb{F}_2)$

Degree

Def M, N oriented closed connected n -mfds, $f: M \rightarrow N$

$$f_*: H_n(M) \rightarrow H_n(N)$$

$$[M] \mapsto \deg(f) \cdot [N] \in \mathbb{Z}$$

Local degree

Lemma If $f^{-1}(y)$ finite, Local map like in chapter 7

$$\text{then } \deg(f) = \sum_{x \in f^{-1}(y)} \deg(f|_{x,*})$$

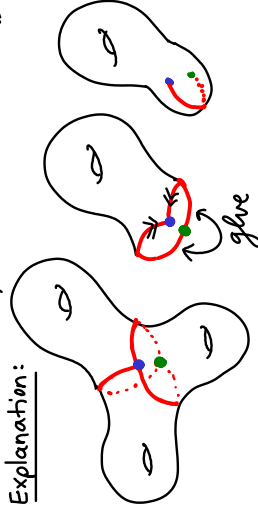
$$\begin{array}{ccc} [M] \in H_n(M) & \xrightarrow{f_*} & H_n(N) \cong [N] \\ \downarrow & & \downarrow \\ \bigoplus_{x \in f^{-1}(y)} H_n(M, M \setminus x) & \xrightarrow{(f_x)_*} & H_n(N, N \setminus y) \cong \mu_y^N \\ \downarrow & & \downarrow \\ \bigoplus_{x \in f^{-1}(y)} \mu_x^M & \xrightarrow{(\sum \deg(f_x)_*) \cdot \mu_y^N} & \mu_y^N \end{array}$$

Examples

$$1) S^1 \rightarrow S^1, z \mapsto z^n, [S^1] \mapsto n \cdot [S^1] \text{ so } \deg = n$$

$$2) \Sigma_3 \xrightarrow{q} \Sigma_3 / \mathbb{Z}_3 \text{-rotation action} = \Sigma_1 \text{ torus}$$

Explanation:



Easy check: $\deg(q) = 3$ (e.g. use local degrees)

Cultural Rmk

For M, N, f smooth, the $\deg f = \#$ (preimages of a generic point of N)
 Idea: $\deg f$ tells you how many times you cover N . (almost all points work)

Poincaré duality

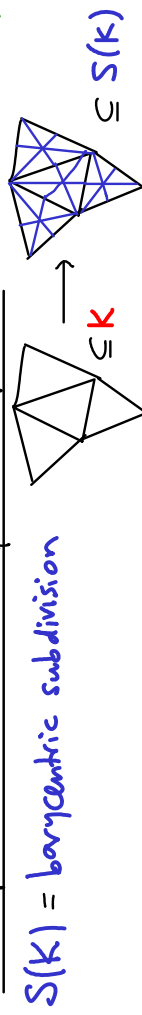
FACT Theorem For M closed n -mfd

$$M \text{ oriented} \Rightarrow H^k(M) \cong H_{n-k}(M)$$

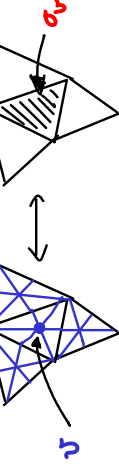
$$\text{s.t. } 1 \leftrightarrow [M] \\ H^0(M) \cong H_n(M)$$

M non-oriented \Rightarrow same holds with \mathbb{F}_2 coefficients

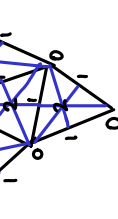
Sketch proof when M is a simplicial complex K (Non-examinable)



1) simplex σ^v of K with barycentre $v \leftrightarrow v^*$ vertex of $S(K)$



$$2) \text{ht}(v) = (\text{height of } v) = \dim \sigma^v$$



3) σ k -simplex of K

dual simplex

$$\hat{\sigma} = \bigcup_{\tau} \tau$$

$\tau \in S(K), v_\sigma \in \tau$
 $\text{ht}(v_\sigma)$ is min of heights of vertices of τ

Rmk: $\bigcup_{v \in \sigma} \tau$ with $\text{ht}(v_\sigma) \max$ will give back σ .

Thus $\hat{\sigma}, \sigma$ intersect transversely at v_σ . One can also describe $\hat{\sigma}$ as

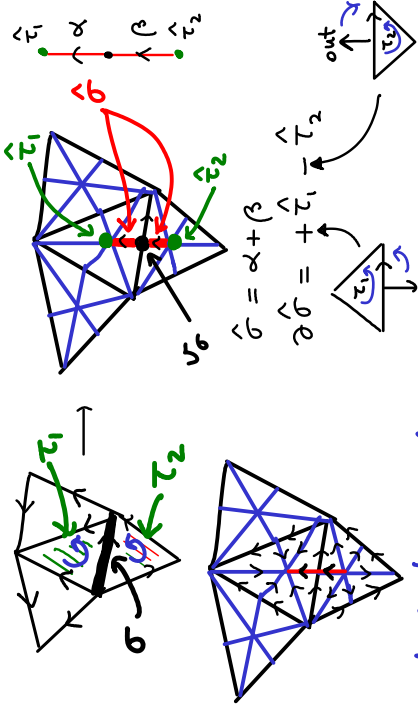
$\hat{\sigma} = \bigcup_{v \in \sigma} \text{Star}(v)$ (closed star is the union of simplices of $S(K)$ having v as a face)

FACTS • $\dim \hat{\sigma} = n - \dim \sigma$ ("polygonal" complex rather than Δ -cx)

• dual cells $\hat{\sigma}$ give a cell decomposition of M

(*) $\partial \hat{\sigma} = \sum_{\substack{\text{facet} \\ \sigma \neq \tau}} \pm \hat{\tau}$

need compare orientations of σ, τ (+ if σ as a facet of τ has boundary orientation)



4) dual chain complex

$D_{n-k} =$ free abelian group on dual chains $\hat{\sigma}$

$H_*(M) \cong H_*(D_*, \partial_*)$ (since $\hat{\sigma}$ give a cell decomp. of M)

5) $\varphi: D_{n-k} \rightarrow C^k(M)$

where $\sigma^*(\alpha) = \begin{cases} 0 & \text{for } k\text{-cells } \alpha \neq \sigma \\ 1 & \text{for } \alpha = \sigma \end{cases}$

• φ linear bijection ✓

• chain map: $\xrightarrow{\text{see (*)}}$

$\varphi(\partial \hat{\sigma}) = \varphi \sum \pm \hat{\tau} = \sum \pm \tau^*$

$\partial^* \varphi(\hat{\sigma}) = \partial^* \sigma^* = (\sigma^* \circ \partial : \tau \mapsto \sum \pm \sigma_i \xrightarrow{\text{facet } \sigma} \pm 1 \text{ if one of } \sigma_i = \sigma)$
 $= \sum \pm \tau^* = \varphi(\partial \hat{\sigma}) \checkmark$

UPSHOT φ is chain iso so get iso:

$H_*(M) \cong H_*(D_*, \partial_*) \xrightarrow{\varphi} H^{n-k}(M)$

Cor χ (odd dimensional closed orientable mfd) = 0

Pf Betti numbers $b_i = \text{rank } H_i(M) \stackrel{\text{universal coeff. thm.}}{=} \text{rank } H_i(M) \stackrel{\text{Poincaré duality}}{=} \text{rank } H_{n-i}(M)$

$\chi(M) = b_0 - b_1 + \dots + b_{\dim M-1} - b_{\dim M}$ equal. \square

(Poincaré-)Lefschetz duality

Theorem M compact oriented n -mfd with boundary

$H^k(M) \cong H_{n-k}(M, \partial M)$

$1 \in H^0(M) \mapsto [M, \partial M] \in H_n(M, \partial M)$

relative fundamental class

$H_k(M) \cong H^{n-k}(M, \partial M)$

Non-oriented \Rightarrow same holds with \mathbb{F}_2 coefficients.

Pf basically same as Poincaré duality. \square

either by universal coefficient thm since $H_0(M, \partial M) = 0$ or by hand since given $p \in M, q \in C^1(M, \partial M)$ consider $\langle \partial q, p \rangle$ for δ path from p to any $q \in \partial M$.

Cor M compact, connected, $\partial M \neq \emptyset \Rightarrow \begin{cases} H^n(M) \cong H_0(M, \partial M) = 0 \\ H_n(M) \cong H^0(M, \partial M) = 0 \end{cases}$

Examples

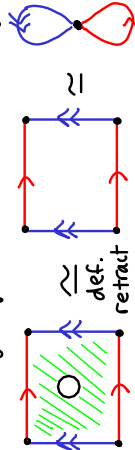
1) $D^n \xrightarrow{\partial D^n = S^{n-1}}$

2) $A = \text{annulus} \subseteq \mathbb{R}^2 \simeq S^1$

$Z \cong H^0 A \cong H_2(A, \partial A)$ generator $D_1 - D_2$
 $Z \cong H^1 A \cong H_1(A, \partial A)$ generator \leftarrow copy of D^1
 $0 \cong H^2 A \cong H_0(A, \partial A)$ (notice $\partial D \rightarrow \partial A$)

Rank notice gen. of $H_1(A)$ is \circlearrowleft which intersects gen. of $H_1(A, \partial A)$ once transversely.

3) $M = T^2 \setminus \text{open ball} = \text{ball} \simeq S^1 \vee S^1$



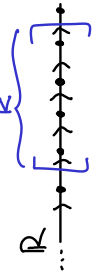
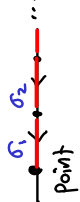
$\Rightarrow H_*^*(M, \partial M) \cong H^{2-*}(S^1 \vee S^1) = \begin{cases} \mathbb{Z}^2 & * = 1 \\ \mathbb{Z} & * = 2 \\ 0 & \text{else} \end{cases}$

What happens in the non-compact case?

Locally finite homology (Borel-Moore)

$C_*^{lf}(X)$ allow infinite sums $\sum_{i \in \mathbb{Z}} n_i \sigma_i$ generators of $C_*(X)$
 s.t. given any compact subset $K \subseteq X$,
 $\#\{n_i \neq 0 : K \cap \text{supp } \sigma_i \neq \emptyset\} < \infty$.

Examples

- $C_1^{lf}(\mathbb{R}) \ni \sum_{m \in \mathbb{Z}} \sigma_m, \sigma_m: I \cong [m, m+1] \subseteq \mathbb{R}$

- $C_0^{lf}(\mathbb{R}) \ni [\text{point}] \ni \partial(\sigma_1 + \sigma_2 + \dots)$ is a boundary:


exercise $H_*^{lf}(\mathbb{R}) = \begin{cases} \mathbb{Z} & * = 1 \\ 0 & \text{else} \end{cases} (\cong H^{1-*}(\mathbb{R}))$

FACT Theorem M orientable n-mfd $\Rightarrow H^*(M) \cong H_{n-*}^{lf}(M)$
 (possibly not compact) depends on ϕ

cohomology with compact supports $H_c^*(X)$

$C_c^*(X)$: only allow cochains $\phi: C_* X \rightarrow \mathbb{Z}$ s.t. \exists compact $K \subseteq X$ with $\phi(C_*(X \setminus K)) = 0$ (vanish on chains in $X \setminus K$)

Example $c \in C_*(X) \Rightarrow \phi(\alpha) = \text{signed \# intersections of } c \text{ with } \alpha$
 (geometric intersection #)
 $\Rightarrow \phi \in C_c^*(X)$ since $\phi(\alpha) = 0$ if $\alpha \subseteq X \setminus \text{Im}(c)$

Thm M orientable n-mfd $\Rightarrow H_*(M) \cong H_c^{n-*}(M)$
 (possibly not compact)

Warning H_*^{lf}, H_c^* are not homotopy invariant (indeed non-trivial for \mathbb{R}^n)
 Caused because they are not functorial. They are however functorial for proper maps (preimages of compact sets are compact)

Fact $H_c^*(X) \cong \varinjlim H^*(X, X \setminus K)$ where compacts $K_1 \subseteq K_2$ give $H^*(M, M \setminus K_1) \rightarrow H^*(M, M \setminus K_2)$
Direct limit $\varinjlim G_i$ via maps $G_i \rightarrow G_j$ means $\sqcup G_i$ / identifying $g \in G_i$ with its images under those maps
 (The indices are partially ordered & directed: $\forall i, j, \exists k > i, j, \exists$ maps $G_i \rightarrow G_k, G_j \rightarrow G_k$)
Fact \varinjlim is an exact functor.

Cap product and Poincaré duality revisited

X space, $k \geq l$ (sometimes write $\emptyset \cap \sigma$)

$n: C_k(X) \times C^l(X) \rightarrow C_{k-l}(X)$ cap product

$(\sigma: \Delta^k \rightarrow X) \cap (\phi: C_l(X) \rightarrow \mathbb{Z}) = \phi(\sigma|_{[e_0, \dots, e_l]}) \cdot \sigma|_{[e_{l+1}, \dots, e_k]}$
 "bottom face" $\in \mathbb{Z}$ "top face" $\cong \Delta^{k-l} \in C_{k-l}(X)$

(easy) Properties

- \cap bilinear
- $\partial(\sigma \cap \phi) = (-1)^l (\partial\sigma \cap \phi - \sigma \cap \partial\phi)$
- cycle \cap cocycle is cycle
- boundary \cap cocycle are boundaries
- cycle \cap coboundary

$\Rightarrow n: H_k(X) \times H^l(X) \rightarrow H_{k-l}(X)$ bilinear

Theorem (Poincaré duality) The map $\phi \mapsto [M] \cap \phi$ gives following isos

- ① For M closed oriented n-mfd $[M] \cap \cdot: H^*(M) \xrightarrow{\cong} H_{n-*}(M)$
- ② For M non-compact oriented n-mfd, $[M] \cap \cdot: H_c^*(M) \xrightarrow{\cong} H_{n-*}(M)$
- ③ $[M] \cap \cdot: H^*(M) \xrightarrow{\cong} H_{n-*}^{lf}(M)$

Sketch Pf of ③ for smooth mfd (Non-examinable)

If M smooth $\Rightarrow \exists$ "good cover" U_i of M meaning open cover s.t.
 FACT from Riemannian geometry ("Convex neighbourhoods") $U_i \cong \mathbb{R}^n$
 $U_{i_1} \cap \dots \cap U_{i_k} \cong \mathbb{R}^n$ or \emptyset
 Then compute $H_c^*(\mathbb{R}^n) \cong \begin{cases} \mathbb{Z} & * = n \\ 0 & \text{else} \end{cases}$ and ③ holds for \mathbb{R}^n .
 \Rightarrow holds $\forall U_i$
 \Rightarrow by naturality of ③ and of Mayer-Vietoris get ③ for $\cup_{i \in \text{finite}} U_i$
 \Rightarrow ③ for M , which is ①. \square (use 5-lemma)

General Pf of Poincaré duality ← **Non-examinable**

Step 1: holds for \mathbb{R}^n

Pf $H_c^k(\mathbb{R}^n) \cong H^k(D^n, S^{n-1}) \cong \begin{cases} 0 & k \neq n \\ \mathbb{Z} & k = n \end{cases} \cong H_{n-k}(\mathbb{R}^n)$

(recall fact: $H_c^k(X) \cong \lim_{\leftarrow} H^k(X, \mathbb{R} \setminus K)$ can make K larger by picking $K = \text{large ball}$)
 then $H^k(\mathbb{R}^n, \mathbb{R}^n \setminus K) \cong H^k(K, \partial K)$

Pick Δ -cx structure for \mathbb{R}^n . So $[\mathbb{R}^n] = \sum \pm \sigma_i \leftarrow$ sum over n -simplices.
 Say \exists simplex $\sigma_0: \Delta^n \rightarrow \mathbb{R}^n$. Define $\phi: C_c^k(\mathbb{R}^n) \rightarrow \mathbb{Z}, \phi(\sigma_0) = \pm 1$ (other simplices) = 0

$[\mathbb{R}^n] \cap \phi = \sum \phi(\pm \sigma_i) = \phi(\pm \sigma_0) = 1$ (pick sign in \oplus)

Step 2 holds for $A, B, A \cup B \Rightarrow$ holds for $A \cup B$

Pf Mayer-Vietoris for H_c^* , naturality, 5-lemma \checkmark

Step 3 holds for A_i , and $A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots \Rightarrow$ holds for $\cup A_i$

Pf By applying ling: both sides of P.D. iso commute with limits \checkmark

Step 4 holds for open subsets in \mathbb{R}^n

Pf Every such set is a union of convex open sets (e.g. balls)

By Step 3 enough to consider case of finite union.

By induction on # convex open sets:



2 convex sets: KEY TRICK convex set \cap convex set is convex in \mathbb{R}^n !

\Rightarrow use Step 2 & previous case

$k+1$ convex sets: $A = \cup \{\text{first } k \text{ convex sets}\}, B = \text{last convex set} \Rightarrow$ use step 2 & inductive hypothesis

Step 5 holds for mfd M

Consider open sets in M for which it holds.

By a Zorn's Lemma argument we get a maximal open subset U where holds.

If $U \neq M$ pick $p \in M \setminus U$ and nbhd $V \cong \mathbb{R}^n$ of p . Then holds for $U, V, U \cup V$

(note $U \cup V \subseteq V \cong \mathbb{R}^n$ open, so Step 4 applies) so by Step 2 holds for $U \cup V$

Contradicts maximality. $\checkmark \square$

Recall there is a well-defined evaluation of H^* -classes on H_* :

$\langle \cdot, \cdot \rangle: H_k(M; \mathbb{R}) \otimes H^k(M; \mathbb{R}) \rightarrow \mathbb{R}$
 $\subset \otimes \alpha \mapsto \langle c, \alpha \rangle = \langle \varphi(c), \alpha \rangle$
 any representative cocycle φ for α

Easy exercise $\langle c, \alpha \cup \beta \rangle = \langle c \cap \alpha, \beta \rangle$ any $\alpha, \beta \in H^*, c \in H_*$

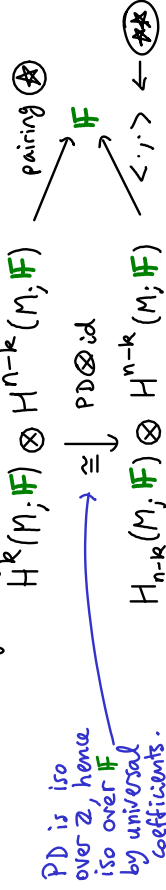
Corollary of Poincaré duality

M compact oriented n -mfd, \mathbb{F} field.

$H^k(M; \mathbb{F}) \otimes H^{n-k}(M; \mathbb{F}) \xrightarrow{\otimes} H^k(M; \mathbb{F}) \otimes H^{n-k}(M; \mathbb{F}) \xrightarrow{\langle \cdot, \cdot \rangle} \langle [M], \alpha \cup \beta \rangle$
 is a non-singular bilinear form.

Pf. By exercise, $\langle [M], \alpha \cup \beta \rangle = \langle [M] \cap \alpha, \beta \rangle = \langle PD(\alpha), \beta \rangle$

So the following diagram commutes:



By universal coefficients, $H^*(M; \mathbb{F}) \cong \text{Hom}(H_*(M; \mathbb{F}), \mathbb{F})$ via $\beta \mapsto \langle \beta, \cdot \rangle$

Hence \otimes is a non-degenerate bilinear pairing $\left(\text{using that for any } \mathbb{F}\text{-vector-space } V \otimes V^* \rightarrow \mathbb{F}, v \otimes \varphi \mapsto \varphi(v) \right)$

Hence so is the pairing \otimes in the diagram. \square (Hom(V, F) is non-deg. pairing.)

Remark For M non-orientable, the same holds for \mathbb{F} of characteristic 2, e.g. \mathbb{Z}_2 . For \mathbb{Z} coefficients it can fail if $H^*(M) \neq \text{Hom}(H_*(M), \mathbb{Z})$. So we define:

Betti group $B^k(M) = H^k(M) / \text{torsion}(H^k(M))$
 $B_k(M) = H_k(M) / \text{torsion}(H_k(M))$

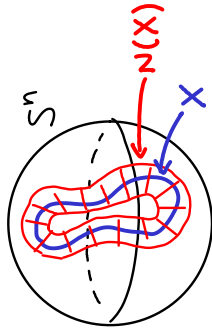
By what we proved in the section on universal coefficients, $B^1(M) \cong \text{Hom}(B_0(M), \mathbb{Z})$ whenever $H_{q-1}(M)$ is finitely generated (which we know holds for compact mfd's) The iso is given by $\langle \cdot, \cdot \rangle$ again: this descends to quotients since $\langle c, \alpha \rangle = 0 \in \mathbb{Z}$ if c or α has finite order (i.e. torsion). The same proof as above yields:

M compact oriented n -mfd $\Rightarrow B^k(M) \otimes B^{n-k}(M) \rightarrow \mathbb{Z}, \alpha \otimes \beta \mapsto \langle [M], \alpha \cup \beta \rangle$
 is non-degenerate bilinear form.

Also the Remark holds.

Example Use this to prove ex. 4(c) sheet 3. (Hint: $H^{2k}(\mathbb{C}P^n) \cup H^{2n-2k}(\mathbb{C}P^n) = H^{2n}(\mathbb{C}P^n)$)

Alexander duality



(in fact, enough to assume \$X\$ is locally contractible)

\$\emptyset \neq X \subseteq S^n\$ compact subset s.t.

\$\exists\$ open neighbourhood \$N(X)\$ which deformation retracts to \$X\$

such that \$\overline{N(X)} \subseteq S^n\$ is an \$n\$-mfd with boundary.

Theorem $\tilde{H}_*(X) \cong \tilde{H}^{n-*}(\overline{S^n \setminus X})$

Pf later

Example \$X \subseteq S^3\$ knot (i.e. \$X = \text{image}(S^1 \xrightarrow{\text{homeomorphism}} S^3)\$ onto the image)

\$\Rightarrow N(X) \cong \text{solid torus} \cong S^1\$

\$\Rightarrow \tilde{H}_0(X) = 0 = \tilde{H}^2(S^3 \setminus X)\$

\$\tilde{H}_1(X) = \mathbb{Z} = \tilde{H}^1(S^3 \setminus X)\$

\$\tilde{H}_2(X) = 0 = \tilde{H}^0(S^3 \setminus X)\$

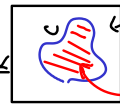
so the homology of a knot complement does not tell knots apart (always same)

Theorem (Jordan curve theorem)

\$C \cong S^1\$ closed curve in \$\mathbb{R}^2 \subseteq S^2\$

\$\Rightarrow \mathbb{R}^2 \setminus C\$ has 2 path-components (= connected components)

Similarly for \$S^n \cong C \subseteq \mathbb{R}^{n+1} \subseteq S^{n+1}\$.



\$\text{Pf } S^n \cong C \subseteq \mathbb{R}^{n+1} \subseteq S^{n+1} \Rightarrow \mathbb{Z} \cong \tilde{H}_n(S^n) \cong \tilde{H}^0(S^{n+1} \setminus C)\$

\$\Rightarrow H^0(S^{n+1} \setminus C) \cong \mathbb{Z}^2\$

\$\Rightarrow S^{n+1} \setminus C\$ has 2 path components. \$\square\$

Alexander duality

Proof Alexander duality

\$Y := S^n \setminus N \cong S^n \setminus X\$

for \$* \leq n-1\$

\$\tilde{H}^{n-*}(\overline{Y}) = H^{n-*}(\overline{Y})\$

\$\cong H_{*+1}(Y, \partial Y)\$

Lefschetz

\$\cong H_{*+1}(S^n, \overline{N})\$

exc.

\$\cong \tilde{H}_*(\overline{N})\$

LES using \$* < n-1\$

for \$* = n-1\$

\$\tilde{H}^0(Y) \oplus \mathbb{Z} \cong H^0(Y)\$

\$\cong H_n(Y, \partial Y)\$

\$\cong H_n(S^n, \overline{N})\$

\$\cong \tilde{H}_{n-1}(\overline{N}) \oplus \mathbb{Z}\$

Explanation of \$\mathbb{Z}\$:

LES: \$0 \rightarrow \tilde{H}_n(S^n) \rightarrow H_n(S^n, \overline{N}) \rightarrow \tilde{H}_{n-1}(\overline{N}) \rightarrow 0\$ is SES

\$\downarrow\$ quotient

\$H_n(S^n, S^n \setminus \infty) \cong \mathbb{Z}\$

\$\tilde{H}_n(\overline{N}) = H_n(\overline{N}) = 0\$ (see Cor. to Poincaré-Lefschetz) using: each (path-) connected component of the manifold \$\overline{N}\$ has non-empty boundary.



Hence that quotient map gives a splitting of the SES.

for \$* = n\$ \$H^{n-*}(\overline{Y}) = H^{-1}(Y) = 0\$

\$H_n(X) \cong H_n(N) \cong H^0(N, \partial N) = 0. \square\$

\$\uparrow\$ Lefschetz duality (see \$\star\$)