Homological Algebra

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Based on lectures by Kobi Kremnitzer

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Contents

1 Some Random Comments

Here are some comments that would have been helpful for me when I was learning Homological Algebra. They're not important at all, so feel free to ignore them.

I will sometimes use the word "obvious" in contexts like "this is defined in the obvious way". This doesn't mean that the thing is easy, but rather that it's defined in the only way that makes sense. It might still take some work to see what that way is.

Of the algebra-related courses in Michaelmas of fourth year, I think Homological Algebra and Category Theory are the two hardest. However, a lot of the difficulty is shared. Homological algebra is made much harder if you aren't familiar with categories and functors, and working with the concepts in homological algebra trivialises a fair bit of the category theory course. Therefore, I recommend taking both. Also, if you want to do a PhD in anything related to algebra, category theory is invaluable.

Homological algebra generally takes place on two different levels of abstraction, namely the less abstract R-modules vs the more abstract "abelian category". The distinction between these settings is often blurred (indeed the Freyd-Mitchell Embedding Thereom tells us that they are equivalent), and we will often work implicitly in R-modules, since it is much simpler to do so.

2 Exact Sequences and Homology

The concept of an exact sequence seems at first a little arbitrary, but it turns out to be incredibly useful. There is another type of sequence, called a **chain complex**, that is in some sense close to being exact. From an algebraic standpoint, **homology** measures the failure of a chain complex to be exact. The usual motivation for homology comes from Algebraic Topology, and I recommend learning at least the definitions of simplicial and singular homology, since they provide very useful context.

For now, we will work exclusively with R-modules. In the following section, we will generalise the notions to the setting of **abelian categories**.

2.1 Exactness

Definition 2.1. A **sequence** of R-modules is a collection $(A_n, f_n)_{n\in\mathbb{Z}}$, where the A_n are R-modules and the $f_n : A_n \to A_{n-1}$ are module homomorphisms.

Definition 2.2. A **cosequence**[1] of R-modules is a collection $(A^n, f^n)_{n\in\mathbb{Z}}$, where the A^n are R-modules, and the $f^n : A^n \to A^{n+1}$ are module homomorphisms.

Definition 2.3. The sequence

$$\ldots \to A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \to \ldots$$

is **exact at** A_n if $\operatorname{im} f_{n-1} = \ker f_n$.

Definition 2.4. A sequence is **exact** if it is exact at every term.

Exactness is defined in the same way for cosequences.

Example 2.5. Consider the sequence[2]

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to 0.$$

[1]This terminology is nonstandard, and a little silly, but I think it helps the flow of these notes.

[2]Being pedantic, we defined sequences to be indexed by $\mathbb{Z}$. This sequence, as written, has only four terms, but we can imagine there being infinitely many zeros on either side, so it does fit our definition of sequences.

This sequence is exact at the first copy of $\mathbb{Z}$, but not the second.

Example 2.6. More generally, a sequence

$$0 \to A \xrightarrow{f} B \to 0$$

is exact at A if and only if f is injective, and it is exact at B if and only if f is surjective. Therefore, the statement that

$$0 \to A \to B \to 0$$

is exact is equivalent to the statement that $A \to B$ is an isomorphism.

The most recent example hints at the power of exact sequences. They give us a new language for rephrasing familiar statements about algebra. The fact that we specify the maps involved allows us to discuss these concepts with more precision. For instance, the exact sequences

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z}, \qquad 0 \to \mathbb{Z} \xrightarrow{3} \mathbb{Z}$$

both tell us that $\mathbb{Z}$ has a subgroup isomorphic to $\mathbb{Z}$. However, the sequences are talking about different subgroups (namely $2\mathbb{Z}$ and $3\mathbb{Z}$ respectively), and they tell us exactly **where** those subgroups are.

Definition 2.7. Let $\{A_n\}$ and $\{B_n\}$ be sequences of R-modules. A **morphism** from $\{A_n\}$ to $\{B_n\}$ is a collection of maps $A_n \to B_n$ such the diagram

$$\begin{array}{ccccccccc} \dots & \longrightarrow & A_{n+1} & \longrightarrow & A_n & \longrightarrow & A_{n-1} & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \dots & \longrightarrow & B_{n+1} & \longrightarrow & B_n & \longrightarrow & B_{n-1} & \longrightarrow & \dots \end{array}$$

commutes.

Sequences form a category with these morphisms, and the identity morphism is given by the identity map $A_n \to A_n$ for each n. It is easy to check that a morphism of sequence is an isomorphism[1] if and only if each map $A_n \to B_n$ is an isomorphism.

[1] In the sense of category theory.

2.2 Short Exact Sequences

In algebra, we often take quotients of modules by submodules. The "exact sequence version" of this concept is the short exact sequence.

Definition 2.8. A **short exact sequence** is an exact sequence of the form

$$0 \to A \xrightarrow{i} B \xrightarrow{\pi} C \to 0.$$

In rough terms, up to isomorphism, a short exact sequence just tells us that A is a submodule of B and C is the quotient. The following lemma makes this precise.

Lemma 2.9. *Let*

$$0 \to A \xrightarrow{i} B \xrightarrow{\pi} C \to 0$$

be a short exact sequence. There is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\ i\ } & B & \xrightarrow{\ \pi\ } & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\cong} & & \| & & \downarrow{\scriptstyle\cong} & & \\ 0 & \longrightarrow & A' & \hookrightarrow & B & \twoheadrightarrow & B/A' & \longrightarrow & 0, \end{array}$$

where A' is a submodule of B and the maps on the bottom are the obvious maps.

Proof. By exactness at A, the map i is injective, so we may set $A' = i(A)$. Let $c \in C$. Since π is surjective (by exactness at C, there is some $b \in B$ with $\pi(b) = c$. Then define $\varphi(c) = b + A' \in B/A'$. The map φ is well-defined, since if $\pi(b) = \pi(b') = c$, then $b - b' \in \ker \pi = \operatorname{im} i = A'$, so $b + A' = b' + A'$. It is easy to check that φ is an R-module homomorphism, and that the diagram commutes. To see that φ is an isomorphism, we can define the inverse map

$$b + A' \mapsto \pi(b),$$

and use similar ideas to check that it is a well-defined inverse to φ. $\square$

The kernel and cokernel of a map have the following universal properties from category theory, which we take as definitions.

Definition 2.10 (Categorical kernel)**.** The **kernel** of a map $f : X \to Y$ is a morphism $i : K \to X$ such that

1. $f \circ i = 0$.
2. If $\tilde{i} : \tilde{K} \to X$ is a map with $f \circ \tilde{i} = 0$, then there is a unique morphism $\varphi : \tilde{K} \to K$ with $i \circ \varphi = \tilde{i}$.

In this situation, we say that i **exhibits** K **as the kernel of** f.

Definition 2.11 (Categorical cokernel)**.** The **cokernel** of a map $f : X \to Y$ is a morphism $q : Y \to C$ such that

1. $q \circ f = 0$.
2. If $\tilde{q} : Y \to \tilde{C}$ is a map with $\tilde{q} \circ f = 0$, then there is a unique map $\varphi : C \to \tilde{C}$ such that $\varphi \circ q = \tilde{q}$.

In this situation, we say that q **exhibits** C **as the cokernel of** f.

Lemma 2.12. *Let*

$$0 \to A \xrightarrow{i} B \xrightarrow{\pi} C \to 0$$

be a sequence. The following are equivalent:

1. *The sequence is exact.*
2. *The map i exhibits A as the kernel of π, and the map π exhibits C as the cokernel of i.*

Proof. This basically follows from Lemma 2.9, together with the explicit constructions of kernels and cokernels in R-**mod** (namely $\ker \pi = \{x \in B : \pi(x) = 0\}$ and $\operatorname{coker} i = B/\operatorname{im} i$). □

Definition 2.13. A short exact sequence is **split** if it is isomorphic to one of the form

$$0 \to A \to A \oplus B \to B \to 0,$$

where the maps are inclusion and projection.

Lemma 2.14 (Splitting Lemma)**.** *Let $0 \to A \xrightarrow{i} B \xrightarrow{\pi} C \to 0$ be a short exact sequence. The following are equivalent:*

1. *The short exact sequence is split.*
2. *There exists a retraction* $r : B \to A$.
3. *There exists a section* $s : C \to B$.

Proof. The proof, as I have presented it, is much longer than any other version I have seen elsewhere. This is because I have been very formalistic, being explicit about lots of details that most authors find "obvious". I'd probably recommend looking elsewhere for an easier proof to follow, and consulting this one if you get stuck on any details.

We will show that (1) implies (2), then that (2) is equivalent to (3), and finally that (2) and (3) implies (1).

Step 1: (1) $\implies$ ((2) and (3)).

Suppose that the short exact sequence is split (i.e. that (1) holds). Then we have an isomorphism of short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\ i\ } & B & \xrightarrow{\ \pi\ } & C & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle \mathrm{id}} & & \big\downarrow{\scriptstyle \varphi} & & \big\downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & A & \xrightarrow{\ i'\ } & A \oplus C & \xrightarrow{\ \pi'\ } & C & \longrightarrow & 0 \end{array}$$

where i' and π' are the natural maps.

Let $b \in B$. Define $r(b)$ to be the projection of $\varphi(b)$ onto A, and for $c \in C$ let $s(c) = \varphi^{-1}(0, c)$.

For $a \in A$, commutativity of the diagram tells us that $i'(a) = \varphi(i(a))$. Also, by definition of r we have $i'r(b) = \varphi(b)$ for all $b \in B$. It follows that

$$i'(a) = \varphi(i(a)) = i'ri(a),$$

so $a = ri(a)$ by injectivity of i', which means that r is a retraction. So (2) holds.

For $c \in C$, we have

$$c = \pi'(0, c) = \pi' \circ \varphi(s(c)) = \pi(s(c))$$

by commutativity of the diagram, so $\pi s = \mathrm{id}_C$, hence $s : C \to B$ is a section. So (3) holds.

Step 2: (2) $\iff$ (3).

Suppose that (2) holds. Let $r : B \to A$ be the retraction.

Let $c \in C$. We would like to define $s(c) = b - ir(b)$, where $b \in \pi^{-1}(c)$. Such a b exists because π is surjective.

To see that s is well-defined, suppose that $b, b' \in \pi^{-1}(c)$. Then $b - b' \in \ker \pi = \text{im } i$ so $b = i(a) + b'$ for some $a \in A$. It follows that

$$\begin{aligned} b - ir(b) &= i(a) + b' - ir(i(a) + b') \\ &= i(a) + b' - i(a) - ir(b') \\ &= b' - ir(b'). \end{aligned}$$

So s is well-defined.

We have $\pi \circ s(c) = \pi(b - ir(b)) = \pi(b)$, since $ir(b) \in \text{im } i = \ker \pi$. So $\pi s = \text{id}_C$, hence s is a section.

Suppose conversely the (3) holds, so a section $s : C \to B$ exists. Let $b \in B$. Then we have

$$\pi(b - s\pi(b)) = 0,$$

so $b - s\pi(b) \in \ker \pi = \text{im } i$, so $b - s\pi(b) = i(a)$ for some $a \in A$. By injectivity of i, we have a well-defined function $r : B \to A, b \mapsto a$. It is easy to check that this is a homomorphism (using the fact that i is a homomorphism).

Let $x \in A$, and write $b = i(x), a = r(b)$. Then we have $b - s\pi(b) = i(a)$, which means that

$$i(a) = i(x) - s\pi i(x) = i(x),$$

since $\pi i = 0$. So $i(a) = i(x)$, hence $a = x$ by injectivity of i, and we have $x = a = ri(x)$. So r is a retraction.

Step 3: (2) and (3) $\implies$ (1).

Suppose that we have a section $s : C \to B$ and a retraction $r : B \to A$. We define maps

$$B \underset{\psi}{\overset{\varphi}{\rightleftarrows}} A \oplus C$$

by $\varphi(b) = (r(b), \pi(b))$ and $\psi(a, c) = i(a) + s(c)$. It is easy to check that these maps are mutually inverse isomorphisms. □

2.3 Chain Complexes and Homology

Definition 2.15. A **chain complex of R-modules** is a sequence $C_* = \{C_n, d_n\}$ such that $d_n \circ d_{n+1} = 0$ for all n. The map d_n is called the **differential** of the chain complex.

Definition 2.16. A **cochain complex**[1] is a cosequenece with the condition that the composition of any two successive maps is zero.

Definition 2.17. A chain complex (resp. cochain complex) is called **acyclic** if it is an exact sequence (resp. cosequence).

In other words, a chain complex is a sequence such that each "double map" is zero. It is easy to see that chain complexes form a full subcategory of sequences, which we denote $\mathbf{Ch}_*(R)$.

Definition 2.18. A morphism of chain complexes (resp. cochain complexes) is called a **chain map** (resp. **cochain map**).

Remark 2.19. It is common to abuse notation and denote all differentials by d. This is justified more formally if we imagine that d is a function from the graded module $\oplus_n C_n$ to itself.

Remark 2.20. Annoyingly, there seems to be no agreed-upon name for the modules C_n that constitute the chain complex. I have seen these referred to in many ways, such as "the n^{th} space", "the degree n part", "the n^{th} term", etc.

The condition that the double differential is zero is equivalent to $\operatorname{im} d_{n+1} \subseteq \ker d_n$ for all n. Clearly then, we have a well-defined quotient module

$$\frac{\ker d_n}{\operatorname{im} d_{n+1}},$$

[1]Unlike the cosequence, this terminology is completely standard.

and the chain complex C_* is acyclic if and only if this quotient vanishes for all n. Therefore, we may view the quotient as some sort of "obstruction to exactness". It turns out that this obstruction is very useful. We call it the n^{th} **homology** of the chain complex C_*, and denote it by $H_n(C_*)$. Since this is so important, we will restate the definition more formally, while introducing some terminology.

Definition 2.21. Let C_* be a chain complex. An element $c \in C_n$ is called a **cycle** if $d_n c = 0$, and a **boundary** if $c = d_{n+1}c'$ for some $c' \in C_{n+1}$. We denote the cycles and boundaries in degree n by Z_nC and B_nC respectively. The n^{th} **homology** of C_* is the quotient module

$$H_n(C_*) = \frac{Z_nC}{B_nC}.$$

Definition 2.22. The corresponding notions for a cochain complex are called **cocycles** and **coboundaries**, and they are denoted Z^nC and B^nC respectively. The n^{th} **cohomology** is then defined to be

$$H^n(C) = \frac{Z^nC}{B^nC}.$$

Remark 2.23. The terms "cycle" and "boundary" are motivated by topology. In topology, we work with a chain complex whose differential takes a "thing" (the thing is kind of, but not really, a subspace of the topological space) to the boundary of the thing. For instance, if the thing is a path, then the differential is the endpoint minus the startpoint. If the thing is a disc, then the differential is (kind of) its boundary circle. By definition, then, an element of the image of the differential is a boundary. We call elements of the kernel "cycles", since in the case of a path, the endpoint minus the startpoint will be zero if and only if the path is a loop (i.e. it "cycles round").

Definition 2.24. The n^{th} **homology** of a chain complex C_* is the module

$$H_n(C_*) = \frac{Z_nC}{B_nC} = \frac{\ker d_n}{\operatorname{im} d_{n+1}}.$$

The slogan is that homology is "cycles modulo boundaries". Homology has the useful property that each H_n is a functor from chain complexes to R-modules. We now make this precise.

Lemma 2.25. *Let $f : C_* \to D_*$ be a morphism of chain complexes. There is a well-*

defined module homomorphism

$$H_n(f) : H_n(C_*) \to H_n(D_*),$$

given by

$$c + B_nC \mapsto f(c) + B_nD.$$

Proof. First of all, we claim that f restricts to a map $Z_nC \to Z_nD$. This follows from commutativity of

$$\begin{array}{ccc} C_n & \xrightarrow{d} & C_{n-1} \\ {\scriptstyle f_n}\downarrow & & \downarrow{\scriptstyle f_{n-1}} \\ D_n & \xrightarrow[d]{} & D_{n-1}, \end{array}$$

since any $c \in Z_nC$ has $d(f_nc) = f_{n-1}(dc) = f_{n-1}(0) = 0$, so $f_n(c) \in Z_nD$. Similarly, f takes boundaries to boundaries (use the same commutative diagram, but replace n with $n+1$), so the composition

$$Z_nC \to Z_nD \to Z_nD/B_nD$$

kills B_nC, hence descends to a well-defined homomorphism

$$H_n(C_*) \to H_n(D_*).$$

□

Definition 2.26. We call the homomorphism $H_n(f)$ the **induced homomorphism** of f.

When n is clear, we often denote the induced map $H_n(f)$ by f_*.

Lemma 2.27. *Let induced homomorphism has the following useful properties.*

1. *If $f, g : C_* \to D_*$ are chain maps, and $r \in R$, then*

$$(f + rg)_* = f_* + rg_*.$$

2. *If $f : C_* \to D_*$ and $g : D_* \to E_*$ are chain maps, then*

$$(g \circ f)_* = g_* \circ f_*.$$

3. For any chain complex C_*, we have $(\mathrm{id}_{C_*})_* = \mathrm{id}_{H_n(C)}$.

Proof. Follows straight from the definitions. □

The above lemma tells us that H_n is a functor $\mathbf{Ch}_*(R) \to R\text{-}\mathbf{mod}$, and also that the maps of Hom-sets are R-linear.

There are several notions of equivalence between chain complexes. The most obvious is isomorphism. Also quite straightforward is the notion of quasi-isomorphism.

Definition 2.28. A **quasi-isomorphism** is a chain map $f : C_* \to D_*$ such that $f_* : H_n(C_*) \to H_n(D_*)$ is an isomorphism for all n.

Remark 2.29. The relation "there exists a quasi-isomorphism from C_* to D_*" is **not** an equivalence relation, because it is not transitive[1]. The transitive closure of the relation is useful, since it allows us to define something called the **derived category** of chain complexes.

2.4 Homotopy

In topology, homotopy is an equivalence relation of maps, and we can use it to define the notion of homotopy equivalence of spaces. We mirror the development, using chain complexes instead of topological spaces.

Definition 2.30. Let $f, g : C_* \to D_*$ be chain maps. A **chain homotopy** from f to g is a collection of maps $h_n : C_n \to D_{n+1}$ such that for each n, we have

$$f_n - g_n = h_{n-1} \circ d_n^{(C)} + d_{n+1}^{(D)} \circ h_n.$$

If there is a chain homotopy from f to g, then we write $f \simeq g$.

[1] See `https://math.stackexchange.com/questions/93273/is-quasi-isomorphism-an-equivalence-relation` for a counterexample.

It is common to represent the situation by the following diagram.

$$\begin{array}{ccccc} C_{n+1} & \longrightarrow & C_n & \longrightarrow & C_{n-1} \\ & \swarrow^{h_n} & f \Big\downarrow \Big\downarrow g & \swarrow^{h_{n-1}} & \\ D_{n+1} & \longrightarrow & D_n & \longrightarrow & D_{n-1} \end{array}$$

Note that this diagram is not commutative. It is just helpful for seeing where the various maps point.

Lemma 2.31. *The relation $\simeq$ is an equivalence relation on chain maps $C_* \to D_*$.*

Proof. For reflexivity, just take $h_n = 0$ for all n. For symmetry, if h is a chain homotopy from f to g, then $-h$ is a homotopy from g to f. For transitivity, if h and h' are chain homotopies from f to g and g to k respectively, then $h + h'$ is a chain homotopy from f to k. □

Lemma 2.32. *Suppose that chain maps $f, g : C_* \to D_*$ are chain homotopic. Then the induced maps $f_*, g_* : H_n(C) \to H_n(D)$ are equal.*

Proof. Let h be a chain homotopy from f to g. We have

$$f_n - g_n = h_{n-1} \circ d_n^{(C)} + d_{n+1}^{(D)} \circ h_n,$$

for each n. Let $x \in H_n(C)$. Then $x = [c]$ for some cycle $c \in Z_n C$. We have

$$\begin{aligned} f_*(x) - g_*(x) &= [f_n(c) - g_n(c)] \\ &= [h_{n-1} \circ d_n^{(C)}(c) + d_{n+1}^{(D)} \circ h_n(c)] \\ &= [d_{n+1}^{(D)} \circ h_n(c)] \\ &= 0, \end{aligned}$$

where the third equality comes from the fact that c is a cycle. □

Now that we have defined homotopy of maps, we can define homotopy equivalence of chain complexes just as we did in topology.

Definition 2.33. Let C_* and D_* be chain complexes. A **chain homotopy equivalence** from C_* to D_* is a tuple (f, g, h_1, h_2), where

$$C_* \underset{g}{\overset{f}{\rightleftarrows}} D_*$$

are chain maps, h_1 is a homotopy from gf to id_{C_*}, and h_2 is a homotopy from fg to id_{D_*}. If a chain homotopy equivalence enxists, then we say that C_* and D_* are **chain homotopy equivalent**.

When it is clear what we mean (which is almost always), we drop the word "chain" from these terms, just referring to homotopies and homotopy equivalences.

Lemma 2.34. *If C_* and D_* are homotopy equivalent chain complexes via (f, g, h_1, h_2), then f_* and g_* are mutually inverse isomorphisms.*

Proof. We have

$$g_* \circ f_* = (g \circ f)_* = (\mathrm{id}_{C_*})_* = \mathrm{id}_{H_n(C)},$$

and similarly $f_* \circ g_* = \mathrm{id}_{H_n(D)}$. □

Corollary 2.35. *Homotopy equivalent chain complexes have isomorphic homology.*

Proof. Immediate. □

3 Abelian Categories

As was the case with Galois's Galois Theory, freshly invented concepts are often messy. It can take decades, or even centuries, for mathematicians to hammer the theory into a more elegant shape. This was not the case for Grothendieck, who spewed out modern homological algebra, more or less fully-formed, in his legendary "Tohoku Paper" of 1957.

Central to Grothendieck's treatment is a type of category called an "abelian category". This is basically a generalisation of R-**mod**. It retains the nice properties we took advantage of in the previous section, but allows us to extend the methods to many exotic areas of mathematics.

3.1 Roadmap

To reach abelian categories, we need a few definitions. These can be difficult to absorb, so we have broken the section up as much as possible. It may help to keep the following diagram in mind. Its meaning will become clear as the definitions are introduced.

$$
\begin{array}{c}
\text{Category} \\
\big\downarrow {\scriptstyle \text{Abelian group structure on Hom-sets}} \\
\text{Ab-enriched category} \\
\big\downarrow {\scriptstyle \text{0 object; Finite coproducts}} \\
\text{Additive category} \\
\big\downarrow {\scriptstyle \text{Kernels and cokernels}} \\
\text{Pre-abelian category} \\
\big\downarrow {\scriptstyle \text{Every mono is kernel of its cokernel; Every epi is cokernel of its kernel}} \\
\text{Abelian category}
\end{array}
$$

3.2 Some Terminology from Category Theory

We rattle off some definitions from category theory. Hopefully, these are familiar.

Definition 3.1. Let $\mathcal{C}$ be a category, and let $x \in \mathcal{C}$. We say that x is **terminal** if for every $c \in \mathcal{C}$, there is exactly one morphism $c \to x$. Dually, we say that x is **initial** if for every $c \in \mathcal{C}$, there is exactly one morphism $x \to c$.

Definition 3.2. A **zero object** in a category is an object that is both initial and terminal.

Definition 3.3. A **monomorphism** is a morphism f such that $fg_1 = fg_2 \implies g_1 = g_2$ for any morphisms g_1, g_2. Dually, an **epimorphism** is a morphism f such that $g_1 f = g_2 f \implies g_1 = g_2$ for any g_1, g_2.

3.3 Ab-enriched Categories

Definition 3.4. A **pre-additive** or **Ab-enriched** category is a category in which every hom-set is equipped with the structure of an abelian group, such that the composition

$$\mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$$

is $\mathbb{Z}$-bilinear.

Proposition 3.5. *In an* **Ab***-enriched category, any initial object is also terminal.*

Proof. Let $*$ be initial. Then id_* is the unique element of $\mathrm{Hom}(*, *)$, so id_* is zero in this group. Then since composition respects the group structures, we have for any map $f : * \to A$,

$$f = f \circ \mathrm{id}_* = f \circ 0 = 0$$

so $*$ is terminal. □

Proposition 3.6. *If $\mathcal{C}$ is an* **Ab***-enriched category, then so is its opposite category $\mathcal{C}^{\mathrm{op}}$.*

Proof. For $X, Y \in \mathcal{C}^{\mathrm{op}}$, the sets

$$\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$$

are already endowed with the structure of an abelian group. Thus, we have only to prove that composition is bilinear. Let $X, Y, Z \in \mathcal{C}$ and let

$$f, f' \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y), \quad g \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(Y, Z).$$

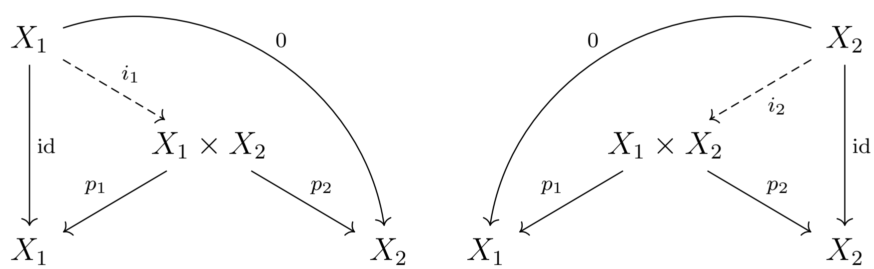
Then

$$g \circ_{\mathrm{op}} (f + f') = (f + f') \circ g = f \circ g + f' \circ g = g \circ_{\mathrm{op}} f + g \circ_{\mathrm{op}} f'.$$

Similarly composition is bilinear in the other argument as well. □

Proposition 3.7. *In an* **Ab**-*enriched category, a binary product is also a binary coproduct.*

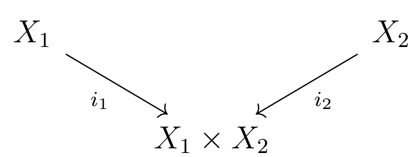
Proof. Let X_1, X_2 be elements of an **Ab**-enriched category $\mathcal{C}$. Suppose that X_1 and X_2 have a product $X_1 \times X_2$ in $\mathcal{C}$, with projections $p_k : X_1 \times X_2 \to X_k$. By definition of products, there are unique morphisms $i_k : X_k \to X_1 \times X_2$ such that the following diagrams commute.



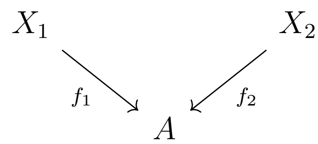
Then we have

$$p_1 \circ (i_1 p_1 + i_2 p_2) = p_1, \quad p_2 \circ (i_1 p_1 + i_2 p_2) = p_2.$$

By defintion of products, id : $X_1 \times X_2 \to X_1 \times X_2$ is the unique morphisms with $p_k \circ \mathrm{id} = p_k$ for each k, so $i_1 p_1 + i_2 p_2 = \mathrm{id}_{X_1 \times X_2}$. We claim that



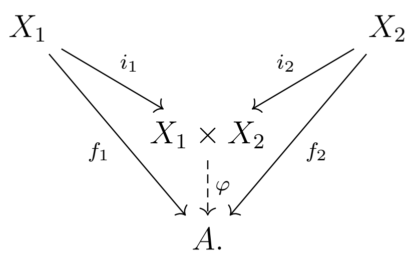
is a universal cocone, so that $X_1 \times X_2 = X_1 \coprod X_2$. Suppose that



is another cocone. Then we have a map

$$\varphi = f_1 \circ p_1 + f_2 \circ p_2 : X_1 \times X_2 \to A,$$

which is easily seen to give a commutative diagram



It remains to show that φ is unique. To see this, note that for any such φ we have

$$\begin{aligned}\varphi &= \varphi \circ \mathrm{id}_{X_1 \times X_2} \\ &= \varphi \circ (i_1 p_1 + i_2 p_2) \\ &= \varphi i_1 \circ p_1 + \varphi i_2 \circ p_2 \\ &= f_1 \circ p_1 + f_2 \circ p_2.\end{aligned}$$

□

Proposition 3.8. *In an* **Ab***-enriched category, all binary coproducts are also binary products.*

Proof. This is dual to Proposition 3.7. We will explain the duality explicitly.

Let $\mathcal{C}$ be **Ab**-enriched, and let $X_1, X_2 \in \mathcal{C}$ have a coproduct $X_1 \coprod X_2$. Then the object $X_1 \coprod X_2$ is also an object of $\mathcal{C}^{\mathrm{op}}$ (since $\mathcal{C}$ and $\mathcal{C}^{\mathrm{op}}$ have the same objects), and it is a product of X_1 and X_2 in the category $\mathcal{C}^{\mathrm{op}}$. Since $\mathcal{C}$ is **Ab**-enriched, so is $\mathcal{C}^{\mathrm{op}}$, which means that all binary products are binary coproducts in $\mathcal{C}^{\mathrm{op}}$. Since $X_1 \coprod X_2$ is a product in $\mathcal{C}^{\mathrm{op}}$, it is therefore is also a coproduct in $\mathcal{C}^{\mathrm{op}}$, which makes it a product of

X_1 and X_2 in $\mathcal{C}$. □

By Propositions 3.7 and 3.8, binary products and binary coproducts are the same object in an **Ab**-enriched category. This motivates the following definition.

Definition 3.9. Let $\mathcal{C}$ be an **Ab**-enriched category, and let $x, y \in \mathcal{C}$. If x and y have a product in $\mathcal{C}$, then it is called the **biproduct** of x and y, which we denote by $x \oplus y$.

Definition 3.10. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor between **Ab**-enriched categories. Then F is said to be **additive** if it preserves finite biproducts.

Lemma 3.11. *For any ring R, the category R-**mod** is **Ab**-enriched.*

Proof. For any two left R-modules A and B, the group $\mathrm{Hom}_R(A, B)$ is naturally an abelian group under pointwise addition. It is easy to check that composition is bilinear. □

3.4 Additive Categories

Definition 3.12. A category is **additive** if it is Ab-enriched and admits finite coproducts.

Lemma 3.13. *Let $\mathcal{A}$ be an additive category. Suppose that $i : a \to b$ is a monomorphism in $\mathcal{A}$ and $i \in \mathrm{Hom}(a, b)$ is the zero morphism. Then $a = 0$.*

Proof. Let $x \in \mathcal{A}$. Since $\mathrm{Hom}(a, x)$ is an abelian group, it contains at least one morphism (zero). Let $f : a \to x$ be any morphism. Then

$$\alpha \circ 0 = 0 = \alpha \circ f.$$

Since α is a monomorphism, we have $f = 0$. Therefore a is initial, hence it is the zero object. □

Lemma 3.14. *Let $\mathcal{A}$ be an additive category. Suppose that $q : a \to b$ is an epimorphism in $\mathcal{A}$. If $q = 0$, then $b = 0$.*

Proof. Since $\mathcal{A}$ is additive, the opposite category $\mathcal{A}^{\text{op}}$ is too. The map q is a monomorphism $q : b \rightarrowtail a$ in $\mathcal{A}^{\text{op}}$, and it is still the zero morphism. Therefore by Lemma 3.13, b is the zero object in $\mathcal{A}^{\text{op}}$, hence in $\mathcal{A}$. □

Lemma 3.15. *For any ring R, the category R-**mod** is additive.*

Proof. We know that the direct sum exists and is a coproduct in R-**mod**. □

3.5 Pre-abelian Categories

Definition 3.16. An additive category is **pre-abelian** if every morphism has a kernel and cokernel.

Lemma 3.17. *Let $\mathcal{A}$ be a pre-abelian category. Every monomorphism has kernel 0, and every epimorphism has cokernel 0.*

Proof. Let $i : a \rightarrowtail b$ be a monomorphism in $\mathcal{A}$. Let

$$\operatorname{Ker} i \xrightarrow{\ker i} a$$

be the kernel of i. Then $i \circ \ker i = 0 = i \circ 0$, so $\ker i$ is the zero morphism (since i is a monomorphism). Since $\ker i$ is a monomorphism, we have $\operatorname{Ker} i = 0$. □

Lemma 3.18. *For any ring R, the category R-**mod** is pre-abelian.*

Proof. Let $f : A \to B$ be a morphism in R-**mod**. It is easy to check that $\operatorname{Ker} f = \{a \in A : f(a) = 0\}$ is a kernel in the categorical sense. Similarly, $\operatorname{Coker} f = B/\operatorname{Im} f$ is a categorical cokernel. □

3.6 Abelian Categories

Definition 3.19. An pre-abelian category is **abelian** if every monomorphism is the kernel of its cokernel and every epimorphism is the cokernel of its kernel.

We explain a bit more what this definition means. Let $i : x \to y$ be a monomorphism in an abelian category $\mathcal{A}$. Then there is a natural map $q : y \to \operatorname{Coker} i$. The content of (the first part of) the definition is that

$$x \xrightarrow{i} y \overset{q}{\underset{0}{\rightrightarrows}} \operatorname{Coker} i$$

is a universal cone, so that we may say $x = \operatorname{Ker} q$. The statement that every epimorphism is the cokernel of its kernel is similar.

Lemma 3.20. *The category of left R-modules is an abelian category.*

Proof. Let $i : A \to B$ be a monomorphism of R-modules. Then $\operatorname{Coker} i = B/i(A)$ and the cokernel map is the quotient $q : B \to i(A)$ with $q(b) = b + i(A)$. It is clear that $i(A) = \operatorname{Ker} q$ in the set-theoretic sense, so i exhibits A as the kernel of q.

Let $q : A \to B$ be an epimorphism of R-modules. Let $i : \operatorname{Ker} q \to A$ be the inclusion. Then $\operatorname{Coker} i = A/\operatorname{Ker} q \cong B$, so q exhibits B as the cokernel of i. □

Lemma 3.21. *If $\mathcal{A}$ is abelian, then so is $\mathcal{A}^{\mathrm{op}}$.*

Proof. The notions of kernel and cokernel are dual, as are monomorphisms and epimorphisms. In particular, if i is a monomorphism in $\mathcal{A}^{\mathrm{op}}$, then it is an epimorphism in $\mathcal{A}$, so it is the kernel of its cokernel in $\mathcal{A}$, which means that it is cokernel of its kernel in $\mathcal{A}^{\mathrm{op}}$. □

Remark 3.22. In my opinion, the preceding proof is not a rigorous argument, since there are a lot of things to check that I have taken to be obvious. This sort of argument "by duality" is pretty common, and checking all the details is often hard.

Lemma 3.23. *If $\mathcal{A}$ is an abelian category and $\mathcal{C}$ is any category, then Fun$(\mathcal{C}, \mathcal{A})$ is abelian.*

Proof. See `https://math.stackexchange.com/questions/3042724/the-functor-category-aj-is-abelian-category-if-a-is-abelian`. □

3.7 Connection with $R-$mod

Lemma 3.24. *The category R-**mod** is an abelian category.*

Proof. It is easy to check that R-**mod** is pre-abelian. Let $i : M \to N$ be a monomorphism (i.e. an injective homomorphism of R-modules).

The cokernel of i is the quotient map $\pi : N \to N/\operatorname{im} i$. Then we have a short exact sequence

$$0 \to M \xrightarrow{i} N \xrightarrow{\pi} N/\operatorname{im} i \to 0,$$

which means that i exhibits M as the kernel of π, so i is the kernel of its cokernel. Showing that every epimorphism is the cokernel of its kernel is similar. □

There are certainly abelian categories other than R-**mod** but, miraculously, there is a partial converse to Lemma 3.24. In the statement of the following theorem, we refer to an **exact functor**. We haven't defined this properly yet, but it basically just means that if a sequence is exact, then so is its image under the functor.

Theorem 3.25 (Freyd-Mitchell Embedding Theorem)**.** *Let $\mathcal{A}$ be a small abelian category. Then there is a ring R and an exact, fully faithful functor $F : \mathcal{A} \to R$-**mod**. This functor embeds $\mathcal{A}$ as a full subcategory in $R-mod$, by which we mean that for all $M, N \in \mathcal{A}$, we have*

$$\operatorname{Hom}_{\mathcal{A}}(M, N) \cong \operatorname{Hom}_R(F(M), F(N)).$$

Proof. See Weibel, Page 25, Theorem 1.6.1. □

Lemma 3.26. *The Freyd-Mitchell embedding preserves kernels and cokernels.*

Proof. Let $f : x \to y$ be a morphism in an abelian category $\mathcal{A}$, and let $F : \mathcal{A} \to R$-**mod** be the Freyd-Mitchell embedding. Consider the sequence

$$0 \to \operatorname{Ker} f \xrightarrow{i} x \xrightarrow{f} y \xrightarrow{q} \operatorname{Coker} f \to 0.$$

□

Lemma 3.27. *Let $\mathcal{A}$ be an abelian category and let $F : \mathcal{A} \to R$-**mod** be the embedding from Theorem 3.25. Then $F(0) = 0$.*

Proof. We have $\operatorname{Hom}_{R\text{-}\mathbf{mod}}(F(0), F(0)) \cong \operatorname{Hom}_{\mathcal{A}}(0, 0)$, so there is only one R-module homomorphism $F(0) \to F(0)$. For any R-module M, we have homomorphisms $0 : M \to M$ and $\operatorname{id}_M : M \to M$, which means that $\operatorname{id}_{F(0)} = 0$, so $F(0) = 0$. □

The following proof is a powerful application of the Freyd-Mitchell Embedding Theorem.

Theorem 3.28 (Snake Lemma)**.** *Let $\mathcal{A}$ be an abelian category, and suppose that we have a diagram in A of the form*

$$\begin{array}{ccccccc} & & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \\ 0 & \longrightarrow & A' & \xrightarrow[f']{} & B' & \xrightarrow[g']{} & C' & & \end{array}$$

where the rows are exact. Then there is a morphism $\delta : \operatorname{Ker} c \to \operatorname{Coker} a$ such that the following sequence is exact

$$\operatorname{Ker} a \to \operatorname{Ker} b \to \operatorname{Ker} c \xrightarrow{\delta} \operatorname{Coker} a \to \operatorname{Coker} b \to \operatorname{Coker} c.$$

Furthermore, if f is a monomorphism, then so is the natural map $\operatorname{Ker} a \to \operatorname{Ker} b$, and if g' is an epimorphism then so is the natural map $\operatorname{Coker} b \to \operatorname{Coker} c$.

Proof. Suppose first that $\mathcal{A} = R$-**mod**. Let $z \in \operatorname{Ker} c$. Since g is onto, there is some $y \in B$ such that $z = g(y)$. We have $g'b(y) = cg(y) = c(z) = 0$, so $b(y) \in \operatorname{Ker} g' = \operatorname{Im} f'$, so there is some $x' \in A'$ such that $f'(x') = b(y)$.

Suppose that y_1, y_2 and x_1', x_2' are choices for y, x' above. Then we have $g(y_1) = g(y_2) = z$, so $y_1 - y_2 \in \operatorname{Ker} g = \operatorname{Im} f$, so $y_1 - y_2 = f(x)$ for some $x \in A$. Now $f'(x_1' - x_2') = b(y_1 - y_2) = bf(x) = f'a(x)$. The map f' is injective, so $a(x) = x_1' - x_2'$, hence $x_1' + \operatorname{Im} a = x_2' + \operatorname{Im} a$ as elements of $A'/\operatorname{Im} a = \operatorname{Coker} a$. Therefore we have a well-defined map $\delta : \operatorname{Ker} c \to \operatorname{Coker} a$ given by

$$\delta(z) = x + \operatorname{Im} a, \quad \text{where } f'(x) = b(y) \text{ for some } y \in B \text{ with } g(y) = z.$$

We claim that δ is a an R-module homomorphism. Let $z_1, z_2 \in \operatorname{Ker} c$ and for each i let (x_i, y_i) be the pair in the defition of $\delta(x_i)$. Then we have $g(y_1 - y_2) = z_1 - z_2$ and $f'(x_1 - x_2) = b(y_1 - y_2)$, so $\delta(z_1 - z_2) = x_1 - x_2$. Similarly, for $r \in R$ we have $\delta(rx_1) = r\delta(x_1)$. Therefore δ is indeed a module homomorphism.

The final part of the theorem follows immediately from the definitions of the natural maps. Therefore the theorem is proved in R-**mod**.

Now let $\mathcal{A}$ be any abelian category. Let $F : \mathcal{A} \to R$-**mod** be the embedding from Theorem 3.25. Then

$$\begin{array}{ccccccc} & & F(A) & \xrightarrow{F(f)} & F(B) & \xrightarrow{g} & F(C) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle F(a)} & & \downarrow{\scriptstyle F(b)} & & \downarrow{\scriptstyle F(c)} & & \\ 0 & \longrightarrow & F(A') & \xrightarrow[F(f)']{} & F(B') & \xrightarrow[F(g)']{} & F(C') & & \end{array}$$

is a diagram in R-**mod** with exact rows. Then by the case in R-**mod**, there is a morphism $\delta_* : \operatorname{Ker} F(c) \to \operatorname{Coker} F(a)$ such that

$$\operatorname{Ker}(F(a)) \to \operatorname{Ker}(F(b)) \to \operatorname{Ker}(F(c)) \xrightarrow{\delta_*} \operatorname{Coker}(F(a)) \to \operatorname{Coker}(F(b)) \to \operatorname{Coker}(F(c))$$

is exact. □

Definition 3.29. A **short exact sequence of chain complexes** is a sequence

$$0 \to A_* \to B_* \to C_* \to 0$$

of chain complexes, such that each

$$0 \to A_n \to B_n \to C_n \to 0$$

is a short exact sequence.

Corollary 3.30 (Sometimes also called the Snake Lemma)**.** *Suppose that $0 \to A_* \to B_* \to C_* \to 0$ is a short exact sequence of chain complexes in an abelian category $\mathcal{A}$. Then for each n, there is a **connecting map** $\delta_n : H_n(C) \to H_{n-1}(A)$ such that we have a long exact sequence*

$$\ldots \to H_n(C) \xrightarrow{\delta_n} H_{n-1}(A) \to H_{n-1}(B) \to H_{n-1}(C) \to \ldots .$$

Proof. This follows from Lemma 3.28, applied to the diagram

$$\begin{array}{ccccccccc} & & A_n & \longrightarrow & B_n & \longrightarrow & \frac{C_n}{\partial_{n+1}C_{n+1}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Z_{n-1}A & \longrightarrow & Z_{n-1}B & \longrightarrow & \frac{Z_{n-1}B}{Z_{n-1}A}. & & \end{array}$$

It is somewhat nontrivial what the maps in this diagram are, but if you think about it, you'll see that there is only one thing they could be, and that they are actually well-defined with exact rows. □

Remark 3.31. We can also prove this directly (and I think it makes more sense to do so) using a diagram chase in R-**mod**, and then applying the Freyd-Mitchell Embedding Theorem to transfer the result to general abelian categories.

Remark 3.32. It is actually possible to avoid the FM Embedding Theorem and do diagram chases rigorously in any abelian category. For more information, see `https://unapologetic.wordpress.com/2007/09/28/diagram-chases-done-right/`.

4 Exact Functors

Since we care about exact sequences, it seems reasonable to study functors that preserve exactness. In fact, it will be fruitful to study slightly more general functors as well, that only preserve exactness on the left or on the right.

4.1 Left- and Right- Exact Functors

Definition 4.1. A functor F is left-exact if for every short exact sequence $0 \to A \to B \to C \to 0$, the sequence

$$0 \to F(A) \to F(B) \to F(C)$$

is exact. Similarly, F is right-exact if instead

$$F(A) \to F(B) \to F(C) \to 0$$

is always exact.

Lemma 4.2. *If $F : \mathcal{A} \to \mathcal{B}$ is left-exact, and i is a monomorphism in $\mathcal{A}$, then $F(i)$ is a monomorphism in $\mathcal{B}$.*

Proof. If $i : A \to B$ is a monomorphism, then we have a short exact sequence

$$0 \to A \to B \to \operatorname{coker} i \to 0.$$

Therefore, $0 \to F(A) \to F(B)$ is exact, so $F(i)$ is a monomorphism. □

Lemma 4.3. *Let $F : \mathcal{A} \to \mathcal{B}$ be a functor between abelian categories. The following are equivalent:*

1. *F is left-exact.*
2. *For any exact sequence $0 \to A \to B \to C$, the corresponding sequence $0 \to F(A) \to F(B) \to F(C)$ is also exact.*

Proof. It is trivial that (2) $\implies$ (1). Suppose that (1) holds. Let $0 \to A \xrightarrow{i} B \xrightarrow{\pi} C$ be exact. Then we have a short exact sequence $0 \to A \to B \to \operatorname{im}\pi \to 0$, and therefore the sequence

$$0 \to F(A) \to F(B) \to F(\operatorname{im}\pi)$$

is exact. Now, $\operatorname{im}\pi \to C$ is a monomorphism, so $F(\operatorname{im}\pi) \to F(C)$ is too by Lemma 4.2. Therefore

$$\begin{aligned}\ker\big(F(B) \to F(C)\big) &= \ker\big(F(B) \to F(\operatorname{im}\pi) \to F(C)\big)\\ &= \ker\big(F(B) \to F(\operatorname{im}\pi)\big)\\ &= \operatorname{im}\big(F(A) \to F(B)\big).\end{aligned}$$

□

Corollary 4.4. *Let $F : \mathcal{A} \to \mathcal{B}$ be a functor between abelian categories. The following are equivalent:*

1. *F is right-exact.*
2. *For any exact sequence $A \to B \to C \to 0$, the corresponding sequence $F(A) \to F(B) \to F(C) \to 0$ is exact.*

Proof. This follows from Lemma 4.3 by duality. □

Lemma 4.5. *Let $\mathcal{A}$ be an abelian category, and consider maps*

$$A \xrightarrow{f} B \xrightarrow{g} C$$

in $\mathcal{A}$. Suppose that for all $Z \in \mathcal{A}$, the sequence

$$\operatorname{Hom}(A, Z) \xleftarrow{-\circ f} \operatorname{Hom}(B, Z) \xleftarrow{-\circ g} \operatorname{Hom}(C, Z) \leftarrow 0$$

is exact. Then $A \to B \to C \to 0$ is exact.

Proof. We need to show that g exhibits C as the cokernel of f. Suppose that $\alpha : B \to Z$ is some map with $\alpha \circ f = 0$. Then

$$\alpha \in \ker(- \circ f) = \operatorname{im}(g \circ -),$$

so $\alpha = \varphi \circ g$ for a unique map $\varphi : C \to Z$. This is precisely the universal property of the cokernel. ☐

Lemma 4.6. *Suppose we have an adjunction*

$$\mathcal{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{B}$$

of additive functors between abelian categories, where F is the left adjoint. Then F is right-exact.

Proof. Let

$$0 \to A \to B \to C \to 0$$

be a short exact sequence in $\mathcal{A}$, and let $Z \in \mathcal{B}$. Then $G(Z) \in \mathcal{A}$, so

$$\mathrm{Hom}(A, G(Z)) \overset{-\circ i}{\leftarrow} \mathrm{Hom}(B, G(Z)) \overset{-\circ \pi}{\leftarrow} \mathrm{Hom}(C, G(Z)) \leftarrow 0$$

is exact by left-exactness of Hom. Therefore,

$$\mathrm{Hom}(F(A), Z) \overset{-\circ i}{\leftarrow} \mathrm{Hom}(F(B), Z) \overset{-\circ \pi}{\leftarrow} \mathrm{Hom}(F(C), Z) \leftarrow 0$$

is exact, so

$$F(A) \to F(B) \to F(C) \to 0$$

is exact by Lemma 4.5. ☐

Corollary 4.7. *If F, G are as above, then G is left exact.*

Proof. We apply some hand-wavy duality. It might be worth working through the details here to convince yourself that the claims are true. Since G is a right adjoint, the functor $G : \mathcal{D}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ is a left adjoint, so it is right exact. Therefore, $G : \mathcal{D} \to \mathcal{C}$ is left exact. ☐

4.2 Exact Functors

Definition 4.8. A functor is **exact** if it is left-exact and right-exact.

Lemma 4.9. *Suppose that we have a long exact sequence*

$$\ldots \to A_{n-1} \overset{f_{n-1}}{\to} A_n \overset{f_n}{\to} A_{n+1} \to \ldots$$

and and exact functor F. Then

$$\ldots \to F(A_{n-1}) \to F(A_n) \to F(A_{n+1}) \to \ldots$$

is also exact.

Proof. Since we only have to check exactness at each term, it suffices to show that for an exact sequence

$$A \overset{f}{\to} B \overset{g}{\to} C,$$

the sequence

$$F(A) \overset{F(f)}{\to} F(B) \overset{F(g)}{\to} F(C)$$

is also exact. We prove this with a diagram-chase. Note that

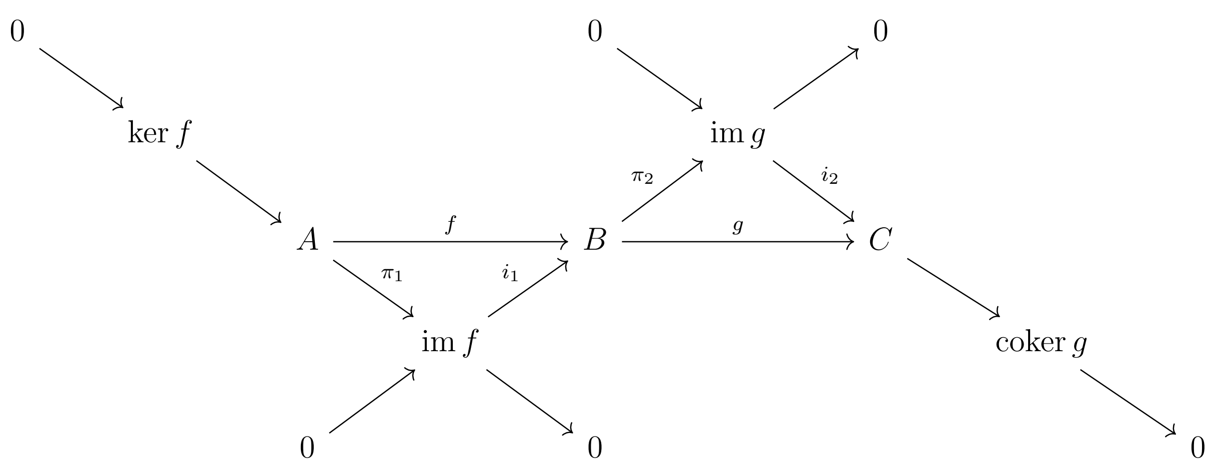
$$0 \to \ker f \to A \to \operatorname{im} f \to 0,$$

$$0 \to \ker g \to B \to \operatorname{im} g \to 0,$$

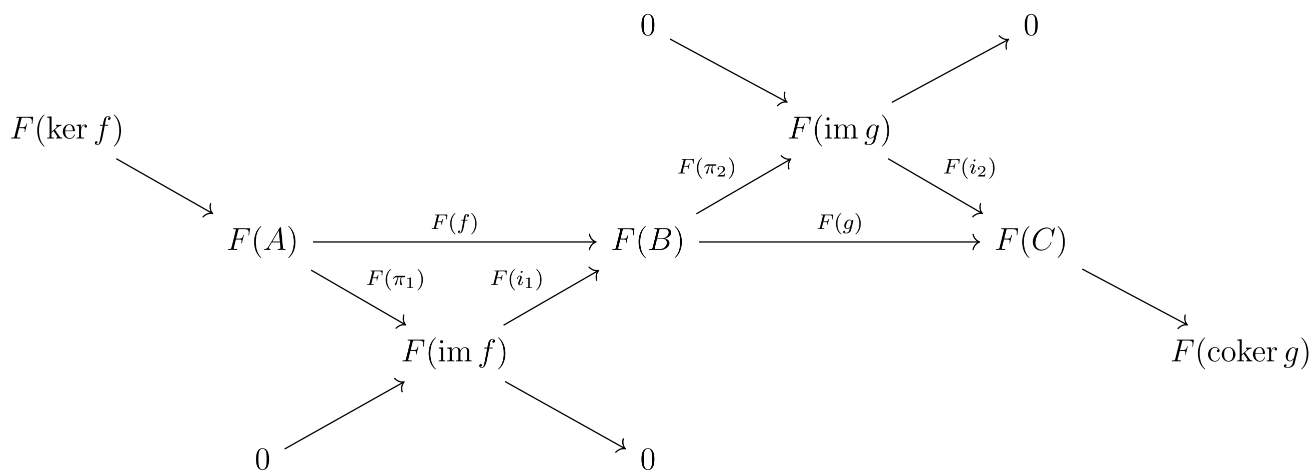
and

$$0 \to \operatorname{im} g \to C \to \operatorname{coker} g \to 0$$

are short exact sequences. We can fit these short exact sequences into the following commutative diagram:



Note that the diagonals are exact. Applying F to the diagram (and removing some redundant terms) gives a commutative diagram:



Again the diagonals are exact. Since $F(\pi_1)$ is surjective, we have $\operatorname{im} F(f) = \operatorname{im} F(i_1) = \ker F(\pi_2)$ by exactness of at $F(B)$. But $F(i_2)$ is injective, so $\ker F(g) = \ker F(\pi_2)$, and it follows that

$$\operatorname{im} F(f) = \ker F(\pi_2) = \ker F(g).$$

□

4.3 Specific Functors

Example 4.10. The following functors are **not** exact:

1. $- \otimes_{\mathbb{Z}} \mathbb{Z}/2$,
2. $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/2, -)$,
3. $\operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{Z}/2)$.

To see this, consider the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/2 \to 0.$$

Each functor takes this to a sequence that is not exact.

Lemma 4.11. *Let $\mathcal{A}$ be an abelian category, and let $M \in \mathcal{A}$ be an object. Then the functor* $\mathrm{Hom}_{\mathcal{A}}(M, -) : \mathcal{A} \to \boldsymbol{Ab}$ *is left-exact.*

Proof. Let $0 \to A \xrightarrow{i} B \xrightarrow{\pi} C \to 0$ be a short exact sequence in $\mathcal{A}$. We have to show that the sequence

$$0 \to \mathrm{Hom}(M, A) \xrightarrow{i\circ -} \mathrm{Hom}(M, B) \xrightarrow{\pi\circ -} \mathrm{Hom}(M, C)$$

is exact. For exactness at $\mathrm{Hom}(M, A)$, suppose that $i \circ \varphi = 0$, where $\varphi : M \to A$ is a map. Since i is a monomorphism and $i \circ \varphi = i \circ 0$, we have $\varphi = 0$. Therefore, the sequence is exact at $\mathrm{Hom}(M, A)$.

Since $\pi \circ i = 0$, we have $\mathrm{im}(i \circ -) \subseteq \ker(\pi \circ -)$. Let $\varphi \in \ker(\pi \circ -)$. Then $\pi \circ \varphi = 0$. Since i exhibits A as the kernel of π, there is a unique map $f : M \to A$ such that $i \circ f = \varphi$. Therefore, the sequence is exact at $\mathrm{Hom}(M, B)$. □

Corollary 4.12. *Let $\mathcal{A}$ be an abelian category and let $M \in \mathcal{A}$ be an object. Then the functor*

$$\mathrm{Hom}_{\mathcal{A}}(-, M) : \mathcal{A}^{op} \to \boldsymbol{Ab}$$

is left-exact.

Proof. This follows from the definition of the opposite category. □

Corollary 4.13. *Let R be a ring and M be an R-module. Then the functors*

$$\mathrm{Hom}_R(M, -) : R\text{-}\mathbf{mod} \to \boldsymbol{Ab}, \quad \mathrm{Hom}_R(-, M) : R\text{-}\mathbf{mod}^{op} \to \boldsymbol{Ab}$$

are left-exact.

Lemma 4.14. *For any ring R, the functor $- \otimes_R N : R$-**mod** $\to R$-**mod** is right-exact.*

Proof. This follows from the adjunction

$$(- \otimes_R N) \dashv \mathrm{Hom}_R(N, -)$$

and Lemma 4.6. □

5 Projectives and Injectives

For an object P of an abelian category $\mathcal{A}$, the functor $\mathrm{Hom}_{\mathcal{A}}(P, -)$ always left exact. In this section, we study objects P such that the functor is also right exact. Such objects are called projective. We also study the dual notion of injective objects.

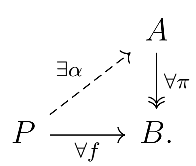
5.1 Projective Objects

Definition 5.1. Let $\mathcal{A}$ be an abelian category. An object $P \in \mathcal{A}$ is said to be **projective** if $\mathrm{Hom}_{\mathcal{A}}(P, -) : \mathcal{A} \to \mathbf{Ab}$ is an exact functor.

We opted for the above definition of projective objects because it is the easiest to state. However, the following equalence is very important, and Weibel uses it to define projective objects.

Lemma 5.2. *Let P be an object of an abelian category $\mathcal{A}$. The following are equivalent:*

1. *The object P is projective.*
2. *The functor $\mathrm{Hom}_{\mathcal{A}}(P, -)$ sends epimorphisms to epimorphisms.*
3. *For every epimorphism $\pi : A \to B$ and every morphism $f : P \to B$, there exists a morphism $\alpha : P \to A$ such that the following diagram commutes:*



Proof. Clearly (1) $\implies$ (2).

For (2) $\implies$ (1), recall from Lemma 4.11 that $F = \mathrm{Hom}_{\mathcal{A}}(P, -)$ is left-exact. Let

$$0 \to A \to B \to C \to 0$$

be a short exact sequence. Then by left-exactness of F, the sequence

$$0 \to F(A) \to F(B) \to F(C) \to 0$$

is exact at $F(A)$ and at $F(B)$. But the map $B \to C$ is an epimorphism, so (2) implies that $F(B) \to F(C)$ is also an epimorphism, which means that we have exactness at $F(C)$ as well.

Condition (3) basically just says that when π is an epimorphism, the natural map

$$\mathrm{Hom}(P, B) \to \mathrm{Hom}(P, A)$$

is surjective (i.e., for every $f \in \mathrm{Hom}(P, B)$, there is some $\alpha \in \mathrm{Hom}(P, A)$ that maps to f). This precisely (2), so (2) $\iff$ (3). □

Definition 5.3. We say that an R-module is **projective** if it is a projective object in R-**mod**.

Lemma 5.4. *Free R-modules are projective.*

Proof. Let $F = \oplus_i Re_i$ be a free R-module with basis $\{e_i : i \in I\}$. Suppose that we have a diagram

$$\begin{array}{ccc} & & A \\ & & \downarrow\!\!\downarrow \pi \\ F & \xrightarrow[f]{} & B. \end{array}$$

Since π is surjective, for each i there is some $a_i \in A$ with $\pi(a_i) = f(e_i)$. Define the map $\alpha : F \to A$ by $\alpha(e_i) = a_i$. □

Lemma 5.5. *An R-module is projective if and only if it is a direct summand of a free R-module.*

Proof. Suppose that P is a direct summand of a free module. Then there is some R-module P' such that $P \oplus P'$ is free. Let $\pi : A \twoheadrightarrow B$ be a surjection and let $f : P \to B$ be some map. Let $f' : P \oplus P' \to B$ be the map $f'(p, p') = f(p)$. Since $P \oplus P'$ is free, hence projective, f' has a lift $\alpha' : P \oplus P' \to A$. Now define $\alpha : P \to A$ by $\alpha(p) = \alpha'(p, 0)$.

Suppose conversely that P is projective. Then we have a natural surjection

$$\pi : \bigoplus_{p \in P} Re_p \to P, \quad e_p \mapsto p,$$

and taking $f : P \to P$ to be the identity gives us a section α of this surjection (since P is projective). Therefore, the result follows by the Splitting Lemma. □

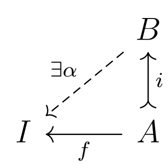
5.2 Injective Objects

Injective objects are dual to projective objects.

Definition 5.6. An object I of an abelian category $\mathcal{A}$ is called **injective** if the object $I \in \mathcal{A}^{\mathrm{op}}$ is projective.

Lemma 5.7. *For an R-module I, the following are equivalent:*

1. *I is injective.*
2. *The functor $\mathrm{Hom}_{\mathcal{A}}(-, I) : \mathcal{A}^{op} \to \boldsymbol{Ab}$ is exact.*
3. *The contravariant functor $\mathrm{Hom}_{\mathcal{A}}(-, I) : \mathcal{A} \to \boldsymbol{Ab}$ sends monomorphisms to epimorphisms.*
4. *for every monomorphism $i : A \to B$ and every map $f : A \to I$, there exists a $g : B \to I$ such that the following diagram commutes:*



Proof. This is the dual of Lemma 5.2. □

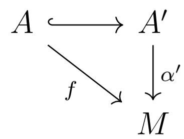
Theorem 5.8 (Baer's Criterion)**.** *Let M be a right R-module. The following are equivalent.*

1. *M is injective.*
2. *For every right ideal I of R, every module homomorphism $I \to M$ can be extended to a module homomorphism $R \to M$.*

Proof. The (1) $\implies$ (2) is immediate from Lemma 5.7. The other direction is significantly harder.

Assume that (2) holds. Fix some injection $i : A \rightarrowtail B$ of R-modules, and some map $f : A \to M$. Without loss of generality, assume that $A \subseteq B$ and i is the inclusion. Let

Σ be the set whose elements are R-module maps $\alpha' : A' \to M$, where $A \subseteq A' \subseteq B$ and α' extends f (that is, the diagram



commutes). We may give this set a partial order by saying that $\alpha' \leq \alpha''$ when $A' \subseteq A''$ and α'' extends α'. Suppose that $\alpha_1 \leq \alpha_2 \leq \ldots$ is an ascending chain in Σ, with corresponding modules $A_1 \subseteq A_2 \subseteq \ldots$. Let $A' = \cup A_n$, and define $\alpha' : A' \to M$ by $\alpha'(a) = \alpha_i(a)$ for $a \in A_i$. It is easy to check that α' is a well-defined element of Σ, and it is an upper bound on the chain (n.b. we are really just taking the colimit of the chain).

Since Σ is a partially ordered set in which every ascending chain has an upper bound, Zorn's Lemma tells us that it has a maximal element, which we will call $\alpha' : A'toM$. To show that M is injective, we need to show that $A' = B$, since we then have an extension α of f to B.

Suppose that $A' \neq B$. Let $b \in B \setminus A'$, and define $A'' = A' + Rb \subseteq B$. Let $I = \{r \in R : br \in A'\}$. Then I is a right ideal of R, and we have a map

$$I \to M, \quad r \mapsto \alpha'(br).$$

Since (2) holds, this extends to a map $\varphi : R \to M$. We claim that there is a well-defined map

$$\alpha'' : A'' \to M, \quad a + br \mapsto \alpha'(a) + \varphi(r),$$

where $a \in A'$ and $r \in R$. To see that this is well-defined, suppose that

$$a + br = a' + br'.$$

Then

$$a - a' = b(r' - r) \in A' \cap bR.$$

We have

$$\alpha'(a - a') = \alpha'(b(r' - r)) = \varphi(r' - r),$$

since $r - r' \in I$. Therefore, it follows that

$$\alpha'(a) + \varphi(r) = \alpha'(a') + \varphi(r'),$$

so α'' is well-defined. But then α'' strictly extends α', contradicting maximality of α'. Therefore, $A' = B$, so we are done. $\square$

Corollary 5.9. *If R is a PID, then an R-module I is injective if and only if it is* ***divisible****. That is, for all $x \in I$ and $r \in R \setminus \{0\}$ there exists $q \in I$ such that $x = rq$.*

The details of the proof can get in the way of the intuitive idea, which is quite simple. Maybe try proving it yourself before reading on (use Baer's Criterion).

Proof. Let I be an injective R-module, and let $x \in I$ and $r \in r \setminus \{0\}$. Set $J = rR$ and define $f : J \to I$ by $f(r) = x$. By Baer's Criterion, we may extend f to a homomorphism $\tilde{f} : R \to I$. Then $x = f(r) = \tilde{f}(r \cdot 1) = r \cdot \tilde{f}(1)$. So taking $q = \tilde{f}(1)$, we see that I is divisible.

Suppose conversely that I is a divisible R-module. Let J be an ideal of R and let $f : J \to I$ be a module homomorphism. If J is the zero ideal, then trivially we may extend f to the zero homomorphism $R \to I$. Assume that J is nonzero.

Since R is a PID, we have $I = rR$ for some nonzero r. Let $x = f(r)$. Then since I is divisible, there is some $q \in I$ such that $x = rq$. Define $\tilde{f} : R \to I$ by $\tilde{f}(1) = q$. Clearly $\tilde{f}$ is an extension of f, so I is injective by Baer's Criterion. $\square$

Corollary 5.10. *The $\mathbb{Z}$-module $\mathbb{Q}$ is injective.*

Proof. Clearly $\mathbb{Q}$ is a divisible $\mathbb{Z}$-module. $\square$

We will give a vast generalisation of Corollary 5.10, but it requires a bit more machinery.

Lemma 5.11. *Let I be an injective right R-module and let I' be a direct summand of I. Then I' is injective.*

Proof. Write $I = I' \oplus M$ for some right R-module M. We will use Baer's Criterion.

Let J be a right ideal of R, and let $\varphi : J \to I'$ be a module homomorphism. Then let $\tilde{\varphi}$ be the composition $J \xrightarrow{\varphi} I' \hookrightarrow I$. Since I is injective, Baer's Criterion tells us that $\tilde{\varphi}$

extends to a homomorphism $\tilde{\alpha} : R \to I$. Let α be the composition $R \xrightarrow{\tilde{\alpha}} I \overset{\pi}{\twoheadrightarrow} I'$, where π is the projection onto I' along M.

Then for $x \in J$ we have $\alpha(x) = \pi(\tilde{\alpha}(x)) = \pi(\tilde{\varphi}(x)) = \varphi(x)$ by definition of $\tilde{\varphi}$, so $\alpha : R \to I'$ is an extension of φ, as required by Baer's Criterion. $\square$

Lemma 5.12. *Let R be a PID, and let $p \in R$ be prime. Then the R-module $M = R[\frac{1}{p}]/R$ is injective.*[1]

Proof. Note that R is a PID, hence a UFD. Let $x + R \in M$ and $r \in R \setminus \{0\}$. Since R is a UFD, we have $x = \frac{\alpha}{p^k}$ for $k \geq 0$ and $\alpha \in R$. We also have $r = p^l r'$ for $l \geq 0$ and $r' \in R \setminus pR$. Since r' is coprime to p^{k+l}, there is a $q' \in R$ such that $r'q' \equiv \alpha (\text{mod } p^k)$ Let $q = \frac{q'}{p^{k+l}} \in R[\frac{1}{p}]$. Then

$$rq - x = \frac{p^l r' q'}{p^{k+l}} - \frac{\alpha}{p^k} = \frac{r'q' - \alpha}{p^k},$$

is in R since $p^k \mid r'q' - \alpha$ by definition of q'. Therefore $r(q + R) = x + R \in M$, so M is divisible, hence injective. $\square$

Lemma 5.13. *A $\mathbb{Z}$-module is injective if and only if it is a direct sum of copies of $\mathbb{Q}$ and $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$ for prime p.*

Sketch Proof. Let I be an injective $\mathbb{Z}$-module, and let Σ be the set of $\mathbb{Z}$-submodules of I that can be expressed as direct sums of $\mathbb{Q}$ and $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$ for prime p. For $M, N \in \Sigma$, we say that $M \leq N$ if for any basis $\{e_a : a \in A\}$ of M, there is a basis $\{e_a : a \in A'\}$ of N extending it (that is, $A' \supseteq A$).

Clearly $(\Sigma, \leq)$ is a partially ordered set. Suppose that $M_1 \leq M_2 \leq \ldots$ is a chain in Σ. Let $\{e_a : a \in A_1\}$ be any basis for M_1, and repeatedly extend it to bases $\{e_a : a \in A_i\}$ for M_i. Then $M = \cup_i M_i$ is free with basis $\{e_a : a \in \cup_i A_i\}$, so it is in Σ. Since the basis $\{e_a : a \in A_1\}$ for M_1 was arbitrary, we have $M_1 \leq M$. Similarly, $M_i \leq M$ for all i. Therefore, any chain in Σ has an upper bound, so by Zorn's Lemma, Σ has a maximal element, which we will call M.

[1] I have tried to write the proof as concisely as possible, but this makes it seem a lot less intuitive than it is. I would recommend trying to prove it yourself first. Maybe try the special case $R = \mathbb{Z}$ before generalising to arbitrary PIDs.

Suppose that $M \neq I$. Then we have a short exact sequence

$$0 \to M \to I \to I/M \to 0.$$

Let I be an injective $\mathbb{Z}$-module, and let Σ be the set of $\mathbb{Z}$-submodules M of I equipped with decompositions into direct sums of $\mathbb{Q}$ and modules of the form $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$ for prime p. To be clear, an element of Σ is not just the module M, but the module M together with the data of a specific direct sum decomposition.

We define a partial ordering on Σ by saying that $M \leq N$ if $M \subseteq N$ and the inclusion is compatible with the direct sum decompositions. Let $M_1 \leq M_2 \leq \ldots$ be a chain in the partially ordered set $(\Sigma, \leq)$. Then $M = \cup_i M_i$ is also in Σ, taking the union of the direct sum decompositions.

Therefore, by Zorn's Lemma, the partially ordered set $(\Sigma, \leq)$ has a maximal element, M. Since M is a direct sum of divisible modules, it is divisible, hence injective. Therefore, the short exact sequence

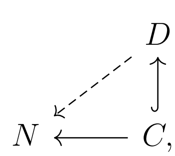
$$0 \to M \to I \to I/M \to 0$$

splits, so

$$I \cong M \oplus I/M.$$

By maximality of M, the module I/M cannot have any submodules isomorphic to $\mathbb{Q}$ or $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$ for any prime p. Let $N = I/M$, and suppose that $N \neq 0$. Let $x \in N$ be nonzero, and let C_0 be the subgroup generated by x. Then C_0 is a cyclic group. If it is infinite, it is isomorphic to $\mathbb{Z}$. If it is finite, then it has a subgroup of the form $\mathbb{Z}/p$ for some prime p. Either way, we can take $C \subseteq C_0$ to be a subgroup isomorphic either to $\mathbb{Z}$ or to $\mathbb{Z}/p$ for prime p.

If $C \cong \mathbb{Z}/p$, define $D = \mathbb{Z}[\frac{1}{p}]/p\mathbb{Z} \cong \mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$. If $C \cong \mathbb{Z}$, define $D = \mathbb{Q}$. We have a natural injection $C \hookrightarrow D$. Since N is a direct summand of I, it is injective, so by the lifting property of injective modules,



we have a map $\varphi : D \to N$ extending $C \hookrightarrow N$. We claim that φ is injective. Let $d \in D$ have $\varphi(d) = 0$. Then there is some $n \in \mathbb{N}$ such that $nd \in C$. Since $\varphi|_C$ is injective and $\varphi(nd) = 0$, we have $nd = 0$. But then $d = 0$, so φ is injective. So we have constructed a submodule of N isomorphic to $\mathbb{Q}$ or $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$. Either way, we have a contradiction, so $N = 0$, and therefore $I = M$. □

Lemma 5.14. *For an algebraically closed field k, a $k[x]$-module is injective if and only it is a direct sum of copies of $k(x)$ and $k[x][\frac{1}{(x-a)}]/k[x]$ for $a \in k$.*

Proof. Similar to the previous Lemma. In fact, I'm pretty sure we can extend the result to UFDs in general without much work, although I haven't checked in detail. □

5.3 Resolutions

Definition 5.15. Let M be an object of an abelian category $\mathcal{A}$. A **projective (resp. free) resolution** of M is a chain complex

$$\ldots \to P_2 \to P_1 \to P_0 \to 0,$$

together with a map $\varepsilon : P_0 \to M$, called the **augmentation**, such that the sequence

$$\ldots \to P_2 \to P_1 \to P_0 \xrightarrow{\varepsilon} M \to 0$$

is exact, and each P_i is projective (resp. free).

Remark 5.16. Since free modules are projective, a free resolution is a special type of projective resolution.

Example 5.17. The $\mathbb{Z}$-module $\mathbb{Z}/5$ has free resolution

$$\mathbb{Z} \xrightarrow{5} \mathbb{Z},$$

since we have an exact sequence

$$\mathbb{Z} \xrightarrow{5} \mathbb{Z} \to \mathbb{Z}/5 \to 0.$$

The augmentation in this case is the quotient map $\mathbb{Z} \to \mathbb{Z}/5$.

Definition 5.18. An **injective resolution** $N \to I^\bullet$ of an R-module N is a cochain complex $I^\bullet$ of injective R-modules that fits into an exact sequence

$$0 \to N \xrightarrow{\eta} I^0 \to I^1 \to \dots.$$

The map η is called the **augementation**.

Definition 5.19. Let $\mathcal{A}$ be an abelian category. We say that $\mathcal{A}$ **has enough projectives** if for every $M \in \mathcal{A}$, there is a projective object P and an epimorphism $P \twoheadrightarrow M$.

Lemma 5.20. *Let $\mathcal{A}$ be an abelian category with enough projectives. Then every object of $\mathcal{A}$ has a projective resolution.*

Proof. Let $M \in \mathcal{A}$. Since $\mathcal{A}$ has enough projectives, we may take a projective object P_0 together with an epimorphism $P_0 \overset{\partial_0}{\twoheadrightarrow} M$. Now we may construct our resolution inductively as follows.

Suppose that we already have an exact sequence

$$P_k \xrightarrow{\partial_k} P_{k-1} \xrightarrow{\partial_{k-1}} P_{k-2} \to \dots \to P_0 \to M \to 0,$$

where the P_i are projective. Then there is a monomorphism $\ker \partial_k \rightarrowtail P_k$, and since $\mathcal{A}$ has enough projectives there is a surjection $P_{k+1} \twoheadrightarrow \ker \partial_k$ where P_{k+1} is projective. Then let ∂_{k+1} be the composition $P_{k+1} \twoheadrightarrow \ker \partial_k \rightarrowtail P_k$. Then $\operatorname{im} \partial_{k+1} = \ker \partial_k$, so the sequence

$$P_{k+1} \xrightarrow{\partial_{k+1}} P_k \xrightarrow{\partial_k} P_{k-1} \to \dots \to P_0 \to M \to 0,$$

is exact at P_k, and hence it is exact everywhere. By induction the sequence

$$\dots \to P_2 \to P_1 \to P_0 \to M \to 0$$

is exact at every term, hence it is a projective resolution. □

Definition 5.21. Let $\mathcal{A}$ be an abelian category. We say that $\mathcal{A}$ **has enough injectives** if for every $M \in \mathcal{A}$, there is an injective object I and a monomorphism $M \rightarrowtail I$.

Lemma 5.22. *If an abelian category $\mathcal{A}$ has enough injectives, then every object admits an injective resolution.*

Proof. Probably the "right" way to think about this is to note that an object $I \in \mathcal{A}$ is injective if and only if $I \in \mathcal{A}^{\text{op}}$ is projective, and use the fact that left- and right-resolutions are dual to each other. However, we will give a direct proof, similar to that of Lemma 5.20.

Let $M \in \mathcal{A}$. Then there is an injective object I^0 and a monomorphism $\partial^{-1} : M \rightarrowtail I^{-1}$, so the sequence $0 \to M \overset{\partial^{-1}}{\to} I^0$ is exact. Suppose that we have already constructed an exact sequence

$$0 \to M \overset{\partial^{-1}}{\to} I^0 \to \ldots \to I^{k-1} \overset{\partial^{k-1}}{\to} I^k,$$

where the I^i are injective objects. Then we have an epimorphism $I^k \twoheadrightarrow \operatorname{coker} \partial^k$, and (since $\mathcal{A}$ has enough injectives) a monomorphism $\operatorname{coker} \partial^k \rightarrowtail I^{k+1}$. Let ∂^{k+1} be the composition $I^k \twoheadrightarrow \operatorname{coker} \partial^k \rightarrowtail I^{k+1}$. Then $\ker \partial^{k+1} = \ker \operatorname{coker} \partial^k = \operatorname{im} \partial^k$. We conclude in the same manner as we did in Lemma 5.20. □

Lemma 5.23. *The category of left R-modules has enough projectives.*

Proof. Since free modules are projective, it suffices to show that every R-module is the image of some homomorphism from a free module.

To do this, let M be an R-module and let F be the free module with basis $\mathcal{B} = \{v_m : m \in M\}$. By the universal property of free modules, there is a homomorphism $\varphi : F \to M$ with $\varphi(v_m) = m$ for each $m \in M$. Then is clearly a surjective homomorphism, so we are done. □

5.4 R-mod Has Enough Injectives

Proving that R-**mod** has enough injectives is hard enough to warrant a whole subsection to itself.

Definition 5.24 (Adjoint Functors)**.** Let $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ be categories and functors. We say that the functors F and G are **adjoint** if for all $x \in \mathcal{C}$ and $y \in \mathcal{D}$, there are mutually inverse bijections bijections

$$\operatorname{Hom}_{\mathcal{C}}(x, G(y)) \underset{\Psi_{x,y}}{\overset{\Phi_{x,y}}{\rightleftarrows}} \operatorname{Hom}_{\mathcal{D}}(F(x), y)$$

satisfying the so-called "naturality conditions":

1. For any map $x \xrightarrow{f} x'$ in $\mathcal{C}$, the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(x', G(y)) & \xrightarrow{\Phi_{x',y}} & \mathrm{Hom}_{\mathcal{D}}(F(x'), y) \\ \downarrow{\scriptstyle -\circ f} & & \downarrow{\scriptstyle -\circ F(f)} \\ \mathrm{Hom}_{\mathcal{C}}(x, G(y)) & \xrightarrow{\Phi_{x,y}} & \mathrm{Hom}_{\mathcal{D}}(F(x), y) \end{array}$$

2. For any map $y \xrightarrow{g} y'$ in $\mathcal{D}$, the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(x, G(y)) & \xrightarrow{\Phi_{x,y}} & \mathrm{Hom}_{\mathcal{D}}(F(x), y) \\ \downarrow{\scriptstyle G(g)\circ -} & & \downarrow{\scriptstyle g\circ -} \\ \mathrm{Hom}_{\mathcal{C}}(x, G(y')) & \xrightarrow{\Phi_{x,y'}} & \mathrm{Hom}_{\mathcal{D}}(F(x), y') \end{array}$$

In particular, we say that F is **left adjoint** to G and G is **right adjoint** to F, and we call the relationship an **adjunction**. This situation is represented by the notation $F \dashv G$.

Lemma 5.25. *Suppose that we have abelian categories and functors* $\mathcal{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{B}$ *, where $F \dashv G$ and F is exact. Then G preserves injective objects.*

Proof. Let $I \in \mathcal{B}$ be injective. Then we want to show that $G(I) \in \mathcal{A}$ is injective. By Lemma 5.7, it suffices to show that the functor $\mathrm{Hom}_{\mathcal{A}}(-, G(I)) : \mathcal{A} \to \mathbf{Ab}$ sends monomorphisms to epimorphisms. Recall that for a map $\varphi : A \to A'$ in $\mathcal{A}$, this functor sends φ to the morphism

$$- \circ \varphi : \mathrm{Hom}_{\mathcal{A}}(A', G(I)) \to \mathrm{Hom}_{\mathcal{A}}(A, G(I)).$$

Let $i : A \to A'$ be a monomorphism in $\mathcal{A}$. Since the adjuncion $F \dashv G$ is natural in x, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(A', G(I)) & \xrightarrow{\Phi_{A',I}} & \mathrm{Hom}_{\mathcal{D}}(F(A'), I) \\ \downarrow{\scriptstyle -\circ i} & & \downarrow{\scriptstyle -\circ F(i)} \\ \mathrm{Hom}_{\mathcal{C}}(A, G(I)) & \xrightarrow{\Phi_{A,I}} & \mathrm{Hom}_{\mathcal{D}}(F(A), I) \end{array}$$

Since F is exact, $F(i)$ is a monomorphism, which means that $- \circ F(i) = \mathrm{Hom}_{\mathcal{B}}(F(i), I)$

is an epimorphism. So the arrow on the right is a surjective function. Since the top arrow is a bijection, the top path of the commutative diagram is a surjection, hence the bottom path is too. Since the bottom arrow is a bijection, the left arrow must be onto. Therefore $- \circ i$ is an epimorphism, as required. □

Recall that an abelian group A is **divisible** if and only if for all $a \in A$ and nonzero $n \in \mathbb{Z}$, there is some $q \in A$ with $a = nq$.

Lemma 5.26. *We have*

1. *If $\{A_i : i \in I\}$ is a family of divisible abelian groups, then $\bigoplus_{i\in I} A_i$ is a divisible abelian group.*
2. *If A is a divisible abelian group and $K \subseteq A$ is any subgroup, then A/K is divisible.*

Proof. Easy. □

Lemma 5.27. *The category of abelian groups has enough injectives.*

Proof. Let A be an abelian group. By the proof of Lemma 5.23, we may express A as a quotient of a free abelian group. In particular, we have

$$A = \Big(\bigoplus_{s\in S} \mathbb{Z}\Big)/K$$

where K is a submodule of $\bigoplus_{s\in S} \mathbb{Z}$. There is an obvious embedding of $\mathbb{Z}$-modules $\bigoplus_{s\in S} \mathbb{Z} \hookrightarrow \bigoplus_{s\in S} \mathbb{Q}$, which induces an embedding

$$A = \Big(\bigoplus_{s\in S} \mathbb{Z}\Big)/K \hookrightarrow \Big(\bigoplus_{s\in S} \mathbb{Q}\Big)/K =: I.$$

Since $\mathbb{Q}$ is divisible, Lemma 5.26 tells us that I is divisible, hence injective, so we are done. □

The following theorem is the technical workhorse of this subsection. We will develop some adjoint functor theory from scratch, making the proof longer than it needs to be if you already know category theory. For a cleaner version, see `https://ncatlab.org/nlab/show/injective+module`.

Theorem 5.28. *Let $F \dashv G$ be a pair of additive adjoint functors $\mathcal{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{B}$ between abelian categories $\mathcal{A}, \mathcal{B}$ such that F is an exact, faithful functor. If $\mathcal{B}$ has enough injectives, then so does $\mathcal{A}$.*

Proof. Let $A \in \mathcal{A}$. Then $F(A) \in \mathcal{B}$ so there is an injective object $I \in \mathcal{B}$ together with a monomorphism $i : F(A) \rightarrowtail I$. Since $i \in \mathrm{Hom}_{\mathcal{B}}(F(A), I)$, we have $\tilde{i} := \Psi_{A,I}(i) \in \mathrm{Hom}_{\mathcal{A}}(A, G(I))$. Since F is exact, Lemma 5.25 tells us that $G(I) \in \mathcal{A}$ is injective. Therefore it suffices to show that $\tilde{i} : A \to G(I)$ is a monomorphism.

By definition of adjunctions, we have a map

$$\mathrm{Hom}_{\mathcal{A}}(G(I), G(I)) \xrightarrow{\Phi_{G(I),I}} \mathrm{Hom}_{\mathcal{B}}(FG(I), I)$$

and we define $\varepsilon = \Phi_{G(I),I}(\mathrm{id}_{G(I)}) : FG(I) \to I$. By definition of adjunctions, the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{A}}(G(I), G(I)) & \xrightarrow{\Phi_{G(I),I}} & \mathrm{Hom}_{\mathcal{B}}(FG(I), I) \\ \Big\downarrow{\scriptstyle -\circ\tilde{i}} & & \Big\downarrow{\scriptstyle -\circ F(\tilde{i})} \\ \mathrm{Hom}_{\mathcal{A}}(A, G(I)) & \xrightarrow{\Phi_{A,I}} & \mathrm{Hom}_{\mathcal{B}}(F(A), I) \end{array}$$

Therefore we have

$$\begin{aligned} \varepsilon \circ F(\tilde{i}) &= \Phi_{G(I),I}(\mathrm{id}_{G(I)}) \circ F(\tilde{i}) \\ &= \Phi_{A,I}(\mathrm{id}_{G(I)} \circ \tilde{i}) \\ &= \Phi_{A,I}(\tilde{i}) \\ &= i, \end{aligned}$$

where the final two equalities come respectively from commuativity of the diagram and from the definition of $\tilde{i}$. Clearly then $\varepsilon \circ F(\tilde{i})$ is a monomorphism, which implies that $F(\tilde{i})$ is a monomorphism (check this).

Now consider the exact sequence

$$\ker \tilde{i} \longrightarrow A \xrightarrow{\tilde{i}} G(I).$$

Since F is an exact functor, we have an exact sequence

$$F(\ker \tilde{i}) \longrightarrow F(A) \xrightarrow{F(\tilde{i})} FG(I),$$

Since $F(\tilde{i})$ is a monomorphism, exactness implies that the map $F(\ker \tilde{i}) \to F(A)$ is the zero morphism, so by faithfulness of F the map $\ker \tilde{i} \to A$ is zero, which means that $\tilde{i} : A \to G(I)$ is a monomorphism, so we are done. □

Definition 5.29. Let A be an abelian group and B be a left R-module. Then we give $\mathrm{Hom}_{\mathbf{Ab}}(B, A)$ a natural right R-module structure via $(f \cdot r)(b) = f(rb)$.

Lemma 5.30. *Define functors* $R\text{-}\mathbf{mod} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{Ab}$ *by*

1. *F is the forgetful functor.*
2. *$G(A) = \mathrm{Hom}_{\mathbf{Ab}}(R, A))$.*

Then $F \dashv G$.

Proof. Let $M \in R\text{-}\mathbf{mod}$ and $A \in \mathbf{Ab}$. Then define

$$\Phi_{M,A} : \mathrm{Hom}_{R\text{-}\mathbf{mod}}(M, A) \to \mathrm{Hom}_{\mathbf{Ab}}(M, G(A))$$

by $\Phi_{M,A}(f)(m) = f(mr)$ and

$$\Psi_{M,A} : \mathrm{Hom}_{\mathbf{Ab}}(M, G(A)) \to \mathrm{Hom}_{R\text{-}\mathbf{mod}}(M, A)$$

by $\Psi_{M,A}(g)(m) = g(m)(1)$. It is an unpleasant exercise in abstract nonsense to check that these are mutual inverses, and that they are natural in M and A. □

Theorem 5.31. *For any ring R, the category R-**mod** has enough injectives.*

Proof. Let F, G be as in Lemma 5.30. It is clear that the forgetful functor F is exact and faithful. We know that $\mathbf{Ab}$ has enough injectives, so since $F \dashv G$, Lemma 5.30 gives the result. □

6 Derived Functors

Suppose that $F : \mathcal{A} \to \mathcal{B}$ is a left exact functor between abelian categories, and that $0 \to X \to Y \to Z \to 0$ is a short exact sequence in $\mathcal{A}$. Then the sequence

$$0 \to F(X) \to F(Y) \to F(Z)$$

is exact at $F(X)$ and $F(Y)$. It would be nice if we could find a long exact sequence that extends this sequence. Derived functors give us a way of doing that.

6.1 δ-functors

Definition 6.1. Let $\mathcal{A}, \mathcal{B}$ be abelian categories. A **homological delta functor** between $\mathcal{A}$ and $\mathcal{B}$ is a collection $T = \{T_n : \mathcal{A} \to \mathcal{B} : n \geq 0\}$ of additive functors T_n such that for any short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$ there are morphisms

$$\delta_n : T_n(C) \to T_{n-1}(A), \quad n \in \mathbb{Z},$$

(where we write $T_n = 0$ for $n < 0$) satisfying

1. There is a long exact sequence

$$\ldots \to T_{n+1}(C) \stackrel{\delta_{n+1}}{\to} T_n(A) \to T_n(B) \to T_n(C) \stackrel{\delta_n}{\to} T_{n-1}(A) \to \ldots$$

2. For each morphism of short exact sequences from $0 \to A \to B \to C \to 0$ to $0 \to A' \to B' \to C' \to 0$, the following diagram commutes:

$$\begin{array}{ccc} T_n(C) & \xrightarrow{\delta_n} & T_{n-1}(A) \\ \downarrow & & \downarrow \\ T_n(C') & \xrightarrow{\delta'_n} & T_{n-1}(A') \end{array}$$

Definition 6.2. Let $\mathcal{A}, \mathcal{B}$ be abelian categories. A **cohomological delta functor** between $\mathcal{A}$ and $\mathcal{B}$ is a collection $T = \{T^n : \mathcal{A} \to \mathcal{B} : n \geq 0\}$ of additive functors T^n such that for any short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$ there are morphisms

$$\delta^n : T^n(C) \to T^{n+1}(A), \quad n \in \mathbb{Z},$$

(where we write $T^n = 0$ for $n < 0$) satisfying

1. There is a long exact sequence

$$\ldots \to T^{n-1}(C) \xrightarrow{\delta^{n-1}} T^n(A) \to T^n(B) \to T^n(C) \xrightarrow{\delta^n} T^{n+1}(A) \to \ldots$$

2. For each morphism of short exact sequences from $0 \to A \to B \to C \to 0$ to $0 \to A' \to B' \to C' \to 0$, the following diagram commutes:

$$\begin{array}{ccc} T^n(C) & \xrightarrow{\delta^n} & T^{n-1}(A) \\ \downarrow & & \downarrow \\ T^n(C') & \xrightarrow{(\delta^n)'} & T^{n-1}(A') \end{array}$$

By the Snake Lemma, homology is a homological δ-functor from the category of chain complexes $\mathbf{Ch}_{\geq 0}(\mathcal{A})$ to $\mathcal{A}$, for any abelian category $\mathcal{A}$. Similarly, cohomology gives a cohomological δ-functor from $\mathbf{Ch}^{\geq 0}(\mathcal{A})$ to $\mathcal{A}$.

Definition 6.3. A **morphism** $S \to T$ of homological (resp. cohomological) δ-functors is a collection of natural transformatinos $S_n \to T_n$ (resp. $S^n \to T^n$) that commute with δ.

Definition 6.4. A homological δ-functor T is **universal** if for any homological δ-functor S and natural transformation $f_0 : S_0 \to T_0$, there is a unique morphism of δ-functors $S \to T$ exending f.

Definition 6.5. A cohomological δ-functor T is **universal** if for any cohomological δ-functor S and natural transformation $f^0 : T^0 \to S^0$, there is a unique morphism $T \to S$ of δ-functors extending f^0.

6.2 Building Projective Resolutions

Our theory of left derived functors will depend heavily on projective resolutions. In this subsection, we develop the tools we will need to manipulate projective resolutions for our purposes.

Theorem 6.6. *Let $f' : M \to N$ be a map of R-modules. Suppose that we have chain complexes of the form*

$$\begin{array}{ccccccccccc}
\dots & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 & \xrightarrow{\epsilon} & M & \longrightarrow & 0, \\
& & & & & & & & \Big\downarrow{\scriptstyle f'} & & \\
\dots & \longrightarrow & Q_2 & \longrightarrow & Q_1 & \longrightarrow & Q_0 & \xrightarrow{\eta} & N & \longrightarrow & 0,
\end{array}$$

where all P_i are projective and the bottom row is exact.

Then there is a chain map $f : P_\bullet \to Q_\bullet$ extending f', by which we mean that the following diagram commutes:

$$\begin{array}{ccccccccccc}
\dots & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 & \xrightarrow{\epsilon} & M & \longrightarrow & 0, \\
& & \Big\downarrow{\scriptstyle f_2} & & \Big\downarrow{\scriptstyle f_1} & & \Big\downarrow{\scriptstyle f_0} & & \Big\downarrow{\scriptstyle f'} & & \\
\dots & \longrightarrow & Q_2 & \longrightarrow & Q_1 & \longrightarrow & Q_0 & \xrightarrow{\eta} & N & \longrightarrow & 0.
\end{array}$$

Furthermore, this chain map is unique up to homotopy equivalence.

Proof. **(Existence.)** We construct the f_i by induction.

For the base case, since $\eta : Q_0 \to M$ is surjective and Q_0 is projective we may lift $f' \circ \varepsilon : P_0 \to N$ to a map $f_0 : P_0 \to Q_0$.

Assume that we already have maps $f_0, f_1, \dots, f_n$ that commute with the differentials as in the diagram. Let $P_\bullet, Q_\bullet$ have differentials $\partial_\bullet, \widetilde{\partial}_\bullet$ respectively. By assumption, we have the following commutative diagram:

$$\begin{array}{ccccc}
P_{n+1} & \xrightarrow{\partial_{n+1}} & P_n & \xrightarrow{\partial_n} & P_{n-1} \\
& & \Big\downarrow{\scriptstyle f_n} & & \Big\downarrow{\scriptstyle f_{n-1}} \\
Q_{n+1} & \xrightarrow{\widetilde{\partial}_{n+1}} & Q_n & \xrightarrow{\widetilde{\partial}_n} & Q_{n-1}
\end{array}$$

We claim that $f_n(\operatorname{im} \partial_{n+1}) \subseteq \operatorname{im} \widetilde{\partial}_{n+1}$. To see this, let $p \in \operatorname{im} \partial_{n+1}$. Clearly then $\partial_n p = 0$, so by commutativity we have

$$\widetilde{\partial}_n \circ f_n(p) = f_{n-1} \circ \partial_n(p) = f_{n-1}(0) = 0,$$

which means that $f(p) \in \ker \widetilde{\partial}_n = \operatorname{im} \widetilde{\partial}_{n+1}$ by exactness of $Q_\bullet$, so $f_n(\operatorname{im} \partial_{n+1}) \subseteq \operatorname{im} \widetilde{\partial}_{n+1}$

as we claimed. It follows that the composition $f_n \circ \partial_{n+1}$ gives a well-defined map $P_{n+1} \to \operatorname{im} \widetilde{\partial}_{n+1}$, so we have a diagram

$$\begin{array}{ccc} & & Q_{n+1} \\ & & \big\downarrow\!\!\!\!\downarrow \, \widetilde{\partial}_{n+1} \\ P_{n+1} & \xrightarrow{f_n \circ \partial_{n+1}} & \operatorname{im} \widetilde{\partial}_{n+1} \end{array}$$

where the vertical arrow is a surjection. Since P_{n+1} is a projective module, we may lift to a map $f_{n+1} : P_{n+1} \to Q_{n+1}$ such that the following diagram commutes:

$$\begin{array}{ccc} & & Q_{n+1} \\ & \overset{f_{n+1}}{\nearrow} & \big\downarrow \, \widetilde{\partial}_{n+1} \\ P_{n+1} & \xrightarrow{f_n \circ \partial_{n+1}} & Q_n \end{array}$$

Clearly we may redraw this diagram as:

$$\begin{array}{ccc} P_{n+1} & \xrightarrow{\partial_{n+1}} & P_n \\ {\scriptstyle f_{n+1}} \big\downarrow & & \big\downarrow {\scriptstyle f_n} \\ Q_{n+1} & \xrightarrow[\widetilde{\partial}_{n+1}]{} & Q_n \end{array}$$

which is exact what we need.

(Uniqueness.) Suppose first that $f' = 0$. Then we will inductively construct a nullhomotopy $s = \{s_n : P_n \to P_{n+1}\}$, that is $f_n = s_{n-1} \circ \partial_n + \widetilde{\partial}_{n+1} \circ s_n$ for each n. Let $P_0 = M, Q_0 = N$ and $P_i = Q_i = 0$ for $i < 0$. Also let $f_0 = f' : P_0 \to Q_0$ and $f_i = 0 : P_i \to Q_i$ for $i < 0$.

For the base case, take $s_i = 0$ for $i < 0$. Then we have

$$f_{-1} = 0 = \eta \circ s_{-1} + s_{-2} \circ \partial_{-1},$$

so the claim is true for $n = -1$. Now assume that we have constructed s_i for $i \leq n-1$, so that we have a diagram

$$
\begin{array}{ccccccccc}
P_{n+1} & \xrightarrow{\partial_{n+1}} & P_n & \xrightarrow{\partial_n} & P_{n-1} & \xrightarrow{\partial_{n-1}} & P_{n-2} & \longrightarrow & \dots \\
\downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & \swarrow{\scriptstyle s_{n-1}} & \downarrow{\scriptstyle f_{n-1}} & \swarrow{\scriptstyle s_{n-2}} & \downarrow{\scriptstyle f_{n-2}} & & \\
Q_{n+1} & \xrightarrow{\widetilde{\partial}_{n+1}} & Q_n & \xrightarrow{\widetilde{\partial}_n} & Q_{n-1} & \xrightarrow{\widetilde{\partial}_{n-1}} & Q_{n-2} & \longrightarrow & \dots
\end{array}
$$

where the solid arrows commute and

$$f_{n-1} = \widetilde{\partial}_n \circ s_{n-1} + s_{n-2} \circ \partial_{n-1}.$$

Composing this last equation with ∂_n gives

$$
\begin{aligned}
f_{n-1} \circ \partial_n &= \widetilde{\partial}_n \circ s_{n-1} \circ \partial_n + s_{n-2} \circ \partial_{n-1} \circ \partial_n \\
&= \widetilde{\partial}_n \circ s_{n-1} \circ \partial_n,
\end{aligned}
$$

and by commutativity of the solid arrows we have $\widetilde{\partial}_n \circ f_n = f_{n-1} \circ \partial_n$, so

$$\widetilde{\partial}_n \circ f_n = \widetilde{\partial}_n \circ s_{n-1} \circ \partial_n,$$

which means that the map $(f_n - s_{n-1} \circ \partial_n)$ takes P_n into $\ker \widetilde{\partial}_n = \operatorname{im} \widetilde{\partial}_{n+1}$, so we have a diagram:

$$
\begin{array}{ccc}
 & & P_n \\
 & & \downarrow{\scriptstyle (f_n - s_{n-1} \circ \partial_n)} \\
Q_{n+1} & \xrightarrow{\widetilde{\partial}_{n+1}} & \operatorname{im} \widetilde{\partial}_{n+1}
\end{array}
$$

By projectivity of P_n, this gives us a map $s_n : P_n \to Q_{n+1}$ such that

$$\widetilde{\partial}_{n+1} \circ s_n = f_n - s_{n-1} \circ \partial_n,$$

as required.

Finally now consider the general case, where f' is not necessarily zero. Then suppose that $f = \{f_n\}$ and $g = \{g_n\}$ are chain maps extending f'. Then $f - g = \{f_n - g_n\}$ is chain map extending $f' - f' = 0$, hence $f - g \simeq 0$ so $f \simeq g$. □

Corollary 6.7 (Comparison Theorem). *Let $P_\bullet \to M$ be a projective resolution of M and let $f : M \to N$ be a map in R-**mod**. Then for any resolution $Q_\bullet \to N$ (not* *necessarily projective!), there is a chain map $P_\bullet \to Q_\bullet$ lifting f, and this chain map is unique up to homotopy.*

Proof. Take $P_\bullet$ to be exact in the theorem. $\square$

Theorem 6.8 (Horseshoe Lemma)**.** *Let $P'_\bullet \xrightarrow{\varepsilon'} A'$ and $P''_\bullet \xrightarrow{\varepsilon''} A''$ be projective resolutions in an abelian category $\mathcal{A}$, and suppose that $0 \to A' \xrightarrow{i} A \xrightarrow{\pi} A'' \to 0$ is a short exact sequence such that the following diagram commutes:*

$$
\begin{array}{ccccccccccc}
 & & & & & & & & 0 & & \\
 & & & & & & & & \downarrow & & \\
\dots & \longrightarrow & P'_2 & \longrightarrow & P'_1 & \longrightarrow & P'_0 & \xrightarrow{\varepsilon'} & A' & \longrightarrow & 0 \\
 & & & & & & & & \downarrow{\scriptstyle i_A} & & \\
 & & & & & & & & A & & \\
 & & & & & & & & \downarrow{\scriptstyle \pi_A} & & \\
\dots & \longrightarrow & P''_2 & \longrightarrow & P''_1 & \longrightarrow & P''_0 & \xrightarrow{\varepsilon''} & A'' & \longrightarrow & 0 \\
 & & & & & & & & \downarrow & & \\
 & & & & & & & & 0 & &
\end{array}
$$

Set $P_i = P'_i \oplus P''_i$ for each i. Then there are maps $P_i \to P_{i-1}$ such that $P_\bullet \to A$ is a projective resolution for A, and the short exact sequence $0 \to A' \xrightarrow{i} A \xrightarrow{\pi} A'' \to 0$ lifts to a short exact sequence of chain complexes $0 \to P'_\bullet \to P_\bullet \to P''_\bullet \to 0$.

Sketch Proof. We will prove the result in R mod , leaving the general abelian category as an exercise in the Freyd-Mitchell Embedding Theorem. Consider the diagram

$$
\begin{array}{ccc}
P'_0 & \xrightarrow{\varepsilon'} & A' \\
{\scriptstyle \begin{pmatrix} 1 \\ 0 \end{pmatrix}}\downarrow & & \downarrow{\scriptstyle i} \\
P'_0 \oplus P''_0 & & A \\
{\scriptstyle \begin{pmatrix} 0 & 1 \end{pmatrix}}\downarrow & & \downarrow{\scriptstyle \pi} \\
P''_0 & \xrightarrow[\varepsilon'']{} & A''.
\end{array}
$$

Since π is surjective and P''_0 is projective, there is a map $\alpha : P''_0 \to A$ such that the triangle

$$\begin{array}{ccc} & & A \\ & \overset{\alpha}{\nearrow} & \Big\downarrow{\scriptstyle \pi} \\ P_0'' & \xrightarrow[\varepsilon'']{} & A'' \end{array}$$

commutes. Define $\varepsilon : P_0' \oplus P_0'' \to A$ in matrix form by

$$\varepsilon = \begin{pmatrix} i \circ \varepsilon' & \alpha. \end{pmatrix}$$

In other words,

$$\varepsilon(p', p'') = i \circ \varepsilon'(p') + \alpha(p'').$$

We claim that ε is surjective. Let $a \in A$. Then $\pi(a) \in A''$, so $\pi(a) = \varepsilon''(p'')$ for some $p'' \in P_0''$. Then $a - \alpha(p'') \in \ker \pi = \operatorname{im} i$, so

$$a = i(a') + \alpha(p'')$$

for some $a' \in A'$. Since ε' is surjective, we have $a' = \varepsilon(p')$ for some $p' \in P_0'$. Then $a = \varepsilon(p', p'')$, so indeed ε is surjective. Now we have a diagram

$$\begin{array}{ccc} P_1' & \xrightarrow{\partial_1'} \!\!\!\!\rightarrow & \ker \varepsilon' \\ {\scriptstyle \begin{pmatrix} 1 \\ 0 \end{pmatrix}} \Big\downarrow & & \Big\downarrow {\scriptstyle \begin{pmatrix} 1 \\ 0 \end{pmatrix}} \\ P_1' \oplus P_1'' & & \ker \varepsilon \\ {\scriptstyle \begin{pmatrix} 0 & 1 \end{pmatrix}} \Big\downarrow & & \Big\downarrow {\scriptstyle \begin{pmatrix} 0 & 1 \end{pmatrix}} \\ P_1'' & \xrightarrow[\partial_1'']{} \!\!\!\!\rightarrow & \ker \varepsilon''. \end{array}$$

This is of the same form as the diagram we started with, so we get a surjection $\partial_1 : P_1' \oplus P_1'' \to \ker \varepsilon$. Continuing inductively, we get a projective resolution

$$\ldots \to P_2' \oplus P_2'' \to P_1' \oplus P_1'' \to P_0' \oplus P_0'' \to A \to 0.$$

It may be worth going through a few steps of this inductive procedure to convince yourself that it does actually work. $\square$

6.3 Left Derived Functors

Definition 6.9. Let $F : \mathcal{A} \to \mathcal{B}$ be a a right exact functor between abelian categories, and suppose that $\mathcal{A}$ has enough projectives. For an object $A \in \mathcal{A}$, take a projective resolution $P_\bullet \to A$, and define the n^{th} **left derived functor**

$$L_n F(A) = H_n(F(P_\bullet)).$$

Lemma 6.10. *For any choice of projective resolution in Definition 6.9, we have* $L_0 F(A) \cong F(A)$.

Proof. By definition, $L_0 F(A)$ is the zeroth homology of the chain complex

$$\ldots \to F(P_2) \stackrel{F(\partial_2)}{\to} F(P_1) \stackrel{F(\partial_1)}{\to} F(P_0) \to 0,$$

so we have

$$\begin{aligned} L_0 F(A) &= H_0(F(P_*)) \\ &= \operatorname{coker} F(\partial_1). \end{aligned}$$

Now, the sequence

$$P_1 \stackrel{\partial_1}{\to} P_0 \to \operatorname{coker} \partial_1 \to 0$$

is exact, so by Lemma 4.4, the sequence

$$F(P_1) \stackrel{F(\partial_1)}{\to} F(P_0) \to F(\operatorname{coker} \partial_1) \to 0$$

is also exact. In particular, we have

$$\operatorname{coker} F(\partial_1) \cong F(\operatorname{coker} \partial_1) \cong F(A),$$

and the result follows our earlier observation that $L_0 F(A) = \operatorname{coker} F(\partial_1)$. □

Lemma 6.11. *If $P_\bullet \to A$ and $Q_\bullet \to A$ are projective resolutions of $A \in \mathcal{A}$, then*

$$H_n(F(P_\bullet)) \cong H_n(F(Q_\bullet))$$

for all n. In other words, the left derived functor is well-defined.

Proof. We addressed the case $n = 0$ in Lemma 6.10. By the Comparison Theorem, we may lift $\mathrm{id} : A \to A$ to a chain map $f : P_\bullet \to Q_\bullet$ so that the following diagram commutes:

$$\begin{array}{ccccccccc} \dots & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow f_2 & & \downarrow f_1 & & \downarrow f_0 & & \downarrow \mathrm{id} & & \\ \dots & \longrightarrow & Q_2 & \longrightarrow & Q_1 & \longrightarrow & Q_0 & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

Let $f_* : H_\bullet F(P_\bullet) \to H_\bullet F(Q_\bullet)$ be the map induced by f_*. Similarly there is a chain map $g : Q_\bullet \to P_\bullet$ lifting $\mathrm{id} : A \to A$, and we have $g_* f_* = (gf)_*$. Since gf and $\mathrm{id} : P_\bullet \to P_\bullet$ are both chain maps lifting $\mathrm{id} : A \to A$, by the Comparison Theorem they are chain homotopic. The functor F takes this chain homotopy to a chain homotopy between $F(gf)$ and $F(\mathrm{id}_A) = \mathrm{id}_{F(P_\bullet)}$ so $(gf)_* = (\mathrm{id})_* : H_\bullet F(P_\bullet) \to H_\bullet F(Q_\bullet)$ is the identity on homology.

Similarly $f_* g_*$ is is the identity on $H_\bullet F(Q_\bullet)$, so f_*, g_* are isomorphisms. □

Corollary 6.12. *If A is projective, then $L_n F(A) = 0$ for $n \neq 0$.*

Proof. Trivial. □

Lemma 6.13. *Let $f : A \to A'$ be a map in an abelian category $\mathcal{A}$. Then there is a canonical map $L_n F(f) : L_n F(A) \to L_n F(A')$ for all n.*

Proof. Take projective resolutions $P_\bullet \to A$ and $P'_\bullet \to A'$. By the Comparison Theorem, we may lift f to a chain map $P_\bullet \to P'_\bullet$, which induces a map $L_n F(A) = H_n(P_\bullet) \to H_n(P'_\bullet) = L_n F(A')$. Furthermore, any two such chain maps $P_\bullet \to P'_\bullet$ are chain homotopic (by the Comparison Theorem), so they induce the same map on homology. Therefore the map $L_n F(A) \to L_n F(A')$ is natural. □

Corollary 6.14. *The n^{th} left derived functor is a functor $\mathcal{A} \to \mathcal{B}$.*

Proof. Let $A \in \mathcal{A}$ have projective resolution $P_\bullet \to A$. Then id_P lifts id_A, so $L_n F(\mathrm{id}_A) = (\mathrm{id}_P)^* = \mathrm{id}_{L_n F(A)}$.

Let $A \xrightarrow{f} B \xrightarrow{g} C$ be a diagram in $\mathcal{A}$. Then we need to show that $L_n F(g \circ f) = L_n F(g) \circ L_n F(f)$. Let

$$P_\bullet \to A, \quad Q_\bullet \to B, \quad R_\bullet \to C$$

be projective resolutions in $\mathcal{A}$. Then the chain maps from the proof of Lemma 6.13 give a commutative diagram

$$\begin{array}{ccccc} P_\bullet & \longrightarrow & Q_\bullet & \longrightarrow & R_\bullet \\ \downarrow & & \downarrow & & \downarrow \\ A & \xrightarrow[f]{} & B & \xrightarrow[g]{} & C, \end{array}$$

and the result follows by naturality of L_nF on maps. □

Corollary 6.15. *The* n^{th} *left derived functor is an* ***additive*** *functor* $\mathcal{A} \to \mathcal{B}$*.*

Proof. Let $A \underset{g}{\overset{f}{\rightrightarrows}} B$ be a diagram in $\mathcal{A}$. Let $P_\bullet \xrightarrow{\varepsilon} A$ and $Q_\bullet \xrightarrow{\eta} B$ be projective resolutions. By the Comparison Theorem, lift to chain maps $P_\bullet \underset{\widetilde{g}}{\overset{\widetilde{f}}{\rightrightarrows}} Q_\bullet$ such that the following diagrams commute.

$$\begin{array}{ccc} P_\bullet & \xrightarrow{\widetilde{f}} & Q_\bullet \\ {\scriptstyle\varepsilon}\downarrow & & \downarrow{\scriptstyle\eta} \\ A & \xrightarrow[f]{} & B \end{array} \qquad \begin{array}{ccc} P_\bullet & \xrightarrow{\widetilde{g}} & Q_\bullet \\ {\scriptstyle\varepsilon}\downarrow & & \downarrow{\scriptstyle\eta} \\ A & \xrightarrow[g]{} & B \end{array}$$

We also have a commutative diagram

$$\begin{array}{ccc} P_\bullet & \xrightarrow{\widetilde{f}+\widetilde{g}} & Q_\bullet \\ {\scriptstyle\varepsilon}\downarrow & & \downarrow{\scriptstyle\eta} \\ A & \xrightarrow[f+g]{} & B, \end{array}$$

so

$$L_nF(f+g) = H_n(\widetilde{f}+\widetilde{g}) = H_n(\widetilde{f}) + H_n(\widetilde{g}) = L_nF(f) + L_nF(g).$$

□

Lemma 6.16. *Let* $F : \mathcal{A} \to \mathcal{B}$ *be a right-exact functor, and let* $A \in \mathcal{A}$*. Suppose that*

$$M_1 \xrightarrow{\partial_1} M_0 \xrightarrow{\varepsilon} A \to 0$$

is an exact sequence. Then

$$F(M_0)/\operatorname{im} F(\partial_1) \cong F(A).$$

Proof. This follows immediately from the fact that

$$F(M_1) \to F(M_0) \to F(A) \to 0$$

is exact, by Lemma 4.4. □

Corollary 6.17. *If F is a right-exact functor, then $L_0F = F$.*

Proposition 6.18. *Let $F : \mathcal{A} \to \mathcal{B}$ be a right exact functor between abelian categories, and let $U : \mathcal{B} \to \mathcal{C}$ be an exact functor between abelian categories. Then*

$$U(L_nF) = L_n(UF).$$

Theorem 6.19. *The collection $\{L_iF\}_i$ forms a homological δ-functor.*

Proof. We shall only prove that the long exact sequence exists. For functoriality, see Weibel, Page 45, Theorem 2.4.6.

Let $0 \to A' \to A \to A'' \to 0$ be a short exact sequence in an abelian category $\mathcal{A}$, and let $F : \mathcal{A} \to \mathcal{B}$ be a right exact functor. Let $P'_\bullet \to A'$ and $P''_\bullet \to A''$ be projective resolutions. Then by the Horseshoe Lemma, there is a projective resolution $P_\bullet \to A$ fitting into the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P'_\bullet & \longrightarrow & P_\bullet & \longrightarrow & P''_\bullet & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0, \end{array}$$

where both rows are exact. Since the P''_n are projective (or by construction in the proof of the Horseshoe Lemma), the short exact sequence $0 \to P'_n \to P_n \to P''_n \to 0$ is split for each n. Since F is an additive functor it preserves split exact sequences, which means that

$$0 \to F(P'_n) \to F(P_n) \to F(P''_n) \to 0$$

is split[1] exact for each n. Therefore we have a short exact sequence

$$0 \to F(P'_\bullet) \to F(P_\bullet) \to F(P''_\bullet) \to 0$$

of chain complexes in $\mathcal{B}$, and by the Snake Lemma this gives us a long exact sequence

$$\ldots \to L_nF(A'') \stackrel{\delta_n}{\to} L_{n-1}F(A') \to L_{n-1}F(A) \to L_{n-1}F(A'') \stackrel{\delta_{n-1}}{\to} L_{n-2}F(A') \to \ldots.$$

□

Theorem 6.20. *If $\mathcal{A}$ has enough projectives, then the collection $\{L_iF\}_i$ forms a universal homological δ-functor.*

Proof. See Weibel, Page 47, Theorem 2.4.7. □

6.4 F-acyclic Objects

Definition 6.21. Let $F : \mathcal{A} \to \mathcal{B}$ be a left-exact functor between abelian categories. An object $P \in \mathcal{A}$ is F**-acyclic** if $L_iF(P) = 0$ for all $i \geq 1$.

Lemma 6.22. *Projective modules are F-acyclic for any F.*

Proof. Compute the derived functor using the constant projective resolution $\ldots \to P \to P \to P \to 0$ with differential id_P. □

Lemma 6.23. *Let $0 \to M \to P \to A \to 0$ be an exact sequence, where P is F-acyclic. Then*

1. *$L_iF(A) \cong L_{i-1}F(M)$ for $i \geq 2$.*
2. *$L_1F(A) \cong \ker((F(M) \to F(P))$.*

Proof. These both follow from the long exact sequence of L_iF. □

Theorem 6.24. *Let $P_* \to A$ be a resolution of A be F-acyclic objects. Then $L_iF(A) = H_iF(P_*)$ for all i.*

[1] The point here is that F is not exact, so we need the splitting to say that it preserves the exactness of this sequence.

Proof. Let $P_* \to A$ have differentials $\partial_i : P_i \to P_{i-1}$ and augmentation $\varepsilon : P_0 \to A$. We proceed by induction on i, with two base cases $i = 0$ and $i = 1$.

For $i = 0$, we know that $L_0 F(A) = F(A)$, and we have $H_0 F(P_*) \cong F(A)$ by Lemma 6.16. Let $M = \ker \varepsilon$. Then $0 \to M \to P_0 \to A \to 0$ is a short exact sequence. Since F is right-exact, it preserves cokernels, so

$$F(M) = F(\operatorname{coker} \partial_2) = \operatorname{coker} F(\partial_2) = \frac{F(P_1)}{\operatorname{im} F(\partial_2)}.$$

Therefore,

$$\begin{aligned} L_1 F(A) &= \ker \left(\frac{F(P_1)}{\operatorname{im} F(P_2)} \to F(P_0) \right) \\ &= H_1 F(P_*). \end{aligned}$$

For the inductive step, let $i \geq 2$. Then $0 \to M \to P_0 \to A \to 0$ is exact, so by Lemma 6.23, we have

$$L_i F(A) \cong L_{i-1} F(M) \cong H_{i-1}(\ldots \to F(P_2) \to F(P_1) \to 0) \cong H_i F(P_*),$$

where the second isomorphism is by induction. ☐

6.5 Flat Modules

Definition 6.25. A module F is **flat** if $- \otimes_R F$ is an exact functor.

Lemma 6.26. *Let B be a left R-module. The following are equivalent:*

1. *B is flat.*
2. *$\operatorname{Tor}_n^R(A, B) = 0$ for all $n \geq 1$ and all left R-modules A.*
3. *$\operatorname{Tor}_1^R(A, B) = 0$ for all left R-modules A.*

Proof. Suppose that B is flat. Let $F_* \to A$ be a free resolution of A. Since $- \otimes_R B$ is exact, the sequence

$$\ldots \to F_2 \otimes_R B \to F_1 \otimes_R B \to F_0 \otimes_R B \to A \otimes_R B \to 0$$

is exact, so the homology of

$$\ldots \to F_2 \otimes_R B \to F_1 \otimes_R B \to F_0 \otimes_R B \to 0$$

vanishes in positive degree. Therefore, we have (1) $\implies$ (2).

The implication (2) $\implies$ (3) is trivial. Finally, (3) $\implies$ (1) follows from the long exact sequence of Tor, since for any short exact sequence $0 \to X \to Y \to Z \to 0$, we have that

$$0 = \mathrm{Tor}_1^R(Z, B) \to X \otimes B \to Y \otimes B \to Z \otimes B \to 0$$

is exact. □

Corollary 6.27. *Every projective module is flat.*

Proof. If P is projective, then $0 \to P$ is a projective resolution, so $\mathrm{Tor}_n^R(A, P) = 0$ for all A and all $n \geq 1$. □

Let I be a partially ordered set. We say that I is **filtered** if for all i, j in I, there exists $k \in I$ such that $i, j < k$. Recall that we may view a partially ordered set as a category, where $\mathrm{Hom}(i, j)$ has precisely one morphism if $i \leq j$, and is empty otherwise.

For categories $\mathcal{C}$ and $\mathcal{D}$, we write $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ for the category of functors from $\mathcal{C}$ to $\mathcal{D}$.

Lemma 6.28. *Let I be a filtered partially ordered set. Then the functor*

$$\mathrm{colim} : \mathrm{Fun}(I, R\text{-}\mathbf{mod}) \to R\text{-}\mathbf{mod}$$

is exact.

Proof. See Weibel, Page 57, Theorem 2.6.15. □

Corollary 6.29. *A filtered colimit of flat R-modules is flat.*

Proof. The tensor-hom adjunction tells us that the functor $- \otimes_R M : R\text{-}\mathbf{mod} \to R\text{-}\mathbf{mod}$ is a left adjoint, so it preserves colimits. Let $F : I \to R\text{-}\mathbf{mod}$ be a filtered diagram of flat R-modules, and let

$$0 \to X \to Y \to Z \to 0$$

be a short exact sequence in R-**mod**. The $F(i)$ are flat, so each sequence

$$0 \to F(i) \otimes X \to F(i) \otimes Y \to F(i) \otimes Z \to 0$$

is exact. Since colim_I is exact, we have a short exact sequence

$$0 \to \mathrm{colim}_I(F(i) \otimes X) \to \mathrm{colim}_I(F(i) \otimes Y) \to \mathrm{colim}_I(F(i) \otimes Z) \to 0.$$

Finally, the result follows from the fact that $- \otimes_R M$ commutes with colimits for all R-modules M. □

We note the two following facts. I'm not sure if they're examinable, and I won't give proofs, but I'm stating them to be on the safe side.

Fact 6.30. A $\mathbb{Z}$-module is flat if and only if it is torsion-free.

Fact 6.31. Let R be a ring and $S \subset Z(R)$ be a central multiplicative set. Then $S^{-1}R$ is a flat R-module.

Corollary 6.32. *The $\mathbb{Z}$-module $\mathbb{Q}$ is flat.*

We already know that Tor can be computed using projective resolutions. We also know that all projective resolutions are flat. In fact, it turns out that we can relax the projective assumption altogether and computer Tor using any flat resolution. This is the content of the following lemma.

Theorem 6.33 (Flat Resolution Lemma)**.** *Let A be an R-module and let $F_\bullet \to A$ be a flat resolution of A. Then for any R-module B, and all n, we have*

$$\mathrm{Tor}_n^R(A, B) \cong H_n(F_* \otimes_R B).$$

Proof. This is immediate from Lemma 6.24, since the flat modules F_i are $- \otimes_R B$-acyclic. □

6.6 Right Derived Functors

Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor between abelian categories, and assume that $\mathcal{A}$ has enough injectives. For an element $A \in \mathcal{A}$, let $A \to I^\bullet$ be a projective resolution.

Definition 6.34. The n^{th} **right derived functor** of F is

$$R^n F(A) = H^n(F(I^\bullet)).$$

Since an injective object of $\mathcal{A}$ is precisely a projective object of $\mathcal{A}^{\text{op}}$, the injective resolution $A \to I^\bullet$ is a projective resolution in $\mathcal{A}^{\text{op}}$. Therefore[1], right derived functors are just left derived functors in the opposite category, and we get lots of results about them for free, by duality. Most significantly, we get that right derived functors are universal cohomological δ-functors. In particular, we have the following result.

Lemma 6.35. *Let $F : \mathcal{A} \to \mathcal{B}$ be left exact, and let*

$$0 \to A \to B \to C \to 0$$

be a short exact sequence in $\mathcal{A}$. Then we get a long exact sequence

$$0 \to F(A) \to F(B) \to F(C) \to R^1F(A) \to R^1F(B) \to R^1F(C) \to R^2F(A) \to \dots.$$

[1]As is often the case with duality, we are sweeping a lot of technicalities under the rug. It is generally fine to do this, as long as we know what needs to be checked.

7 Defining Ext and Tor

The two most famous derived functors are Ext and Tor. Like most homological constructions, these find uses throughout pure mathematics. One of the most obvious (to me) applications is the Universal Coefficient Theorem from algebraic topology, which allows us to compute the homology and cohomology of topological spaces with all sorts of funky coefficients, using only the homology with $\mathbb{Z}$-coefficients.

7.1 Tor

We know that $- \otimes_R B$ is a right exact functor R-**mod** $\to R$-**mod**.

Definition 7.1. For any R-modules A and B, the modules $\mathrm{Tor}_*^R(A, B)$ are defined to be the values of the left derived functor

$$\mathrm{Tor}_*^R(A, B) = L_*(- \otimes_R B)(A).$$

Example 7.2. We will compute

$$\mathrm{Tor}_*^{\mathbb{Z}}(\mathbb{Z}/6, \mathbb{Z}/9).$$

We want the derived functor of $- \otimes_{\mathbb{Z}} \mathbb{Z}/9$, evaluated at $\mathbb{Z}/6$. Therefore we take a projective resolution of $\mathbb{Z}/6$:

$$0 \to \mathbb{Z} \xrightarrow{6} \mathbb{Z} \to \mathbb{Z}/6 \to 0.$$

The modules $\mathrm{Tor}_*^{\mathbb{Z}}(\mathbb{Z}/6, \mathbb{Z}/9)$ are then the homology of

$$0 \to \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/9 \xrightarrow{6 \otimes \mathbb{Z}/9} \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/9 \to 0.$$

By the natural isomorphism $R \otimes_R M \cong M$, this chain complex is isomorphic to

$$0 \to \mathbb{Z}/9 \xrightarrow{6} \mathbb{Z}/9 \to 0.$$

Let $f : \mathbb{Z}/9 \to \mathbb{Z}/9$ be the multiplication-by-6 map. Then

$$\operatorname{Tor}_n^{\mathbb{Z}}(\mathbb{Z}/6, \mathbb{Z}/9) = \begin{cases} \operatorname{coker} f & \text{if } n = 0, \\ \ker f & \text{if } n = 1, \\ 0 & \text{else.} \end{cases}$$

Finally, we have

$$\ker f = \{[n] \in \mathbb{Z}/9 : 9 \mid 6n\} = 3\mathbb{Z}/9\mathbb{Z} \cong \mathbb{Z}/3,$$

and

$$\operatorname{coker} f = \left(\frac{\mathbb{Z}}{9\mathbb{Z}}\right)/\left(\frac{6\mathbb{Z} + 9\mathbb{Z}}{9\mathbb{Z}}\right) \cong \mathbb{Z}/3.$$

Therefore, we have

$$\operatorname{Tor}_n^{\mathbb{Z}}(\mathbb{Z}/6, \mathbb{Z}/9) \cong \begin{cases} \mathbb{Z}/3 & \text{if } n = 0, 1, \\ 0 & \text{else.} \end{cases}$$

Example 7.3. Similarly, if R is a PID and $a, b \in R$, then

$$\operatorname{Tor}_n^R(R/a, R/b) \cong \begin{cases} R/d & \text{if } n = 0, 1, \\ 0 & \text{else,} \end{cases}$$

where $d = \gcd(a, b)$. This is basically exactly the same proof as the previous example.

We also know that, for any R-module A, the functor $\operatorname{Hom}_R(A, -)$ is left exact. Therefore, it has right derived functors.

7.2 Ext

Definition 7.4. For R-module A and B, we define the modules $\operatorname{Ext}_R^*(A, B)$ to be the values of the right derived functors

$$\operatorname{Ext}_R^*(A, B) = R^*(\operatorname{Hom}_R(A, -))(B).$$

Example 7.5. We will compute

$$\mathrm{Ext}^*_{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Z}/4).$$

Since we are evaluating a right derived functor at $\mathbb{Z}/4$, we need to take an injective resolution. We use the injective resolution

$$\mathbb{Z}/4 \overset{1\mapsto[1/4]}{\to} \mathbb{Q}/\mathbb{Z} \overset{4}{\to} \mathbb{Q}/\mathbb{Z} \to 0.$$

Therefore, $\mathrm{Ext}^*_{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Z}/4)$ is the cohomology of the cochain complex

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Q}/\mathbb{Z}) \overset{4\circ-}{\to} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Q}/\mathbb{Z}),$$

which is isomorphic to

$$\mathbb{Z}/2 \overset{0}{\to} \mathbb{Z}/2.$$

Therefore,

$$\mathrm{Ext}^n_{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Z}/4) \cong \begin{cases} \mathbb{Z}/2 & \text{if } n = 0, 1, \\ 0 & \text{else.} \end{cases}$$

8 Balancing Ext and Tor

We defined $\text{Tor}_*^R(-, B)$ as the left derived functor of $-\otimes_R B$. In other words, $\text{Tor}_*^R(A, B)$ is the homology of $P_* \otimes_R B$, where $P_* \to A$ is a projective resolution. It turns out that we can also compute $\text{Tor}_*^R(A, B)$ by taking the homology of $A \otimes_R Q_*$, where $Q_* \to B$ is a projective resolution. The proof of this fact is colloquially known as "balancing Tor".

Similarly, we defined $\text{Ext}_R^*(A, B)$ as the cohomology of $\text{Hom}(A, I^*)$, where $B \to I^*$ is an injective resolution. It turns out that we can also compute Ext by taking the cohomology of $\text{Hom}(P_*, B)$, where $P_* \to A$ is a projective resolution.

To prove these facts, we need some more machinery, which we will develop in this section.

8.1 Mapping Cones

Definition 8.1. Let $f : B_\bullet \to C_\bullet$ be a chain map. We define the **mapping cone** of f to be the chain complex $\text{cone}(f)$ with degree n part

$$\big[\text{cone}(f)\big]_n = B_{n-1} \oplus C_n,$$

and differential

$$\partial(b_{n-1}, c_n) = (-\partial_B(b_{n-1}), \partial_C(c_n) - f(b_{n-1})).$$

The definition of the differential on $\text{cone}(f)$ is very unpleasant. It is easier to think of it as being represented by a matrix

$$\begin{pmatrix} -\partial_B & 0 \\ -f & \partial_C \end{pmatrix},$$

acting on column vectors of the form $(b_{n-1}, c_n)^T$. Yet another way of thinking about the differential is via the following diagram:

$$\begin{array}{ccc} B_{n-1} & \xrightarrow{-\partial} & B_{n-2} \\ & \searrow^{-f} & \\ C_n & \xrightarrow{\partial} & C_{n-1}. \end{array}$$

For the sake of completeness, we also define the dual notion.

Definition 8.2. If $f : B^\bullet \to C^\bullet$ is a cochain map, then the mapping cone of f is the cochain complex cone(f) with degree n part

$$\big[\operatorname{cone}(f)\big]_n = B^{n+1} \oplus C^n,$$

and whose differential is

$$\partial(b_{n+1}, c_n) = (-\partial_B(b_{n+1}), \partial_C(c_n) - f(b_{n+1})).$$

Lemma 8.3. *Let $f : B \to C$ be a chain map. There is a long exact sequence of homology*

$$\ldots \to H_{n+1}(\operatorname{cone}(f)) \xrightarrow{\pi_*} H_n(B) \xrightarrow{f_*} H_n(C) \xrightarrow{i_*} H_n(\operatorname{cone}(f)) \to \ldots,$$

where $f_* = H_\bullet(f)$ is the map on homology induced by f.

Proof. We define a short exact sequence of chain complexes

$$0 \to C \xrightarrow{i} \operatorname{cone}(f) \xrightarrow{\pi} B[-1] \to 0$$

by $i(c) = (0, c)$ and $\pi(b, c) = -b$. It is easy to check that these are chain maps, and that the sequence is exact.

Note that $H_{n+1}(B[-1]) = H_n(B)$. By the Snake Lemma, this induces a long exact sequence of homology

$$\ldots \to H_{n+1}(\operatorname{cone}(f)) \xrightarrow{\pi_*} H_n(B) \xrightarrow{\delta} H_n(C) \xrightarrow{i_*} H_n(\operatorname{cone}(f)) \to \ldots,$$

for a connecting map $\delta : H_n(B) \to H_n(C)$.

(**Note:** *The following paragraph is hard to understand. Probably the best way to under-*

stand it is to work out for yourself what $\delta(\bar{b})$ is using the proof of the Snake Lemma.)

In particular, for a cycle $b \in B_n$, we have that $\delta(\bar{b}) = \bar{c}$ for any cycle $c \in C$ with $i(c) = \partial(\alpha, \beta)$, where (α, β) is an element of cone(f) such that $\pi(\alpha, \beta) = b$.

By definition of π, it suffices to take $(\alpha, \beta) = (-b, 0)$. Then we have $i(c) = \partial(-b, 0) = (\partial b, f(b)) = (0, f(b))$, since b is a cycle. Therefore we have $c = f(b)$ and hence $\delta(\bar{b}) = \overline{f(b)} = f_*(\bar{b})$. So $\delta = f_*$ and the result follows. □

Corollary 8.4. *A chain map $f : B \to C$ is a quasi-isomorphism if and only if the mapping cone* cone(f) *is acyclic.*

Proof. Suppose that f is a quasi-isomorphism. Then we have an exact sequence

$$H_n(B) \overset{\cong}{\to} H_n(C) \overset{i_*}{\to} H_n(\text{cone}(f)) \overset{\pi_*}{\to} H_{n-1}(B) \overset{\cong}{\to} H_{n-1}(C).$$

By exactness at $H_n(C)$ and $H_{n-1}(B)$ we have that $\ker(i_*) = H_n(C)$ and $\text{im}(\pi_*) = 0$.

So i_* kills everything, which means that $\text{im}(i_*) = 0$, and π_* kills everything, so $\ker(\pi_*) = H_n(\text{cone}(f))$. By exactness we have

$$0 = \text{im}(i_*) = \ker(\pi_*) = H_n(\text{cone}(f)),$$

so cone(f) is acyclic. The converse is very straightforward; if cone(f) is acyclic then we have an exact sequence

$$0 \to H_n(B) \overset{f_*}{\to} H_n(\text{cone}(f)) \to 0.$$

□

8.2 Double Complexes

We now define something called a double complex. Annoyingly, there are two very similar but different conventions. We will follow Weibel, but see André Henriques's notes for the other convention.

Definition 8.5. A **double complex** $C_{\bullet\bullet}$ is a set of abelian groups $C_{p,q}$ indexed by $(p, q) \in \mathbb{Z}^2$ together with maps $d_h : C_{p,q} \to C_{p-1,q}$ and $d_v : C_{p,q} \to C_{p,q-1}$ satisfying

$d_h^2 = d_v^2 = 0$ and $d_h d_v = -d_v d_h$.

In words, a double complex is an infinite two-dimensional grid of abelian groups where each row (respectively each column) is a chain complex, and the horizontal and vertical differentials anticommute. The following diagram is a picture of a double complex.

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longleftarrow & C_{-1,1} & \longleftarrow & C_{01} & \longleftarrow & C_{11} & \longleftarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longleftarrow & C_{-1,0} & \longleftarrow & C_{00} & \longleftarrow & C_{10} & \longleftarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longleftarrow & C_{-1,-1} & \longleftarrow & C_{0,-1} & \longleftarrow & C_{1,-1} & \longleftarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

8.3 Total Complexes

The total complex is a way of turning a double complex into a chain complex by taking "diagonal slices".

Definition 8.6. The **total chain complex** $\mathrm{Tot}(C_{\bullet\bullet})$ defined by $C_{\bullet\bullet}$ is the chain complex with

$$\Big[\mathrm{Tot}(C_{\bullet\bullet})\Big]_n = \bigoplus_{p+q=n} C_{p,q},$$

together with differential

$$d^{\mathrm{Tot}} : \Big[\mathrm{Tot}(C_{\bullet\bullet})\Big]_n \to \Big[\mathrm{Tot}(C_{\bullet\bullet})\Big]_{n-1}, \quad d^{\mathrm{Tot}} = d_h + d_v.$$

The total complex is illustrated by the colours in the following diagram; each "diagonal slice" is given a different colour. For example, $\mathrm{Tot}(C_{\bullet\bullet})_0$ is the direct sum of all the red terms.

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longleftarrow & C_{-1,1} & \longleftarrow & C_{01} & \longleftarrow & C_{11} & \longleftarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longleftarrow & C_{-1,0} & \longleftarrow & C_{00} & \longleftarrow & C_{10} & \longleftarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longleftarrow & C_{-1,-1} & \longleftarrow & C_{0,-1} & \longleftarrow & C_{1,-1} & \longleftarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

There is also a version of the total chain complex where everything is the same, but we take the direct product instead of the direct sum. This is denoted by $\text{Tot}^{\prod}(C_{\bullet\bullet})$. To avoid confusion, we will often write $\text{Tot}^{\oplus}(C_{\bullet\bullet})$ for $\text{Tot}(C_{\bullet\bullet})$.

Let $C_{\bullet\bullet}$ be a double complex. We say that $C_{\bullet\bullet}$ is an **upper half plane** complex if there is some q_0 such that $C_{pq} = 0$ whenever $q < q_0$. Similarly $C_{\bullet\bullet}$ is a **right half plane** complex if there is some p_0 such that $C_{pq} = 0$ whenever $p < p_0$.

The following lemma tell us that in certain situations, the total complex is not only a chain complex, but actually an exact sequence in R-**mod**. We defer to Weibel for the proof.

Lemma 8.7 (Acyclic Assembly Lemma)**.** *Let $C_{\bullet\bullet}$ be a double complex in R-**mod**. Consider the four following situations.*

1. *$C_{\bullet\bullet}$ is an upper half plane complex with exact columns.*
2. *$C_{\bullet\bullet}$ is a right half plane complex with exact rows.*
3. *$C_{\bullet\bullet}$ is an upper half plane complex with exact rows.*
4. *$C_{\bullet\bullet}$ is a right half plane complex with exact columns.*

If (1) *or* (2) *holds, then* $\text{Tot}^{\prod}(C_{\bullet\bullet})$ *is exact. If* (3) *or* (4) *holds, then* $\text{Tot}^{\oplus}(C_{\bullet\bullet})$ *is exact.*

Proof. See Weibel, Page 59, Lemma 2.7.3. □

We also note part of Henriques's formulation. The proof is similar.

Lemma 8.8 (Henriques's AAL)**.** *Let $C_{\bullet\bullet}$ be a double complex such that for every n, there exist only finitely many pairs (p, q) such that $p + q = n$ and $C_{pq} \neq 0$. If $C_{\bullet\bullet}$ has exact rows, then $\mathrm{Tot}^{\oplus}(C_{\bullet\bullet})$ is exact.*

8.4 Non-Canonical Orderings on Double Complexes

As if things weren't confusing enough, we will sometimes write double complexes where the arrows do not point down and to the left. You might see the term "double cochain complex" bandied about, but we will not bother with this. Instead, we call everything a double complex, regardless of where the arrows point.

When the arrows do point down and to the left, we will say that the double complex is **canonically ordered**. Non-canonical orderings get a bit dangerous when we want to apply the Acyclic Assembly Lemma. For instance, the double complex

$$\begin{array}{ccccc} \vdots & & \vdots & & \\ \uparrow & & \uparrow & & \\ C_{01} & \longleftarrow & C_{11} & \longleftarrow & \dots \\ \uparrow & & \uparrow & & \\ C_{00} & \longleftarrow & C_{10} & \longleftarrow & \dots \end{array} \qquad (1)$$

is **not** an upper half plane complex, even though we have drawn it in the upper half plane. This is because, in order to make it into a *canonically ordered* double complex, we would have to reverse the vertical arrows.

Another important point[1] is that in order to take the total complex, strictly speaking you have to put your double complex into canonical order. However, you can usually skip this step and take diagonal slices of the non-canonically ordered double complexes straight away. To do this, note that each arrow must point from one diagonal slice into

[1] This paragraph is hard to understand. It might be best to leave it to one side until you see why it's relevant.

the next. For instance, the complex $C_{\bullet\bullet}$ illustrated in (1) must have diagonal slices in the $\nearrow$ direction. That is, we have

$$\mathrm{Tot}^{\Pi}(C_{\bullet\bullet})_n = \prod_i C_{i,i-n}.$$

8.5 Balancing Tor

For the following definition, let M and N be right and left R-modules respectively, and let $P_\bullet$ and $Q_\bullet$ be projective resolutions for M and N. Let $\partial^{(P)}$ and $\partial^{(Q)}$ be the differentials for $P_\bullet$ and $Q_\bullet$.

Definition 8.9. The double complex $P_\bullet \otimes_R Q_\bullet$ has components

$$(P_\bullet \otimes_R Q_\bullet)_{p,q} = P_p \otimes_R Q_q.$$

The differentials on $P_\bullet \otimes_R Q_\bullet$ are the maps

$$d_h : P_p \otimes_R Q_q \to P_{p-1} \otimes_R Q_q, \quad d_v : P_p \otimes_R Q_q \to P_p \otimes_R Q_{q-1},$$

given by

$$d_h = \partial^{(P)} \otimes 1, \quad d_v = (-1)^p \otimes \partial^{(Q)}.$$

The following lemma tells us that $P_\bullet \otimes_R Q_\bullet$ is a legitimate double complex.

Lemma 8.10. *The differentials in $P_\bullet \otimes_R Q_\bullet$ anticommute.*

Proof. We need to check that the following diagram anticommutes:

$$\begin{array}{ccc} P_p \otimes_R Q_q & \xrightarrow{\partial^{(P)} \otimes Q_q} & P_{p-1} \otimes_R Q_q \\ {\scriptstyle P_p \otimes \partial_q^{(Q)}} \downarrow & & \downarrow {\scriptstyle P_{p-1} \otimes \partial^{Q}} \\ P_p \otimes_R Q_{q-1} & \xrightarrow[\partial^{(P)} \otimes Q_{q-1}]{} & P_{p-1} \otimes_R Q_{q-1} \end{array}$$

By definition of the fuctors $P_m \otimes_R -$ and $- \otimes_R Q_n$, the diagram anticommutes for simple tensors in $P_p \otimes_R Q_q$. Since these simple tensors generate the entire module, the diagram commutes in general. $\square$

Lemma 8.11. *Let $M, N, P_\bullet, Q_\bullet$ be as above. There are natural quasi-isomorphisms*

$$P_\bullet \otimes_R N \leftarrow \mathrm{Tot}(P_\bullet \otimes_R Q_\bullet) \to M \otimes_R Q_\bullet.$$

Proof. View M and N chain complexes with only one nonzero component (of degree 0). Then we can form double complexes $P_\bullet \otimes Q_\bullet$, $P_\bullet \otimes N$ and $M \otimes Q_\bullet$ in the obvious way.

Let $\epsilon : P_\bullet \to M$ and $\eta : Q_\bullet \to N$ be the augmentation maps (that is they send elements all the way down the exact sequence in the obvious way). Then we have maps

$$\epsilon \otimes Q_\bullet : P_\bullet \otimes Q_\bullet \to M \otimes Q_\bullet,$$

and

$$P_\bullet \otimes \eta : P_\bullet \otimes Q_\bullet \to P_\bullet \otimes N.$$

These maps in turn induce chain maps

$$f : \mathrm{Tot}(P_\bullet \otimes Q_\bullet) \to \mathrm{Tot}(M \otimes Q_\bullet) \cong M \otimes Q_\bullet,$$

and

$$g : \mathrm{Tot}(P_\bullet \otimes Q_\bullet) \to \mathrm{Tot}(P_\bullet \otimes N) \cong P_\bullet \otimes N.$$

We claim that f and g are quasi-isomorphisms.

Let $C_{\bullet\bullet}$ be the double complex obtained from $P_\bullet \otimes Q_\bullet$ by adding $M \otimes Q_\bullet$ in the column $p = -1$. To clarify, we represent $C_{\bullet\bullet}$ by the following diagram, where the newly added terms are shown in red.

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\downarrow & & \downarrow & & \downarrow & & \\
M \otimes Q_2 & \longleftarrow & P_0 \otimes Q_2 & \longleftarrow & P_1 \otimes Q_2 & \longleftarrow & \dots \\
\downarrow & & \downarrow & & \downarrow & & \\
M \otimes Q_1 & \longleftarrow & P_0 \otimes Q_1 & \longleftarrow & P_1 \otimes Q_1 & \longleftarrow & \dots \\
\downarrow & & \downarrow & & \downarrow & & \\
M \otimes Q_0 & \longleftarrow & P_0 \otimes Q_0 & \longleftarrow & P_1 \otimes Q_0 & \longleftarrow & \dots
\end{array}$$

One can check that

$$\big[\operatorname{Tot}(C_{\bullet\bullet})\big]_n = (M \otimes Q_{n+1}) \oplus \big[\operatorname{Tot}(P_\bullet \otimes Q_\bullet)\big]_n \cong \big[\operatorname{Tot}(P_\bullet \otimes Q_\bullet)\big]_n \oplus (M \otimes Q_{n+1}).$$

It follows that, at least as sets, we have

$$\operatorname{Tot}(C_{\bullet\bullet})[-1] \cong \operatorname{cone}(f).$$

We will not verify that the isomorphism is a map of double complexes, but this is not too hard to do.

Since each $- \otimes Q_q$ is an exact functor (because the Q_q are projective, hence flat), the rows of C are exact. Since C is an upper half plane complex, the chain complex $\operatorname{Tot}(C_{\bullet\bullet})$ is acyclic by the Acyclic Assembly Lemma. Thus, $\operatorname{cone}(f)$ is an acyclic chain complex, so by Corollary 8.4, we have that f is a quasi-isomorphism.

The proof that g is a quasi-isomorphism is similar, so we omit it. □

Corollary 8.12. *We have an isomorphism*

$$H_n(P_\bullet \otimes N) \cong H_n(M \otimes Q_\bullet),$$

for each n, so Tor *can be computed by*

$$\operatorname{Tor}_n^R(M, N) = L_n(M \otimes -)(N) \cong L_n(- \otimes N)(M).$$

8.6 Balancing Ext

We define two more types of double complex. For illustrations, see the proofs of Lemma 8.5 and Lemma 8.16. Let $C_\bullet$ and $D_\bullet$ be chain complexes. Then we define the double complex $\operatorname{Hom}(C_\bullet, D_\bullet)$ to have terms

$$\operatorname{Hom}(C_\bullet, D_\bullet)_{pq} = \operatorname{Hom}(C_p, D_q),$$

and differentials are

$$d^h_{pq}\varphi = (-1)^q \times \varphi \circ \partial^{(D)}_{p+1}, \quad d^v_{pq} = \partial^{(D)}_q \circ \varphi.$$

This is a non-canonical ordering, with d^h pointing to the right, and d^v pointing down.

If instead $D^\bullet$ is a cochain complex, then we define $\mathrm{Hom}(C_\bullet, D^\bullet)$ to have the same terms (i.e. the (p,q) term is $\mathrm{Hom}(C_p, D^q)$) and the same horizontal differentials, but with vertical differentials

$$d^v_{pq}\varphi = \partial^q_{(D)} \circ \varphi.$$

This is again a non-canonical ordering, with differentials pointing up and to the right.

Theorem 8.13. *Let M and N be R-modules (either both left or both right). Then we have quasi-isomorphisms*

$$\mathrm{Hom}_R(P_\bullet, N) \to \mathrm{Tot}(\mathrm{Hom}_R(P_\bullet, I^\bullet)) \leftarrow \mathrm{Hom}_R(M, I^\bullet),$$

and hence $\mathrm{Ext}^i_R(M,N)$ can be computed by any of

$$H^i(\mathrm{Hom}_R(P_\bullet, N)) \cong H^i(\mathrm{Tot}(\mathrm{Hom}_R(P_\bullet, I^\bullet)) \cong H^i(\mathrm{Hom}_R(M, I^\bullet).$$

Proof. The double complex $\mathrm{Hom}_R(P_\bullet, I^\bullet)$ looks like this.

$$\begin{array}{ccccc}
\vdots & & \vdots & & \\
\uparrow & & \uparrow & & \\
\mathrm{Hom}(P_0, I^1) & \longrightarrow & \mathrm{Hom}(P_1, I^1) & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \\
\mathrm{Hom}(P_0, I^0) & \longrightarrow & \mathrm{Hom}(P_1, I^0) & \longrightarrow & \cdots
\end{array}$$

Let $C_{\bullet\bullet}$ be the double complex obtained by adding $\mathrm{Hom}(M, I^0)$ on the left.

$$\begin{array}{ccccccc}
\color{red}{\vdots} & & \vdots & & \vdots & & \\
\color{red}{\uparrow} & & \uparrow & & \uparrow & & \\
\color{red}{\mathrm{Hom}(M, I^1)} & \longrightarrow & \mathrm{Hom}(P_0, I^1) & \longrightarrow & \mathrm{Hom}(P_1, I^1) & \longrightarrow & \cdots \\
\color{red}{\uparrow} & & \uparrow & & \uparrow & & \\
\color{red}{\mathrm{Hom}(M, I^0)} & \longrightarrow & \mathrm{Hom}(P_0, I^0) & \longrightarrow & \mathrm{Hom}(P_1, I^0) & \longrightarrow & \cdots
\end{array}$$

By the injectivity of the I^n, the double complex $C_{\bullet\bullet}$ has exact columns. If we reversed the arrows, we could turn $C_{\bullet\bullet}$ into a canonically ordered double complex satisfying the hypotheses of Henriques's Acyclic Assembly Lemma. Therefore $\mathrm{Tot}^{\oplus}(C_{\bullet\bullet})$ is acyclic. The n^{th} term of this total complex is

$$\mathrm{Tot}^{\oplus}(C_{\bullet\bullet})_n = \mathrm{Hom}(M, I^{n+1}) \oplus \mathrm{Tot}^{\oplus}\,\mathrm{Hom}(P_\bullet, I^\bullet).$$

Now, the bottom vertical arrows in the most recent diagram give us a chain map $f : \mathrm{Hom}(M, I^\bullet) \to \mathrm{Tot}^{\oplus}(P_\bullet, I^\bullet)$, which has mapping cone

$$\mathrm{cone}(f)_n = \mathrm{Hom}(M, I^{n+1}) \oplus \mathrm{Tot}^{\oplus}\,\mathrm{Hom}(P_\bullet, I^\bullet).$$

Therefore $\mathrm{cone}(f)$ is isomorphic to the total complex, hence it is acyclic, so f is a quasi-isomorphism.

Similarly there is a map $g : \mathrm{Hom}(P_\bullet, N) \to \mathrm{Tot}^{\oplus}(P_\bullet, I^\bullet)$. The corresponding double complex has exact rows because the P_i are projective, and it is not too hard to show that $\mathrm{cone}(g)$ is acyclic, so g is a quasi-isomorphism. □

Corollary 8.14. *The functor* Ext *can be computed by either of*

$$\mathrm{Ext}^n_R(M, N) = R^n(\mathrm{Hom}(-, N))(M) \cong R^n(\mathrm{Hom}(M, -))(N).$$

Since $\mathrm{Hom}(-,)$ is contravariant, its right derived functor is computed using injective resolutions (since they are projective in the opposite category). Therefore, Corollary 8.14 tells us that $\mathrm{Ext}^*_R(A, B)$ is the cohomology of the cochain complex $\mathrm{Hom}_R(P_*, B)$, where $P_* \to A$ is a projective resolution. The following (quite hard) example uses this fact to compute Ext.

Example 8.15. Let $a \geq b \geq c$ be integers. We will compute $\mathrm{Ext}^*_{\mathbb{Z}/2^a}(\mathbb{Z}/2^b, \mathbb{Z}/2^c)$. Let $R = \mathbb{Z}/2^a\mathbb{Z}$. Then $\mathbb{Z}/2^b/\mathbb{Z}$ has projective resolution

$$\ldots \to R \xrightarrow{2^{a-b}} R \xrightarrow{2^b} R \xrightarrow{2^{a-b}} R \xrightarrow{2^b} R \xrightarrow{1} \mathbb{Z}/2^b\mathbb{Z}.$$

Therefore $\mathrm{Ext}^*_R(\mathbb{Z}/2^b, \mathbb{Z}/2^c)$ is the cohomology of

$$\ldots \leftarrow \mathbb{Z}/2^c \xleftarrow{2^{a-b}} \mathbb{Z}/2^c \xleftarrow{2^b} \mathbb{Z}/2^c \xleftarrow{2^{a-b}} \mathbb{Z}/2^c \xleftarrow{2^b} \mathbb{Z}/2^c.$$

Since $b > c$, the map $2^b : \mathbb{Z}/2^c \to \mathbb{Z}/2^c$ is zero, so $\mathrm{Ext}^*_R(\mathbb{Z}/2^b, \mathbb{Z}/2^c)$ is the cohomology of

$$\dots \leftarrow \mathbb{Z}/2^c \overset{2^{a-b}}{\leftarrow} \mathbb{Z}/2^c \overset{0}{\leftarrow} \mathbb{Z}/2^c \overset{2^{a-b}}{\leftarrow} \mathbb{Z}/2^c \overset{0}{\leftarrow} \mathbb{Z}/2^c.$$

Therefore, we have

$$\mathrm{Ext}^i_R(\mathbb{Z}/2^b, \mathbb{Z}/2^c) = \begin{cases} \mathbb{Z}/2^c & \text{if } i = 0, \\ \ker(f) & \text{if } i \geq 1 \text{ is odd}, \\ \mathrm{coker}(f) & \text{if } n \geq 2 \text{ is even}, \end{cases}$$

where f is the map $\mathbb{Z}/2^c \overset{2^{a-b}}{\to} \mathbb{Z}/2^c$. Suppose that $a - b \geq c$. Then $f = 0$, so

$$\mathrm{Ext}^i_R(\mathbb{Z}/2^b, \mathbb{Z}/2^c) = \mathbb{Z}/2^c, \quad \text{for all } i.$$

Suppose that $a - b > c$. Then

$$\begin{aligned} \ker f &= \{\bar{n} : 2^c \mid 2^{a-b} n\} \\ &= \{\bar{n} : 2^{b+c-a} \mid n\} \\ &= 2^{b+c-a}\mathbb{Z}/2^c\mathbb{Z} \\ &= \mathbb{Z}/2^{a-b}. \end{aligned}$$

We also have

$$\mathrm{im}\, f = 2^{a-b}\mathbb{Z}/2^c,$$

so

$$\mathrm{coker}\, f = (\mathbb{Z}/2^c\mathbb{Z})/(2^{a-b}\mathbb{Z}/2^c\mathbb{Z}) \cong \mathbb{Z}/2^{a-b}\mathbb{Z}.$$

Therefore, we have

$$\mathrm{Ext}^i_R(\mathbb{Z}/2^b, \mathbb{Z}/2^c) = \begin{cases} \mathbb{Z}/2^c & \text{if } i = 0, \\ \mathbb{Z}/2^{a-b}\mathbb{Z} & \text{else}. \end{cases}$$

The following result gives us yet another way of computing $\mathrm{Ext}^*_R(M, N)$.

Lemma 8.16. *The modules* $\mathrm{Ext}^*_R(M, N)$ *can be obtained by computing the cohomology of the cochain complex:*

$$\mathrm{Tot}^{\prod}(\mathrm{Hom}_R(P_\bullet, Q_\bullet)).$$

Proof. We start by drawing a picture of the double complex $\mathrm{Hom}_R(P_\bullet, Q_\bullet)$.

$$\begin{array}{ccccc}
\vdots & & \vdots & & \\
\downarrow & & \downarrow & & \\
\mathrm{Hom}(P_0, Q_1) & \longrightarrow & \mathrm{Hom}(P_1, Q_1) & \longrightarrow & \dots \\
\downarrow & & \downarrow & & \\
\mathrm{Hom}(P_0, Q_0) & \longrightarrow & \mathrm{Hom}(P_1, Q_0) & \longrightarrow & \dots
\end{array}$$

Let $C_{\bullet\bullet}$ be the double complex obtained by adding the cochain complex $\mathrm{Hom}(P_\bullet, N)$ as follows.

$$\begin{array}{ccccc}
\vdots & & \vdots & & \\
\downarrow & & \downarrow & & \\
\mathrm{Hom}(P_0, Q_1) & \longrightarrow & \mathrm{Hom}(P_1, Q_1) & \longrightarrow & \dots \\
\downarrow & & \downarrow & & \\
\mathrm{Hom}(P_0, Q_0) & \longrightarrow & \mathrm{Hom}(P_1, Q_0) & \longrightarrow & \dots \\
\downarrow & & \downarrow & & \\
\mathrm{Hom}(P_0, N) & \longrightarrow & \mathrm{Hom}(P_1, N) & \longrightarrow & \dots
\end{array}$$

Since the P_i are projective, the functor $\mathrm{Hom}(P_i, -)$ is exact, so $C_{\bullet\bullet}$ has exact columns. Since $C_{\bullet\bullet}$ is upper half plane, the Acyclic Assembly Lemma tells us that $\mathrm{Tot}^{\Pi}(C_{\bullet\bullet})$ is acyclic. Looking at the picture of $C_{\bullet\bullet}$ we see that

$$\left[\mathrm{Tot}^{\Pi}(C_{\bullet\bullet})\right]_n = \left[\mathrm{Tot}^{\Pi}(\mathrm{Hom}(P_\bullet, Q_\bullet)\right]_n \oplus \mathrm{Hom}(P_{n-1}, N).$$

The vertical maps $\mathrm{Hom}(P_n, Q_0) \to \mathrm{Hom}(P_n, N)$ give a morphism of chain complexes

$$f : \mathrm{Tot}^{\Pi}(\mathrm{Hom}(P_\bullet, Q_\bullet)) \to \mathrm{Hom}(P_\bullet, N),$$

which has mapping cone

$$\mathrm{cone}(f)_n = \left[\mathrm{Tot}^{\Pi}(\mathrm{Hom}(P_\bullet, Q_\bullet)\right]_n \oplus \mathrm{Hom}(P_{n-1}, N) = \left[\mathrm{Tot}^{\Pi}(C_{\bullet\bullet})\right]_n.$$

One can check that the differentials of the mapping cone agree with those of the total complex, so $\mathrm{cone}(f) = \mathrm{Tot}^{\Pi}(C_{\bullet\bullet})$ is acyclic. Therefore f is a quasi-isomorphism and the result follows. $\square$

Example 8.17. We will use Lemma 8.16 to compute $\mathrm{Ext}_{k[x]}(k,k)$ for a field k. Let $R = k[x]$ and take the projective resolution

$$0 \longrightarrow R \xrightarrow{x} R \longrightarrow k \longrightarrow 0.$$

For $0 \leq i, j \leq 1$ we have $\mathrm{Hom}_R(P_i, P_j) = \mathrm{Hom}_R(R,R) \cong R$, and out double complex is:

$$\begin{array}{ccc} R & \xrightarrow{-x} & R \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle x} \\ R & \xrightarrow[x]{} & R \end{array}$$

Then the cochain complex $\mathrm{Tot}^{\Pi}(\mathrm{Hom}(P_\bullet, Q_\bullet))$ is

$$\begin{array}{ccccccccc} & & * = -1 & & * = 0 & & * = 1 & & \\ 0 & \longrightarrow & R & \xrightarrow{1 \mapsto (x,-x)} & R \oplus R & \xrightarrow[(0,1)\mapsto x]{(1,0)\mapsto x} & R & \longrightarrow & 0, \end{array}$$

which has cohomology

$$H^n = \begin{cases} k & \text{if } n = 0 \text{ or } 1, \\ 0 & \text{else.} \end{cases}$$

It is easy to check that this is the same as $\mathrm{Ext}^*_R(k,k)$.

It turns out that the R-module

$$\mathrm{Ext}^*_R(A,A) = \bigoplus_{n=0}^{\infty} \mathrm{Ext}^n_R(A,A)$$

can be given the structure of a graded ring. We start by reinterpreting $\mathrm{Ext}^n_R(A,A)$ the set of degree $-n$ chain maps from P_* to itself (where $P_* \to A$ is a projective resolution), modulo chain homotopy. We then use this characterisation to define a product on $\mathrm{Ext}^*_R(A,A)$ as composition of chain maps.

9 Ring Structure on Ext

So far, we have viewed $\mathrm{Ext}_R^*(A,B)$ as a collection of R-modules. Taking the direct sum, we obtain a single R-module

$$\mathrm{Ext}_R^*(A,B) = \bigoplus_{i=0}^{\infty} \mathrm{Ext}_R^i(A,B).$$

We will define a product $\mathrm{Ext}_R^*(A,B) \times \mathrm{Ext}_R^*(A,B) \to \mathrm{Ext}_R^*(A,B)$, called the **Yoneda product**, which gives $\mathrm{Ext}_R^*(A,B)$ the structure of a graded ring.

9.1 Reinterpreting Ext

Let A and B be R-modules with projective resolutions $P_\bullet$ and $Q_\bullet$ respectively. Then by Lemma 8.16 we have

$$\mathrm{Ext}_R^*(A,B) = H^*(\mathrm{Tot}^{\prod}(\mathrm{Hom}(P_\bullet, Q_\bullet))).$$

Write T for the total complex $\mathrm{Tot}^{\prod}(\mathrm{Hom}(P_\bullet, Q_\bullet))$. Let $\varphi \in T^n$. Then $\varphi = \sum_i \varphi_i$ for $\varphi_i \in \mathrm{Hom}(P_i, Q_{i-n})$. For each i, let $\widetilde{\varphi}_i = \varepsilon_{i-n}\varphi_i$, where

$$\varepsilon_i = \begin{cases} 1 & \text{if } i \equiv 0,3 \pmod 4, \\ -1 & \text{if } i \equiv 1,2 \pmod 4. \end{cases}$$

It is easy to see that $\varphi \mapsto \widetilde{\varphi}$ is an injective module homomorphism from T^n to the module of graded module homomorphisms $P_\bullet \to Q_\bullet[-n]$ (i.e. collections of module homomorphisms $P_i \to Q_{i-n}$, without any assumptions about the chain complex structure).

Proposition 9.1. *The map $\varphi \mapsto \widetilde{\varphi}$ restricts to an isomorphism*

$$Z^n T \cong \mathbf{Ch}_*(R)(P_\bullet, Q_\bullet[-n]),$$

where the left hand side is the cocycles in degree n, and the right hand side denotes chain maps $P_\bullet \to Q_\bullet[-n]$.

Proof. It suffices to show that φ is a cycle if and only if $\widetilde{\varphi} : P_\bullet \to Q_\bullet[-n]$ is a chain map. We have

$$\begin{aligned} d\varphi &= \sum_i (d^h\varphi_i + d^v\varphi_i) \\ &= \sum_i ((-1)^{i-n} \times \varphi_i \circ \partial^P_{i+1} + \partial^Q_{i-n} \circ \varphi_i). \end{aligned}$$

The map $\widetilde{\varphi} : P_\bullet \to Q_\bullet[-n]$ is a chain map if and only if

$$\widetilde{\varphi}_i \circ \partial^P_{i+1} = \partial^Q_{i+1-n} \circ \widetilde{\varphi}_{i+1}$$

for all i. By definition of $\widetilde{\varphi}$, this is equivalent to

$$\varepsilon_{i-n}\varphi_i \circ \partial^P_{i+1} = \varepsilon_{i+1-n}\partial^Q_{i+1-n} \circ \varphi_{i+1},$$

hence

$$\varphi_i \circ \partial^P_{i+1} = (-1)^{i+1-n} \times \partial^Q_{i+1-n} \circ \varphi_{i+1} \tag{2}$$

for all i.

The map $\varphi \in T^n$ is a cycle if and only if

$$d^h\varphi_i + d^v\varphi_{i+1} = 0$$

for all i, which is equivalent to

$$(-1)^{i-n}\varphi_i \circ \partial^P_{i+1} + \partial^Q_{i+1-n} \circ \varphi_{i+1} = 0 \tag{3}$$

for all i. Conditions 2 and 3 are equivalent, so φ is a cycle if and only if $\widetilde{\varphi} : P_\bullet \to Q_\bullet[-n]$ is a chain map. □

Proposition 9.2. *For $\varphi \in Z^nT$, the cycle φ is a boundary if and only if $\widetilde{\varphi} : P_\bullet \to Q_\bullet[-n]$ is chain homotopic to zero.*

Proof. Suppose that φ is a coboundary. Then $\varphi = d\psi$ for some $\psi \in T^{n-1}$. Then we have

$$\varphi_i = (-1)^{i-n} \times \psi_{i-1} \circ \partial^P_i + \times \partial^Q_{i+1-n} \circ \psi_i,$$

so

$$\epsilon_{i-n}\widetilde{\varphi}_i = (-1)^{i-n} \times \varepsilon_{i-n}\widetilde{\psi}_{i-1} \circ \partial_i^P + \varepsilon_{i-n+1}\partial_{i+1-n}^Q \circ \widetilde{\psi}_i.$$

Since $\varepsilon_i^2 = 1$ for all i, this gives

$$\widetilde{\varphi}_i = (-1)^{i-n} \times \widetilde{\psi}_{i-1} \circ \partial_i^P + \varepsilon_{i-n}\varepsilon_{i-n+1} \circ \partial_{i+1-n}^Q\widetilde{\psi}_i,$$

and it is easily checked that $\varepsilon_{i-1}\varepsilon_i = (-1)^i$ for all i, so

$$\widetilde{\varphi}_i = (-1)^{i-n} \times \widetilde{\psi}_{i-1} \circ \partial_i^P + (-1)^{i-n+1} \times \partial_{i+1-n}^Q \circ \widetilde{\psi}_i$$

Set $h_i : P_i \to Q_{i-n+1}$ as $h_i = (-1)^{i+1-n} \times \widetilde{\psi}_i$. Then we have

$$\widetilde{\varphi}_i = h_{i-1} \circ \partial_i^P + \partial_{i+1-n}^Q \circ h_i,$$

hence h is a chain homotopy between $\widetilde{\varphi}$ and 0.

Suppose conversely that $\widetilde{\varphi}$ is chain homotopic to 0. Let h be a the chain homotopy, so that

$$\widetilde{\varphi}_i = h_{i-1} \circ \partial_i^P + \partial_{i+1-n}^Q \circ h_i$$

for all i. Then define $\widetilde{\psi}_i = (-1)^{i+1-n}h_i : P_i \to Q_{i-n+1}$ for each i and

$$\psi_i = \varepsilon_{i-n+1}\widetilde{\psi}_i \in T^{n-1}.$$

Since $\varepsilon_{i-1}\varepsilon_i = (-1)^i$ for all i, we have

$$\psi_i = \varepsilon_{i-n+1}\varepsilon_{i-n}\varepsilon_{i-n+1}h_i = \varepsilon_{i-n}h_i.$$

Then

$$\begin{aligned}
(d\psi)_i &= (-1)^{i-n}\psi_{i-1} \circ \partial_i^P + \partial_{i-n+1}^Q \circ \psi_i \\
&= (-1)^{i-n}\varepsilon_{i-1-n}h_{i-1} \circ \partial_i^P + \varepsilon_{i-n}\partial_{i-n+1}^Q \circ h_i \\
&= \varepsilon_{i-n}h_{i-1} \circ \partial_i^P + \varepsilon_{i-n}\partial_{i-n+1}^Q \circ h_i \\
&= \varepsilon_{i-n}\widetilde{\varphi}_i \\
&= \varphi_i.
\end{aligned}$$

Therefore $\varphi = d\psi$ is a coboundary. □

Therefore $\mathrm{Ext}^n_R(A, B)$ is isomorphic to the quotient of the module of chain maps $P_\bullet \to Q_\bullet[-n]$ by the submodule of nullhomotopic chain maps.

9.2 Yoneda Product

Let A, B be as above, with projective resolutions $P_\bullet$ and $Q_\bullet$. Suppose that C is a third R-module with projective resolution $R_\bullet \to C$. Suppose that we have chain maps

$$\varphi : P_\bullet \to Q_\bullet[-m], \quad \psi : Q_\bullet \to R_\bullet[-n].$$

Then $\psi \circ \varphi$ is a chain map $P_\bullet \to R_\bullet[-m-n]$. Since homotopy commutes with composition of maps, if either of φ and ψ is nullhomotopic, then $\psi \circ \varphi$ is too. Therefore the composition induces a well-defined map

$$\smile : \mathrm{Ext}^m_R(A, B) \times \mathrm{Ext}^n_R(B, C) \to \mathrm{Ext}^{m+n}_R(A, C).$$

One can check that this map is R-bilinear, and in the case $A = B = C$ it is an associative binary operation

$$\smile : \mathrm{Ext}^*_R(A, A) \times \mathrm{Ext}^*_R(A, A) \to \mathrm{Ext}^*_R(A, A),$$

hence it gives $\mathrm{Ext}^*_R(A, A)$ the structure of a graded ring.

Example 9.3. Let $R = k[x]/(x^2)$, and view the field k as the R-module R/xR. Then, as a ring, we have

$$\mathrm{Ext}^\bullet_R(k, k) \cong k[y], \quad |y| = 1.$$

Proof. We will use the projective resolution

$$\dots \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{x} R \longrightarrow 0$$

for k. Call this projective resolution $P_\bullet$.

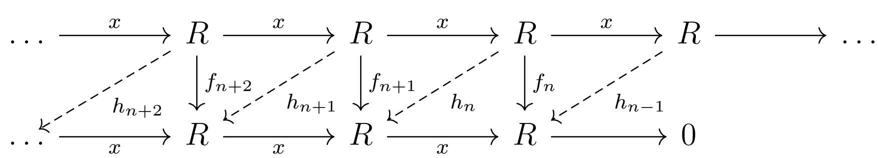
A chain map $P_\bullet \to P_\bullet[-n]$ is a collection of module homomorphisms $f_i : R \to R$ for

$i \geq n$, such that the following diagram commutes.

$$\begin{CD} \cdots @>x>> R @>x>> R @>x>> R @>x>> R @>>> \cdots \\ @. @VV f_{n+2} V @VV f_{n+1} V @VV f_n V \\ \cdots @>>x> R @>>x> R @>>x> R @>>> 0 \end{CD}$$

The commutativity of the diagram is equivalent to $f_i(x) = x f_{i+1}(1)$ for all $i \geq n$, which is equivalent to $(f_i - f_{i+1})(1) \in xR$ for all $i \geq n$. Therefore the $f_i(1)$ are all congruent modulo x.

The chain map $f_\bullet$ is nullhomotopic if and only if there are module homomorphisms $h_i : R \to R$ for $i \geq n-1$ such that $f_i(1) = h_{i-1}(x) + xh_i(1)$ for all $i \geq n$, as illustrated by the following diagram.



In other words, $f_\bullet$ is nullhomotopic if and only if $f_i(1) = x(h_{i-1}(1) + h_i(1))$ for all $i \geq n$. Clearly if $f_\bullet$ is nullhomotopic, then $f_i(1) \in xR$ for all $i \geq n$.

We claim that, conversely, if $f_i(1) \in xR$ for all $i \geq n$, then $f_\bullet$ is nullhomotopic. Suppose that $f_i(1) \in xR$ for all $i \geq n$. Then define $h_{n-1} = 0$, and let $h_n(1) \in R$ be an element with $f_n(1) = xh_n(1)$. Inductively, for $i > n$ define $h_i(1) \in R$ to be an element such that $f_i(1) - xh_{i-1}(1) = xh_i(1)$. Then h is a nullhomotopy, so $f_\bullet$ is nullhomotopic. Therefore "chain maps modulo homotopy" is the same as "chain maps modulo x".

We already showed that for any chain map $f_\bullet : P_\bullet \to P_\bullet[-n]$, all the $f_i(1)$ are congruent modulo x. Modulo homotopy (i.e. modulo x), we may assume that for each i, we have $f_i(1) = a_i$ for some $a_i \in k$. Since the $f_i(1)$ are all congruent modulo x (because $f_\bullet$ is a chain map), the a_i are all equal. Therefore $\mathrm{Ext}^n_R(k, k) = k \cdot f^{(n)}_\bullet$, where $f^{(n)}_\bullet$ is the degree $-n$ chain map

$$\begin{CD} \cdots @>x>> R @>x>> R @>x>> R @>x>> R @>>> \cdots \\ @. @VV 1 V @VV 1 V @VV 1 V \\ \cdots @>>x> R @>>x> R @>>x> R @>>> 0. \end{CD}$$

It is clear that $f_\bullet^{(m)} \circ f_\bullet^{(n)} = f_\bullet^{(m+n)}$, so the Yoneda product gives

$$f_\bullet^{(m)} \smile f_\bullet^{(n)} = f_\bullet^{(m+n)}.$$

Set $y = f_\bullet^{(1)}$. Then for each n, we have

$$\mathrm{Ext}_R^n(k,k) = k \cdot y^n,$$

so, as a ring, we have $\mathrm{Ext}_R^\bullet(k,k) = k[y]$, where y is a degree -1 chain map, so $|y| = 1$. □

10 Universal Coefficient Theorem

Theorem 10.1 (Künneth Formula)**.** *Let $P_\bullet$ be a chain complex of flat R-modules such that each ∂P_n is also flat. Then for all left R-modules M, and all integers n, there is a natural short exact sequence*

$$0 \to H_n(P) \otimes_R M \to H_n(P \otimes_R M) \to \mathrm{Tor}_1^R(H_{n-1}(P), M) \to 0.$$

Proof. For each n, we have a short exact sequence

$$0 \to Z_n P \to P_n \to \partial P_n \to 0.$$

For each $M \in R$-**mod**, the long exact sequence of $\mathrm{Tor}_*^R(-, M)$ has terms

$$\mathrm{Tor}_2^R(\partial P_n, M) \to \mathrm{Tor}_1^R(Z_n P, M) \to \mathrm{Tor}_1^R(P_n, M).$$

Since P_n and ∂P_n are flat, the left and right terms are zero, so $\mathrm{Tor}_1^R(Z_n P, M) = 0$. Since M was arbitrary, $Z_n P$ is flat for all n.

Since ∂P_n is flat, we have $\mathrm{Tor}_1^R(\partial P_n, M) = 0$, so we have a short exact sequence

$$0 \to Z_n P \otimes_R M \to P_n \otimes_R M \to \partial P_n \otimes_R M \to 0,$$

hence we have a short exact sequence of chain complexes

$$0 \to Z_\bullet P \otimes_R M \to P_\bullet \otimes_R M \to \partial P_\bullet \otimes_R M \to 0,$$

which gives us a long exact sequence

$$\begin{aligned} \ldots \to H_{n+1}(\partial P_\bullet \otimes_R M) \to & H_n(Z_\bullet P \otimes_R M) \to H_n(P_\bullet \otimes_R M) \to \\ & H_n(\partial P_\bullet \otimes_R M) \to H_{n-1}(Z_\bullet P \otimes_R M) \to \ldots . \end{aligned}$$

The chain complexes $\partial P_\bullet \otimes_R M$ and $Z_\bullet P \otimes_R M$ both have zero differential. Therefore, this long exact sequence gives an exact sequence

$$\partial P_{n+1} \otimes M \to Z_n P \otimes M \to H_n(P_\bullet \otimes M) \to \partial P_n \otimes_R M \to Z_{n-1} P \otimes_R M \qquad (*)$$

The chain complex

$$\partial P_{n+1} \to Z_n P \to H_n(P_\bullet) \to 0$$

is a projective resolution for $H_n(P_\bullet)$, so $\operatorname{Tor}_*^R(H_*(P_\bullet), M)$ is the homology of the chain complex

$$0 \to \partial P_{n+1} \otimes_R M \xrightarrow{\varphi_n} Z_n P \otimes_R M \to 0,$$

where φ_n is the natural map. Now, (*) gives a short exact sequence

$$\operatorname{coker} \varphi_n \to H_n(P_\bullet \otimes_R M) \to \ker \varphi_{n-1}.$$

This is exactly the result we are trying to prove. □

Theorem 10.2 (Universal Coefficient Theorem)**.** *Let $P_\bullet$ be a chain complex of free $\mathbb{Z}$-modules. Then for all $M \in \mathbb{Z}$-**mod**, we have*

$$H_n(P_\bullet \otimes_\mathbb{Z} M) \cong (H_n(P_\bullet) \otimes_\mathbb{Z} M) \oplus \operatorname{Tor}_1^\mathbb{Z}(H_{n-1}(P_\bullet), M),$$

for all n. However, the decomposition is not natural.

Proof. Since each P_n is a free abelian group, so is the subgroup ∂P_n. Therefore ∂P_n is projective, so the short exact sequence

$$0 \to Z_n P \to P_n \to \partial P_n \to 0$$

splits, and hence $P_n \cong Z_n P \oplus \partial P_n$. Tensoring with M, we get

$$P_n \otimes M \cong (Z_n P \otimes M) \oplus (\partial P_n \otimes M),$$

so the inclusion

$$Z_n P \otimes M \to P_n \otimes M$$

has a retract. This retract restricts to a retract of

$$Z_n P \otimes M \to \ker(\partial_n \otimes \mathrm{id}_M),$$

We have a commutative diagram

$$\begin{array}{ccc} P_{n+1} \otimes M & = & P_{n+1} \otimes M \\ \downarrow & & \downarrow \\ Z_n P \otimes M & \longrightarrow & \ker(\partial_n \otimes \mathrm{id}_M). \end{array}$$

Modding out by the cokernels of the vertical arrows, we get that the natural map

$$H_n(P) \otimes M \to H_n(P_* \otimes M),$$

which has a retract. We know from Theorem 10.1 that there is a short exact sequence

$$0 \to H_n(P) \otimes_R M \to H_n(P \otimes_R M) \to \mathrm{Tor}_1^R(H_{n-1}(P), M) \to 0,$$

which is therefore split.

□

11 Koszul Complexes

Let R be a ring and let $x \in Z(R)$ be a central element. Then we define the **Koszul complex** $K(x)$ of x to be the chain complex

$$0 \to R \xrightarrow{x} R \to 0,$$

concentrated in degrees 1 and 0. We denote the generator of $K(x)_1$ by e_x, so that $K(x)_1 = R \cdot e_x, K(x)_0 = R \cdot 1$, and $de_x = 1$.

Suppose that $\mathbf{x} = (x_1, \ldots, x_n)$ is a finite sequence of central elements in R. Then $K(\mathbf{x})$ is the chain complex

$$K(x_1) \otimes_R \ldots \otimes_R K(x_n).$$

For an R-module A, we define the **Koszul homology** and **Koszul cohomology** to be

$$\begin{aligned} H_p(\mathbf{x}, A) &= H_p(K(\mathbf{x}) \otimes_R A), \\ H^p(\mathbf{x}, A) &= H^p(\mathrm{Hom}(K(\mathbf{x}), A)). \end{aligned}$$

Lemma 11.1. *For $x \in Z(R)$, we have*

$$H_0(x, A) = A/xA, \quad H_1(x, A) = \{a \in A : xa = 0\}.$$

Proof. Easy. □

Lemma 11.2. *The module $K_p(\mathbf{x})$ is free with generators*

$$e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_p}, \quad i_1 < i_2 < \ldots < i_p,$$

and differentials

$$d(e_{i_1} \wedge \ldots \wedge e_{i_p}) = \sum_{k=1}^{p} (-1)^{k+1} x_{i_k} (e_{i_1} \wedge \ldots \wedge \hat{e}_{i_k} \wedge \ldots \wedge e_{i_p}).$$

Corollary 11.3. *We have*

$$H_0(\mathbf{x}, A) = A/(x_1, \ldots, x_n)A, \quad H^0(\mathbf{x}, A) = \mathrm{Hom}(R/(x_1, \ldots, x_n), A).$$

We have a version of a the Künneth Theorem.

Theorem 11.4 (Künneth Formula)**.** *Let $C_\bullet \in \mathbf{Ch}_*(R)$, and let $x \in Z(R)$. Then there is a short exact sequence*

$$0 \to H_0(x, H_q(C)) \to H_q(K(x) \otimes_R C) \to H_1(x, H_{q-1}(C)) \to 0.$$

Proof. Consider the natural short exact sequence

$$0 \to R \to K(x) \to R[-1] \to 0.$$

Tensoring this short exact sequence with C yields a sequence

$$0 \to C \to K(x) \otimes_R C \to C[-1] \to 0.$$

It is easy to check (just consider the chain maps degreewise) that this is a short exact sequence. The Snake Lemma gives us an exact sequence

$$H_{n+1}(C[-1]) \xrightarrow{\delta} H_n(C) \to H_n(K(x) \otimes_R C) \to H_n(C[-1]) \xrightarrow{\delta} H_{n-1}(C).$$

Clearly this is isomorphic to

$$H_n(C) \xrightarrow{\delta} H_n(C) \to H_n(K(x) \otimes_R C) \to H_{n-1}(C) \xrightarrow{\delta} H_{n-1}(C).$$

We claim that $\delta : H_n(C) \to H_n(C)$ is the multiplication by x map. To see this, note that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_n & \longrightarrow & (K(x) \otimes_R C)_n & \longrightarrow & C[-1]_n & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C_{n-1} & \longrightarrow & (K(x) \otimes_R C)_{n-1} & \longrightarrow & C[-1]_{n-1} & \longrightarrow & 0 \end{array}$$

is given explicitly by

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & C_n & \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} & C_n \oplus C_{n-1} & \xrightarrow{\begin{pmatrix} 0 & 1 \end{pmatrix}} & C_{n-1} & \longrightarrow & 0 \\
 & & \Big\downarrow \partial & & \Big\downarrow \begin{pmatrix} \partial & (-1)^n x \\ 0 & \partial \end{pmatrix} & & \Big\downarrow \partial & & \\
0 & \longrightarrow & C_{n-1} & \xrightarrow[\begin{pmatrix} 1 \\ 0 \end{pmatrix}]{} & C_{n-1} \oplus C_{n-2} & \xrightarrow[\begin{pmatrix} 0 & 1 \end{pmatrix}]{} & C_{n-2} & \longrightarrow & 0
\end{array}
$$

Chasing this diagram, we see that that indeed $\delta = (-1)^n x$ as we claimed. Now, it follows that we have a short exact sequence

$$0 \to xH_n(C) \to H_n(K(x) \otimes_R C) \to \{a \in H_{n-1}(C) : xa = 0\} \to 0.$$

By Lemma 11.1, this is the desired result. □

Let A be an R-module. A nonzero element $r \in R$ is called a **zero divisor of** A if there is a nonzero element $a \in A$ such that $ra = 0$. A **regular sequence** on A is a finite sequence $(x_1, \ldots, x_n)$ in R such that the following two conditions hold.

1. The element x_1 is not a zero-divisor of A.

2. For each $i \geq 2$, the element x_i is not a zero-divisor of $A/(x_1, \ldots, x_{i-1})$.

Proposition 11.5. *Let* $\mathbf{x}$ *be a regular sequence on* $A \in R$-**mod**. *Then* $H_q(\mathbf{x}, A) = 0$ *for* $q > 0$.

Proof. We proceed by induction on n. For $n = 1$, the result is clear.

Let $n \geq 2$. Let $\mathbf{y} = (x_1, \ldots, x_{n-1})$ and $\mathbf{x} = (x_1, \ldots, x_n)$. By the Künneth formula for Koszul complexes, we have a short exact sequence

$$0 \to H_0(x_n, H_q(\mathbf{y}, A)) \to H_q(\mathbf{x}, A) \to H_1(x_n, H_{q-1}(\mathbf{y}, A)) \to 0.$$

For $q \geq 2$, the first and last modules are zero, so $H_q(\mathbf{x}, A) = 0$. The case $q = 1$ gives

$$H_1(\mathbf{x}, A) \cong H_1(x_n, H_0(\mathbf{y}, A)),$$

and

$$H_0(\mathbf{y}, A) = \mathbf{y}A,$$

so $H_1(x_n, H_0(\mathbf{y}, A))$ is the first homology of

$$A/\mathbf{y}A \xrightarrow{x_n} A/\mathbf{y}A,$$

and this is zero since x_n is not a zero-divisor of $A/\mathbf{y}A$. □

Proposition 11.6. *If* $\mathbf{x}$ *is a regular sequence on* R*, then* $K(\mathbf{x})$ *is a free resolution of* R/I*, where* $I = (x_1, \ldots x_n)$*, so*

$$\mathrm{Tor}_p^R(R/I, A) = H_p(\mathbf{x}, A), \quad \mathrm{Ext}_R^p(R/I, A) = H^p(\mathbf{x}, A).$$

12 Ext and Extensions

The module $\mathrm{Ext}^1_R(A,B)$ can be thought of as the set of equivalence classes of "extensions of A by B", which we will define in this section. In fact, even the abelian group structure of $\mathrm{Ext}^1_R(A,B)$ can be expressed in terms of these extensions, using somethng called the "Baer Sum".

12.1 Extensions

Let A and B be R-modules. An **extension** of A by B is an exact sequence $0 \to B \to X \to A \to 0$. Extensions ξ and ξ' are **equivalent** if there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & X & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow{=} & & \downarrow{\cong} & & \downarrow{=} & & \\ 0 & \longrightarrow & B & \longrightarrow & X' & \longrightarrow & A & \longrightarrow & 0. \end{array}$$

An extension is **split** if it is equivalent to $0 \to B \stackrel{(0,1)}{\to} A \oplus B \to A \to 0$.

Lemma 12.1. *In* $\mathbb{Z}$-**mod**, *there are exactly* p *equivalence classes of extensions of* $\mathbb{Z}/p$ *by* $\mathbb{Z}/p$.

Proof. Let

$$0 \stackrel{i}{\to} \mathbb{Z}/p \to X \stackrel{\pi}{\to} \mathbb{Z}/p \to 0$$

be an extension of $\mathbb{Z}/p$ by $\mathbb{Z}/p$. If $X =\cong \mathbb{Z}/p \oplus \mathbb{Z}/p$, then the extension is split, so there is only one possible equivalence class.

Suppose that $X \cong \mathbb{Z}/p^2$. Then we have

$$0 \stackrel{i}{\to} \mathbb{Z}/p \to \mathbb{Z}/p^2 \stackrel{\pi}{\to} \mathbb{Z}/p \to 0.$$

The map $\pi : \mathbb{Z}/p^2 \to \mathbb{Z}/p$ is surjective, so $\pi([1])$ is nonzero, which means that $\ker \pi =$

$p\mathbb{Z}/p^2\mathbb{Z}$. Let $\pi([1]) = [a]$ for $a \in \{1, \ldots, p-1\}$. We have a commutative diagram

$$\begin{array}{ccc} \mathbb{Z}/p^2 & \xrightarrow{\pi} & \mathbb{Z}/p \\ \| & & \downarrow{\scriptstyle a^{-1}} \\ \mathbb{Z}/p^2 & \xrightarrow[1]{} & \mathbb{Z}/p. \end{array}$$

Therefore, up to equivalence of extensions, we may assume that $a = 1$. The equivalence class is now determined by $i([1])$, which could be any of $\{[p], [p^2], \ldots, [(p-1)p]\}$.

Therefore there are $1 + (p-1) = p$ possibilities. □

Lemma 12.2. *If* $\mathrm{Ext}^1(A, B) = 0$, *then every extension of* A *by* B *is split.*

Proof. Applying $\mathrm{Ext}(A, -)$ to the extension ξ gives us a short exact sequence

$$0 \longrightarrow \mathrm{Hom}(A, B) \longrightarrow \mathrm{Hom}(A, X) \longrightarrow \mathrm{Hom}(A, A) \longrightarrow 0,$$

so in partuclar there is some $s \in \mathrm{Hom}(A, X)$ such that $A \xrightarrow{s} X \to A$ is the identity. Therefore the extension is split. □

In general, applying $\mathrm{Ext}(A, -)$ to an extension ξ gives an exact sequence

$$0 \to \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, X) \to \mathrm{Hom}(A, A) \xrightarrow{\delta} \mathrm{Ext}^1(A, B) \to \ldots.$$

Let $\Theta(\xi) = \delta(\mathrm{id}_A) \in \mathrm{Ext}^1(A, B)$. Then we have a map

$$\Theta : \{\text{extensions of } A \text{ by } B\} \to \mathrm{Ext}^1(A, B).$$

Lemma 12.3. *The map* Θ *gives a bijection between equivalence classes of extensions of* A *by* B *and* $\mathrm{Ext}^1(A, B)$.

Sketch Proof. We only sketch the proof, because the details are pretty unseemly.

First of all, we claim that Θ is surjective. Let $x \in \mathrm{Ext}^1(A, B)$. We will construct an extension ξ of A by B such that $\Theta(\xi) = x$. Pick some short exact sequence

$$0 \to B \xrightarrow{j} I \xrightarrow{\pi} N \to 0,$$

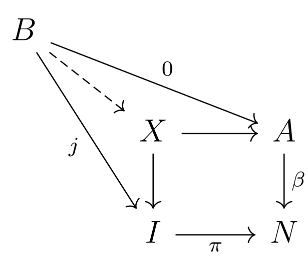
where I is injective. This exists because R-**mod** has enough injectives, and we can just take N to be the cokernel of some monomorphism to an injective object. Since I is injective, we have $\mathrm{Ext}^1(A, I) = 0$, so the long exact sequence of Ext gives an exact sequence

$$0 \to \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, I) \to \mathrm{Hom}(A, M) \xrightarrow{\delta} \mathrm{Ext}^1(A, B) \to 0.$$

Since δ is surjective, there is some $\beta \in \mathrm{Hom}(A, B)$ such that $\delta(\beta) = x$. Define the R-module X to be the pullback

$$\begin{array}{ccc} X & \longrightarrow & A \\ \downarrow & & \downarrow{\scriptstyle \beta} \\ I & \xrightarrow[\pi]{} & N. \end{array}$$

By the universal property of pullbacks, we get a unique map $B \to X$ such that the diagram



commutes. Therefore, we get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & X & \longrightarrow & A & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle \beta} & & \\ 0 & \longrightarrow & B & \xrightarrow[j]{} & I & \xrightarrow[\pi]{} & N & \longrightarrow & 0. \end{array}$$

It turns out that the top row of this diagram is exact. You can prove this using the explicit construction of pullbacks and diagram chasing. By naturality of the long exact sequence of Ext, we get a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(A, A) & \longrightarrow & \mathrm{Ext}^1(A, B) \\ {\scriptstyle \beta \circ -}\downarrow & & \| \\ \mathrm{Hom}(A, N) & \xrightarrow[\delta]{} & \mathrm{Ext}^1(A, B). \end{array}$$

This diagram gives

$$\begin{array}{ccc} \mathrm{id}_A & \longmapsto & \Theta(\xi) \\ \downarrow & & \| \\ \beta & \longmapsto & \delta(\beta) = x, \end{array}$$

where ξ is the short exact sequence

$$0 \to B \to X \to A \to 0.$$

Therefore, Θ is onto. It turns out that the ξ we constructed is unique up to equivalence, so we get a well-defined map

$$\Psi : \mathrm{Ext}^1(A, B) \to \{\text{equivalence classes of extensions}\}$$

with $\Phi \circ \Psi = \mathrm{id}$. It turns out that this is a two-sided inverse for Φ, and the result follows. □

Let

$$\xi_1 : 0 \to B \xrightarrow{i_1} X_1 \xrightarrow{\pi_1} A \to 0, \quad \xi_2 : 0 \to B \xrightarrow{i_2} X_2 \xrightarrow{\pi_2} A \to 0$$

be extensions of A by B. Let

$$X'' = X_1 \times_A X_2 = \{(x_1, x_2) \in X_1 \times X_2 : \pi_1(x_1) = \pi_2(x_2)\},$$

and let $Y = X''/\{(i_1(b), -i_2(b)) : b \in B\}$. We have maps

$$i : B \to Y, b \mapsto (i_1(b), 0)$$

and

$$\pi : Y \to A, (x_1, x_2) \mapsto \pi_1(x_1) + \pi_2(x_2).$$

The sequence

$$0 \to B \xrightarrow{i} Y \xrightarrow{\pi} A \to 0$$

is called the **Baer sum** of ξ and ξ'.

Lemma 12.4. *The Baer sum is a well-defined extension of A by B.*

Lemma 12.5. *The set of equivalence classes of extensions of A by B is an abelian group under the Baer sum, and the map Θ is an isomorphism of abelian groups.*

12.2 Yoneda Ext Groups

Using extensions of A by B, we can define $\mathrm{Ext}^1(A, B)$ in any abelian category (i.e. no need for projectives or injectives). We call this the **Yoneda** Ext **group**.

More generally, we define the **Yoneda** $\mathrm{Ext}^n(A, B)$ to be the equivalence classes of exact sequences

$$\xi : 0 \to B \to X_n \to \ldots \to X_1 \to A \to 0,$$

under the equivalence relation generated by $\xi \sim \xi'$ if there is a diagram

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & B & \longrightarrow & X_n & \longrightarrow & \ldots & \longrightarrow & X_1 & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow{=} & & \downarrow & & & & \downarrow & & \downarrow{=} & & \\ 0 & \longrightarrow & B & \longrightarrow & X'_n & \longrightarrow & \ldots & \longrightarrow & X'_1 & \longrightarrow & A & \longrightarrow & 0. \end{array}$$

Note that the arrows $X_i \to X'_i$ do not have to be isomorphisms. At first glance, this seems different to our definition of equivalence for extensions of A by B. However, by the 5-lemma, this definition does actually generalise the previous one.

We again define a notion of a Baer sum. Let ξ and ξ' be representatives of elements of $\mathrm{Ext}^n(A, B)$. Let X''_1 be the pullback of

$$\begin{array}{ccc} & & X_1 \\ & & \downarrow \\ X'_1 & \longrightarrow & A, \end{array}$$

and let X''_n be the pushout of

$$\begin{array}{ccc} B & \longrightarrow & X_n \\ \downarrow & & \\ X'_n. & & \end{array}$$

Let Y_n be the quotient of X''_n by the antidiagonal. Then the **Baer sum** is

$$0 \to B \to Y_n \to X_{n-1} \oplus X'_{n-1} \to \ldots \to X_2 \oplus X'_2 \to X''_1 \to A \to 0.$$

Suppose that $\mathcal{A}$ has enough projectives and $P_\bullet \to A$ is a projective resolution. Consider the diagram

$$\begin{array}{ccccccccccccccc}
\dots & \longrightarrow & P_{n+1} & \longrightarrow & P_n & \longrightarrow & P_{n-1} & \longrightarrow & \dots & \longrightarrow & P_0 & \longrightarrow & A & \longrightarrow & 0 \\
 & & & & & & & & & & & & \big\downarrow{\scriptstyle =} & & \\
 & & 0 & \longrightarrow & B & \longrightarrow & X_n & \longrightarrow & \dots & \longrightarrow & X_1 & \longrightarrow & A & \longrightarrow & 0.
\end{array}$$

By the comparison theorem, there is a chain map from the top row to the bottom row lifting $\mathrm{id} : A \to A$. Setting $M = \ker \partial_n^{(P)}$ gives a diagram

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & M & \longrightarrow & P_{n-1} & \longrightarrow & \dots & \longrightarrow & P_0 & \longrightarrow & A & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle \beta} & & \big\downarrow & & & & \big\downarrow & & \big\downarrow{\scriptstyle =} & & \\
0 & \longrightarrow & B & \longrightarrow & X_n & \longrightarrow & \dots & \longrightarrow & X_1 & \longrightarrow & A & \longrightarrow & 0
\end{array}$$

with exact rows.

Fact 12.6. There is a natural isomorphism between Yoneda Ext^n and the standard Ext^n.

13 Group (Co)homology

We are often interested in studying mathematical objects by extracting algebraic invariants from them. One such invariant is the homology or cohomology of a group. Group (co)homology is a rich application of the theory we have developed, drawing on tools from throughout these notes.

13.1 Definitions

Definition 13.1. Let G be a group. A G**-module** is an abelian group A together with a group homomorphism $\rho : G \to \mathrm{GL}_n(A)$.

As usual, we abbreviate $\rho(g)(a)$ by $g \cdot a$.

Definition 13.2. A **morphism** $(A, \rho) \to (B, \sigma)$ of G-modules is a $\mathbb{Z}$-linear map $\varphi : A \to B$ such that

$$\varphi(g \cdot a) = g \cdot \varphi(a)$$

for all $g \in G$ and $a \in A$.

These morphisms make G-modules into a category, G-**mod**.

Lemma 13.3. *There is an equivalence of categories*

$$G\text{-}\mathbf{mod} \simeq \mathbb{Z}G\text{-}\mathbf{mod},$$

where $\mathbb{Z}G$-**mod** *is the group algebra over* $\mathbb{Z}$*.*

Definition 13.4. A G-module is **trivial** if $g \cdot a = a$ for all $g \in G$ and $a \in A$.

Definition 13.5. Let $A \in G$-**mod**. Then the **submodule of invariants** if

$$A^G = \{a \in A : g \cdot a = a \forall g \in G\},$$

and the **module of coinvariants** is

$$A_G = A / \langle g \cdot a - a : g \in G, a \in A \rangle.$$

Lemma 13.6. *The assignments $A \mapsto A^G$ and $A \mapsto A_G$ are functorial. That is, we have functors*

$$-^G, -_G : G\text{-}\mathbf{mod} \to \mathbf{Ab}.$$

There is a functor triv : $\mathbf{Ab} \to G$-$\mathbf{mod}$ taking A to the G-module on A with trivial action.

Lemma 13.7. *We have adjunctions $-_G \dashv \mathrm{triv} \dashv -^G$.*

Proof. This is basically because, for any G-module M and abelian group A, the $\mathbb{Z}$-linear homomorphisms $M \to A^G$ are **literally the same thing** as $\mathbb{G}$-linear homomorphisms $M \to A$, and G-linear homomorphisms $\mathrm{triv}(A) \to M$ are precisely the abelian group homomorphisms $A \to M$ with $\langle ga - a : g \in G, a \in A \rangle$ contained in their kernel. □

Corollary 13.8. *The functor $-_G$ is right-exact and the functor $-^G$ is left-exact.*

Lemma 13.9. *We have*

$$A_G = \mathbb{Z} \otimes_{\mathbb{Z}G} A.$$

Proof. Let $N = \langle ga - a : g \in G, a \in A \rangle$. Define $\widetilde{\varphi} : A \to \mathbb{Z} \otimes_{\mathbb{Z}G} A$ by

$$\widetilde{\varphi}(a) = 1 \otimes a.$$

Then $\widetilde{\varphi}$ kills N, so it descends to a $\mathbb{Z}$-module homomorphism

$$\varphi : A_G \to \mathbb{Z} \otimes_{\mathbb{Z}G} A.$$

There is a $\mathbb{Z}G$-bilinear map $B : \mathbb{Z} \times A \to A_G$ given by

$$(n, a) \mapsto na + N.$$

This map induces a map $\psi : \mathbb{Z} \otimes_{\mathbb{Z}G} A \to A_G$ with

$$\psi(n \otimes a) = na + N.$$

Clearly φ and ψ are mutual inverses. □

Lemma 13.10. *We have*

$$A^G = \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, A).$$

Since $-_G$ and $-^G$ are right- and left-exact respectively, we can take their derived functors.

Definition 13.11. The **group homology of G with coefficients in A** is

$$H_*(G, A) = L_*(-_G)(A),$$

and the **group cohomology of G with coefficients in A** is

$$H^*(G, A) = R^*(-^G)(A).$$

Lemma 13.12. *We have*

$$H_*(G, A) \cong \mathrm{Tor}_*^{\mathbb{Z}G}(\mathbb{Z}, A)$$

and

$$H^*(G, A) \cong \mathrm{Ext}^*_{\mathbb{Z}G}(\mathbb{Z}, A).$$

Example 13.13. Let $G = \langle t \rangle$ be infinite cyclic and let A be a G-module. Then $\mathbb{Z}G$ is the Laurent polynomial ring $\mathbb{Z}[t, t^{-1}]$.

We start by computing the group homology $H_*(G, A)$. This is the same as $\mathrm{Tor}_*^{k[t,t^{-1}]}(\mathbb{Z}, A)$. Write $R = \mathbb{Z}[t, t^{-1}]$. Then the trivial R-module $\mathbb{Z}$ has a projective resolution

$$0 \to R \overset{(t-1)}{\to} R \to \mathbb{Z} \to 0,$$

so the group homology is the homology of the chain complex

$$0 \to R \otimes_R A \overset{(t-1))\otimes A}{\to} R \otimes_R A \to 0,$$

which is the same as the homology of

$$0 \to A \overset{(t-1)}{\to} A \to 0.$$

It follows that

$$H_n(G, A) = \begin{cases} \{a \in A : ta = a\} = A^G & \text{if } n = 1, \\ A/(t-1)A = A_G & \text{if } n = 0, \\ 0 & \text{else.} \end{cases}$$

Similarly, we have

$$H^*(G, A) \cong \operatorname{Ext}^*_R(\mathbb{Z}, R).$$

Using the same projective resolution, the group cohomology is the cohomology of the cochain complex

$$0 \leftarrow \operatorname{Hom}_R(R, A) \overset{(t-1)}{\leftarrow} \operatorname{Hom}_R(R, A) \leftarrow 0,$$

which is equivalent to

$$0 \leftarrow A \overset{(t-1)}{\leftarrow} A \leftarrow 0.$$

Therefore $H^*(G, A) = H_{1-*}(G, A)$.

13.2 First Homology

Let $\varepsilon : \mathbb{Z}G \to \mathbb{Z}$ be the augmentation map $g \mapsto 1$ for all $g \in G$. Let $J = \ker \varepsilon \subseteq \mathbb{Z}G$. Then J is a free $\mathbb{Z}$-module with basis $\{g - 1 : g \in G \setminus \{1\}\}$.

Lemma 13.14. *We have*

$$J/J^2 \cong G/[G, G].$$

Proof. Define the map $\theta : G \to J/J^2$ by $\theta(g) = g - 1$. For $a, b \in G$, we have

$$\theta(ab) = ab - 1 + J^2 = (a - 1) + (b - 1) + J^2 = \theta(a) + \theta(b),$$

so θ is a group homomorphism. Since J/J^2 is abelian, we have

$$\theta(aba^{-1}b^{-1}) = \theta(a) + \theta(b) - \theta(a) - \theta(b) = 0,$$

so $[G, G] \subseteq \operatorname{Ker} \theta$, so θ descends to a homomorphism

$$\bar{\theta} : G/[G, G] \to J/J^2.$$

Define $\sigma : J \to G/[G, G]$ to be the unique homomorphism with $g - 1 \mapsto g + [G, G]$. For $a, b \in G$, we have

$$\sigma((a-1)(b-1)) = \sigma(ab-a-b+1) = \sigma(ab-1-(a-1)-(b-1)) = aba^{-1}b^{-1}+[G, G] = 0.$$

So σ descends to a homomorphism $J/J^2 \to G/[G, G]$. Clearly $\bar{\theta}$ and $\bar{\sigma}$ are mutual inverses, so we are done. ∎

Lemma 13.15. *We have*

$$J_G \cong J/J^2.$$

Proof. We have

$$J_G \cong J \otimes_{\mathbb{Z}G} \mathbb{Z} = J \otimes_{\mathbb{Z}G} \mathbb{Z}G/J \cong J/J^2.$$

∎

Theorem 13.16. *We have*

$$H_1(G, \mathbb{Z}) \cong G/[G, G].$$

Proof. We have a short exact sequence

$$0 \to J \to \mathbb{Z}G \to \mathbb{Z} \to 0$$

of G-modules. Since $\mathrm{Tor}_*^{\mathbb{Z}G}$ is a homological δ-functor, we obtain an exact sequence

$$H_1(G, \mathbb{Z}G) \to H_1(G, \mathbb{Z}) \to J_G \to (\mathbb{Z}G)_G \to \mathbb{Z}_G \to 0.$$

Since $\mathbb{Z}G$ is a projective $\mathbb{Z}G$-module, we have $H_1(G, \mathbb{Z}G) = 0$. The right-hand map is the isomorphism

$$\mathbb{Z}G \cong \mathbb{Z}G/J \cong \mathbb{Z},$$

so we have

$$H_1(G, \mathbb{Z}) \cong J_G = J/J^2 \cong G/[G, G].$$

∎

13.3 The Norm Element

Let G be a finite group. The **norm element** of $\mathbb{Z}G$ is

$$N = \sum_{g \in G} g \in \mathbb{Z}G.$$

Clearly $N \in (\mathbb{Z}G)^G$.

Lemma 13.17. *The subgroup*

$$H^0(G, \mathbb{Z}G) = (\mathbb{Z}G)^G$$

is the two-sided ideal $\mathbb{Z}N$ of $\mathbb{Z}G$ generated by N.

Proof. Easy. It's just $\mathbb{Z} \cdot N$. □

13.4 Finite Cyclic Groups

Let $C_m = \langle \sigma : \sigma^m = 1 \rangle$ be the cyclic group of order m. Then the norm element is

$$1 + \sigma + \ldots + \sigma^{m-1}.$$

We have

$$0 = \sigma^m - 1 = (\sigma - 1)N.$$

Lemma 13.18. *The chain complex*

$$\ldots \to \mathbb{Z}C_m \overset{\sigma-1}{\to} \mathbb{Z}C_m \overset{N}{\to} \mathbb{Z}C_m \overset{\sigma-1}{\to} \mathbb{Z}C_m \overset{\varepsilon}{\to} \mathbb{Z} \to 0$$

is a projective resolution for $\mathbb{Z}$ as a $\mathbb{Z}C_m$-module.

Proof. Exactness follows from a diagram chasing, using the fact that the sequences

$$0 \to J \to \mathbb{Z}C_m \overset{N}{\to} \mathbb{Z}N \to 0, \quad 0 \to \mathbb{Z}N \to \mathbb{Z}C_m \overset{\sigma-1}{\to} J \to 0$$

are exact. □

Theorem 13.19. *Let A be a C_m-module. Then*

$$H_n(C_m, A) = \begin{cases} A/(\sigma-1)A & \textit{if } n = 0, \\ A^G/NA & \textit{if } n = 1, 3, 5, \ldots, \\ \{a \in A : Na = 0\}/(\sigma-1)A & \textit{if } n = 2, 4, 6, \ldots. \end{cases}$$

$$H^n(C_m, A) = \begin{cases} A^G & \textit{if } n = 0, \\ \{a \in A : Na = 0\}/(\sigma-1)A & \textit{if } n = 1, 3, 5, \ldots, \\ A^G/NA & \textit{if } n = 2, 4, 6, \ldots. \end{cases}$$

Corollary 13.20. *We have*

$$H_n(C_m, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } n = 0, \\ \mathbb{Z}/m & \text{if } n \geq 1 \text{ is odd}, \\ 0 & \text{else.} \end{cases}$$

$$H^n(C_m, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } n = 0, \\ \mathbb{Z}/m & \text{if } n \geq 2 \text{ is even}, \\ 0 & \text{else.} \end{cases}$$

13.5 Free Groups

Theorem 13.21. *Let G be a free group on a set X. Then J is a free $\mathbb{Z}G$-module with basis $\{x - 1 : x \in X\}$.*

Proof. See, Page 169, Proposition 6.2.6. □

Corollary 13.22. *If G is a free group on X, then $\mathbb{Z}$ has free resolution*

$$0 \to J \to \mathbb{Z}G \to \mathbb{Z} \to 0.$$

Therefore, $H_n(G, A) = H^n(G, A) = 0$ for $n \neq 0, 1$, and $H_0(G, \mathbb{Z}) \cong H^0(G, \mathbb{Z}) \cong \mathbb{Z}$, while

$$H_1(G, \mathbb{Z}) \cong \bigoplus_{x \in X} \mathbb{Z}, \quad H^1(G, \mathbb{Z}) \cong \prod_{x \in X} \mathbb{Z}.$$

13.6 Crossed Homomorphisms

Historically, the maps we are interested in are called “crossed homomorphisms”. In these notes, however, we will adopt the more modern term “derivations”. Many of the proofs from now on are omitted; we have opted just to define the objects and sketch the theory. The proofs can all be found in Weibel.

Definition 13.23. Let G be a group and A be a left G-module. A **derivation** of G in A is a map $D : G \to A$ with $D(gh) = gD(h) + D(g)$ for all $g, h \in G$.

Remark 13.24. For those familiar with derivations more generally, Definition 13.23 is perhaps a little odd. Usually, you would expect

$$D(gh) = gD(h) + D(g)h.$$

The reason we drop the h on the far right is that we are viewing A as having trivial G-action on the right. In other words, we have $D(g)h = D(g)$.

Write $\mathrm{Der}(G, A)$ for the set of derivations. Then $\mathrm{Der}(G, A)$ is an abelian group under pointwise addition. For $a \in A$, let $D_a : G \to A$ be the map $D_a(g) = ga - a$.

Definition 13.25. A derivation of the form D_a is a **principal derivation**.

Write $\mathrm{PDer}(G, A)$ for the set of principal derivations. It is easy to see that $D_a + D_b = D_{a+b}$, so $\mathrm{PDer}(G, A)$ is a subgroup of $\mathrm{Der}(G, A)$. Recall that J is the augmentation ideal of $\mathbb{Z}G$. Let $\varphi : J \to A$ be a G-module homomorphism. Define $D_\varphi : G \to A$ by

$$D_\varphi(g) = \varphi(g - 1).$$

Then $D_\varphi : G \to Ai$ is a G-module map.

Lemma 13.26. *The map $\varphi \mapsto D_\varphi$ is a natural isomorphism*

$$\mathrm{Hom}_G(J, A) \to \mathrm{Der}(G, A)$$

of abelian groups.

Theorem 13.27. *We have*

$$H^1(G, A) = \mathrm{Der}(G, A)/\,\mathrm{PDer}(G, A).$$

Proof. The short exact sequence

$$0 \to J \to \mathbb{Z}G \to \mathbb{Z} \to 0$$

of $\mathbb{Z}G$-modules gives a long exact sequence beginning with

$$0 \to \operatorname{Hom}(\mathbb{Z}, A) \to \operatorname{Hom}(\mathbb{Z}G, A) \to \operatorname{Hom}(J, A) \to H^1(G; A) \to 0,$$

and the natural isomorphism $\operatorname{Hom}(J, A) \cong \operatorname{Der}(G, A)$ takes the image of $\operatorname{Hom}(\mathbb{Z}, A)$ to $\operatorname{PDer}(G, A)$, so

$$H^1(G; A) \cong \operatorname{Der}(G, A) / \operatorname{PDer}(G, A).$$

□

Corollary 13.28. *Let A be a trivial G-module. Then*

$$H^1(G, A) = \operatorname{Der}(G, A) \cong \operatorname{Hom}_{\mathbf{Grp}}(G, A).$$

Theorem 13.29 (Hilbert Theorem 90)**.** *Let L/K be a finite Galois extension with Galois group G. Let L^* be the unit group of L. Then L^* is naturally a G-module, and*

$$H^1(G, L^*) = 0.$$

13.7 Bar Complex

Throughout this section, $\mathbb{Z}$ is a trivial G-module.

Definition 13.30. The **unnormalised bar complex** is the chain complex

$$\ldots \to B_2^u \to B_1^u \to B_0^u \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

with $B_0^u = \mathbb{Z}G$ and $B_n^u = (\mathbb{Z}G)^{\otimes n}$. The differential $d : B_n^u \to B_{n-1}^u$ is given by

$$d = \sum_{i=0}^{n} (-1)^i d_i,$$

where

$$\begin{aligned} d_0(g_1 \otimes \ldots \otimes g_n) &= g_1 \cdot (g_2 \otimes \ldots \otimes g_n), \\ d_i(g_1 \otimes \ldots \otimes g_n) &= g_1 \otimes \ldots \otimes g_i g_{i+1} \otimes \ldots \otimes g_n \quad \text{for } 1 \le i \le n-1, \\ d_n(g_1 \otimes \ldots \otimes g_n) &= g_1 \otimes \ldots \otimes g_{n-1}. \end{aligned}$$

Definition 13.31. The **normalised bar complex** is

$$\ldots \to B_2 \to B_1 \to B_0 \xrightarrow{\epsilon} \mathbb{Z} \to 0,$$

where $B_0 = \mathbb{Z}G$, and for $n \ge 1$, the group B_n is free abelian on basis

$$\{[g_1|\ldots|g_n] : g_i \in G \setminus \{1\}\}.$$

The differential $d : B_n \to B_{n-1}$ is $d = \sum_{i=0}^{n}(-1)^i d_i$, where

$$\begin{aligned} d_0(g_1|\ldots|g_n) &= g_1 \cdot (g_2|\ldots|g_n), \\ d_i(g_1|\ldots|g_n) &= g_1|\ldots|g_i g_{i+1}|\ldots|g_n \quad \text{for } 1 \le i \le n-1, \\ d_n(g_1|\ldots|g_n) &= g_1|\ldots|g_{n-1}. \end{aligned}$$

We write $[]$ for $1 \in B_0 = \mathbb{Z}G$. If any of the g_i is 1, we write $[\ldots|g_i|\ldots]$ for $0 \in B_n$.

Example 13.32. We have

$$d([g|h]) = g[h] - [gh] + [g]$$

and

$$d([f|g|h]) = f[g|h] - [fg|h] + [f|gh] - [f|g].$$

Theorem 13.33. *The normalised and unnormalised bar complexes are free resolutions of $\mathbb{Z}$ as a $\mathbb{Z}G$-module.*

Corollary 13.34. *We have that $H^*(G, A)$ is the cohomology of the chain complexes $\mathrm{Hom}_G(B_*^u, A)$ and $\mathrm{Hom}_G(B_*, A)$.*

This allows us to give an explicit description of group cohomology. Define an n-**cochain** to be a function $f : G^n \to A$. An n-cochain φ is **normalised** if $\varphi(g_1, \ldots, g_n) = 0$

whenever one of the g_i is 1. Then

$$\mathrm{Hom}_G(B_n^u, A) \cong \{n\text{-cochains}\}.$$

Define the differential $d : \{n\text{-cochains}\} \to \{(n+1)\text{-cochains}\}$ by

$$d\varphi(g_0, \ldots, g_n) = g_0\varphi(g_1, \ldots, g_n) + \sum_{i=1}^{n-1}(-1)^i\varphi(\ldots, g_i g_{i+1}, \ldots) + (-1)^n\varphi(g_0, \ldots, g_{n-1}).$$

If $d\varphi = 0$, then φ is an **n-cocycle**, and for all φ, the cochain $d\varphi$ is an **n-coboundary**. Write $Z^n(G, A)$ and $B^n(G, A)$ for the abelian groups of n-cocycles and n-coboundaries respectively. Then

$$H^n(G, A) = Z^n(G, A)/B^n(G, A).$$

Lemma 13.35. *We have*

$$H^1(G, A) = \mathrm{Der}(G, A)/\,\mathrm{PDer}(G, A).$$

13.8 Group Extensions

Let A be an abelian group and let G be a group. An **extension of G by A** is a short exact sequence

$$0 \to A \to E \xrightarrow{\pi} G \to 1.$$

The extension **splits** if π has a section. That is, if there is a group homomorphism $s : G \to E$ such that $\pi \circ s = \mathrm{id}\, G$. Extensions

$$0 \to A \to E_i \xrightarrow{\pi} G \to 1$$

for $i = 1, 2$ are **equivalent** if there is a group isomorphism $E_1 \to E_2$ such that the obvious diagram commutes.

Theorem 13.36. *There is a natural bijection*

$$H^2(G, A) \Longleftrightarrow \{\text{Equivalence classes of extensions of } G \text{ by } A\}.$$