Topological groups, 2024–2025

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Course overview

Groups like the integers, the circle, and general linear groups (over $\mathbb{R}$ or $\mathbb{C}$) share a number of properties naturally captured by the notion of a topological group. Providing a unified framework for these groups and properties was an important achievement of 20th century mathematics, and in this course we shall develop this framework.

Highlights will include the existence and uniqueness of Haar integrals for locally compact topological groups, the topology of the dual group, and the existence of characters in various topological groups. Throughout, the course will use the tools of analysis to tie together the topology and algebra, getting at superficially more algebraic facts by analytic means.

Course synopsis

[6 lectures] Definition of topological and topologized groups and intermediate structures. Examples and non-examples, and basic properties. Subgroups. Quotient groups. The Open Mapping Theorem.
[4 lectures] Complete regularity of topological groups. Continuous partitions of unity and Fubini's Theorem. Existence and uniqueness of Haar integrals.
[6 lectures] The Peter-Weyl Theorem for compact topological groups. Dual groups of topological groups. Local compactness of the dual of a locally compact topological group. Pontryagin duality for compact Hausdorff Abelian topological groups.

References

There are other notes on similar topics with a slightly different focus *e.g.* [Fol95, Kör08, Kra17, Meg17] and [Rud90].

General prerequisites

The course is designed to be pretty self-contained. We assume basic familiarity with groups as covered in Prelims Groups and Group Actions. We shall also assume familiarity with Prelims Linear Algebra and Part A: Metric Spaces and Complex Analysis for material on metric and normed spaces.

Familiarity with topology is essential, though not much is required content-wise. What we use (and more) is covered in Part A: Topology, with the exception of Tychonoff's Theorem. This can be informally summarised as saying that a non-empty product of compact spaces is compact, and there is no harm in taking it as a black box for the course. Those interested in more detail may wish to consult Part C: Analytic Topology.

The Axiom of Choice is sometimes formulated as saying that an arbitrary product of non-empty sets is non-empty, and in this formulation it may be less surprising that it can be used to prove Tychonoff's Theorem. It turns out that the converse is also true, *i.e.* Tychonoff's Theorem (and the other axioms of set theory) can be used to prove the Axiom of Choice[1].

Finally no familiarity with functional analysis is assumed, though there are clear similarities and parallels for those who do have some. See, for example Part B: Functional Analysis, and Part C: Further Functional Analysis.

Teaching

A first draft of these notes is on the website, but they will be updated after each lecture with any resulting changes. This document was compiled on 5$^{\text{th}}$ May, 2025.

Lectures will be supplemented by some tutorial-style teaching where we can discuss the course and also exercises from the sheets. Once I have a list of the MFoCS students attending I shall be in touch to arrange these.

Contact details and feedback

Contact tom.sanders@maths.ox.ac.uk if you have any questions or feedback.

[1] Those unfamiliar and looking for a reference may wish to consult the notes [Ter10].

Group notation

A group G is said to be **written multiplicatively** if the binary operation of the group is written $G \times G \to G; (x, y) \mapsto xy$ and called **multiplication**; the unique inverse is written x^{-1} and the map $G \to G; x \mapsto x^{-1}$ is called **inversion**; and the identity is written 1_G. Given $S, T \subset G$ we write

$$S^{-1} := \{s^{-1} : s \in S\} \text{ and } ST := \{st : s \in S, t \in T\}.$$

For $n \in \mathbb{N}_0$ we define S^n inductively by

$$S^0 := \{1_G\} \text{ and } S^{n+1} := S^n S; \text{ and } S^{-n} := (S^{-1})^n.$$

⚠ This notation means that in general $SS^{-1} \neq S^0$ and $S^2 \neq \{s^2 : s \in S\}$.

It will also be convenient to write $xS := \{x\}S$ and $Sx := S\{x\}$ for $x \in G$, which aligns the the usual notation for left and right cosets when S is a subgroup.

If G is Abelian then it is said to be **written additively** if the binary operation of the group is written $G \times G \to G; (x, y) \mapsto x + y$ and called **addition**; inversion is written $G \to G; x \mapsto -x$ and called **negation**; and the identity is written 0_G. ⚠ All groups written additively are Abelian, but not all Abelian groups will be written additively.

If G is written additively then the above notation changes in the obvious way so we write $-S$ instead of S^{-1}, $S + T$ instead of ST, nS instead of S^n *etc.*

1 Groups with topologies

A group G that is also a topological space is called a **topologized group**. Without any additional assumptions this is no more than its constituent parts: a group and a topological space. When the group inversion $G \to G$ and the group operation $G \times G \to G$ are both continuous we say G is a **topological group**. (Here the product $G \times G$ is given the product topology.)

Example 1.1 (Indiscrete groups)**.** For any group G, we write G_{I} for G endowed with the indiscrete topology. This is a topological group since any map into an indiscrete space is continuous, so in particular both group inversion and the group operation are continuous.

Any indiscrete space is compact since the indiscrete topology is finite, so G_{I} is a compact topological group. G_{I} is Hausdorff if and only if G is the trivial group.

A topological space is **locally compact** if every element is contained in a compact neighbourhood. In the literature sometimes different definitions of local compactness are used – see Remark 2.24 for an example that is relevant to us – although they usually coincide when the space is additionally assumed to be Hausdorff.

Example 1.2 (Discrete groups)**.** For any group G, we write G_{D} for G endowed with the discrete topology. This is a topological group since the product of two copies of the discrete topology is discrete – so both the topological spaces G and $G \times G$ are discrete – and any map from a discrete space is continuous, so in particular both group inversion and the group operation are continuous.

Any discrete space is locally compact since $\{x\}$ is an open neighbourhood of x which is compact, since it is finite; and Hausdorff since $\{x\}$ and $\{y\}$ are disjoint open neighbourhoods of x and y respectively when $x \neq y$. Hence G_{D} is a locally compact Hausdorff topological group. Since the set of singletons in G_{D} is an open cover of G_{D}, G_{D} is compact if and only if it is finite.

The reals under addition may be endowed with the discrete or indiscrete topologies to make them into a topological group as above. However, neither of these is the 'usual' topology which has as open sets unions of intervals without their endpoints.

Example 1.3 (The real line)**.** The additive group $\mathbb{R}$ endowed with its usual topology is a topological group which we call the **real line**, and which we also denote $\mathbb{R}$. The relevant continuity is just the algebra of limits: in particular, if $x_n \to x_0$ then $-(x_n) = (-1)x_n \to (-1)x_0 = -x_0$; and if $x_n \to x_0$ and $y_n \to y_0$, then $x_n + y_n \to x_0 + y_0$.

The compact sets in the real line $\mathbb{R}$ are exactly the closed and bounded sets (this is the Heine-Borel Theorem for $\mathbb{R}$). In particular, $\mathbb{R}$ itself is not bounded and so not compact; and for any $x \in \mathbb{R}$, $[x-1, x+1]$ is a compact neighbourhood of x, so $\mathbb{R}$ is locally compact.

$\mathbb{R}$ is also Hausdorff: certainly if $x \neq y$ then there are two disjoint open intervals, with one containing x and the other y.

In summary, $\mathbb{R}$ is a locally compact Hausdorff topological group that is not compact.

The algebra of limits also apply multiplicatively and to complex numbers:

Example 1.4 (Non-zero complex numbers)**.** The set of non-zero complex numbers, $\mathbb{C}^*$, is a multiplicative group and with the usual topology is a topological group also denoted $\mathbb{C}^*$. Again, the relevant continuity is just the algebra of limits: if $x_n \to x_0$ in $\mathbb{C}^*$ then $x_n^{-1} \to x_0^{-1}$; and if $x_n \to x_0$ and $y_n \to y_0$, then $x_n y_n \to x_0 y_0$.

The compact sets in $\mathbb{C}$ are exactly the closed and bounded sets (this is the Heine-Borel Theorem again, this time for $\mathbb{R}^2$). We can used this as in Example 1.3 to see that $\mathbb{C}^*$ is a locally compact Hausdorff topological group that is not compact.

Maps between topologized groups

The maps which will concern us the most are continuous homomorphisms, and also continuous *open* homomorphisms, that is continuous homomorphisms in which the image of an open set is open.

Example 1.5. The map $\mathbb{R} \to \mathbb{R}; x \mapsto \alpha x$ for $\alpha \in \mathbb{R}$ is a continuous homomorphism of the real line, and in fact these are the only continuous homomorphisms of the real line. For $\alpha = 0$ this map is *not* open; for $\alpha \neq 0$, this map has an inverse of the same form and so is open and in fact is a homeomorphic isomorphism.

Example 1.6. For a topologized group G, the identity map $G \to G_{\mathrm{I}}$ is a continuous isomorphism, because the identity map is an isomorphism and any map to an indiscrete space is continuous.

⚠ G need not be a topological group despite the fact that G_{I} is a topological group. That being said we do have the following:

Proposition 1.7. *Suppose that $\theta : H \to G$ is a homomorphism and G is a topological group. Then H with the initial topology w.r.t. θ (that is the topology $\{\theta^{-1}(U) : U \text{ is open in } G\}$) is a topological group.*

Proof. Suppose U is an open set in H so that there is W, open in G, such that $U = \theta^{-1}(W)$. For continuity of the inverse, note $U^{-1} = (\theta^{-1}(W))^{-1} = \theta^{-1}(W^{-1})$, but W^{-1} is open in G and so U^{-1} is open in H. For continuity of multiplication let $\mathcal{S}$ be a set of products of open sets in G such that $\{(x, y) \in G \times G : xy \in W\} = \bigcup \mathcal{S}$. Then

$$\begin{aligned}\{(x, y) \in H \times H : xy \in U\} &= \{(x, y) \in H \times H : \theta(x)\theta(y) \in W\} \\ &= \{(x, y) \in H \times H : (\theta(x), \theta(y)) \in S \times T \text{ for some } S \times T \in \mathcal{S}\} \\ &= \bigcup \{\theta^{-1}(S) \times \theta^{-1}(T) : S \times T \in \mathcal{S}\},\end{aligned}$$

and this last set is a union of open sets and so open. The result is proved. □

Remark 1.8. In particular, if G is a topological group and H is a subgroup of G, then H with the subspace topology is a topological group since the subspace topology is exactly the initial topology on H w.r.t. the inclusion $H \to G; x \mapsto x$.

Example 1.9 (The rationals)**.** We write $\mathbb{Q}$ for the topological group of rationals with the subspace topology inherited from the topological group $\mathbb{R}$.

$\mathbb{Q}$ is Hausdorff since the real line is Hausdorff, but *not* locally compact (and so certainly not compact) – this is exactly why one constructs the real line! To see this, suppose K were a compact neighbourhood of 0. Then by definition of the subspace topology, there would be $\epsilon > 0$ such that $(-\epsilon, \epsilon) \cap \mathbb{Q} \subset K$. The interval $(-\epsilon, \epsilon)$ contains an irrational α, and then $\{(-\infty, \alpha - 1/n) \cap \mathbb{Q}, (\alpha + 1/n, \infty) \cap \mathbb{Q} : n \in \mathbb{N}^*\}$ would be an open cover of K without a finite subcover – a contradiction.

Example 1.10 (The positive reals)**.** We write $\mathbb{R}_{>0}$ for the topological group of positive reals with the subspace topology inherited from the topological group $\mathbb{C}^*$.

In fact this is in a sense 'the same' as Example 1.3 because there is a homeomorphic isomorphism between the two; part of Exercise II.1 asks for a proof of this.

Example 1.11 (The circle group)**.** We write S^1 for the topological group of complex numbers of modulus 1 with the subspace topology inherited from the topological group $\mathbb{C}^*$; we call it the **circle group**. S^1 is compact (as a closed and bounded subset of $\mathbb{C}$) and Hausdorff.

Example 1.12 (Universal covering of the circle group)**.** The map $\theta : \mathbb{R} \to S^1; x \mapsto \exp(2\pi i x)$ from the real line to the circle group is a surjective continuous open homomorphism. ⚠ The topology on the real line is *not* the initial topology w.r.t θ, and so Proposition 1.7 applied to θ does *not* give us a new way of deducing that the real line is a topological group. We shall revisit the topology we do get later in Example 2.27.

Group actions

Groups often arise with actions, and topological groups are no exception to this. ⚠ Our actions will all be *left* actions.

Example 1.13 (Homeomorphisms of topological spaces)**.** For a topological space X and group G of homeomorphisms of X under composition, the map $G \times X \to X; (g, x) \mapsto g(x)$ is an action called the **evaluation action**.

Observation 1.14. Given an action of a group G on a topological space X, the maps $X \to X; x \mapsto g.x$ are continuous for all $g \in G$ if and only if they are homeomorphisms since $g^{-1}.(g.x) = x = g.(g^{-1}.x)$ for all $x \in X$ and $g \in G$.

Example 1.15. The space $X := [0, 1]$ with its usual topology and G the set of increasing bijections $[0, 1] \to [0, 1]$. This is a group of continuous functions, and so of homeomorphisms by the preceding observation.

Given an action of a group G on a topological space X, the **topology of pointwise converge on G w.r.t. this action** is the weakest topology on G such that the maps $G \to X; g \mapsto g.x$ are continuous for all $x \in X$. (In other words it is the initial topology on G w.r.t. the family of functions $G \to X; g \mapsto g.x$ for $x \in X$.) In particular, given a base $\mathcal{B}$ for X, the sets

$$U(x_1, \ldots, x_n; U_1, \ldots, U_n) := \{g \in G : g.x_1 \in U_1, \ldots, g.x_n \in U_n\}$$

with $x_1, \ldots, x_n \in X$ and $U_1, \ldots, U_n \in \mathcal{B}$ form a base for the topology of pointwise convergence w.r.t. the given action.

Proposition 1.16. *Suppose that X is a topological space with topology given by a metric d, and G is a group acting on X such that $d(g.x, g.y) = d(x, y)$ for all $x, y \in X$ and all $g \in G$. Then G with the topology of pointwise convergence w.r.t. this action is a topological group.*

Proof. Write $B_\epsilon(x) := \{y \in X : d(x, y) < \epsilon\}$ so that $\{B_\epsilon(x) : x \in X, \epsilon > 0\}$ is a base for the topology on X. If $f_0 \in U(x_1, \ldots, x_n; U_1, \ldots, U_n)$ then there is $\epsilon > 0$ such that

$$U(x_1, \ldots, x_n; B_\epsilon(f_0.x_1), \ldots, B_\epsilon(f_0.x_n)) \subset U(x_1, \ldots, x_n; U_1, \ldots, U_n).$$

Hence the preimage under inversion of $U(x_1, \ldots, x_n; U_1, \ldots, U_n)$ contains the preimage of $U(x_1, \ldots, x_n; B_\epsilon(f_0.x_1), \ldots, B_\epsilon(f_0.x_n))$, but for $f \in G$ we have

$$d(f^{-1}.(f_0.x_i), x_i) = d(f_0.x_i, f.x_i) = d(f.x_i, f_0.x_i).$$

Hence this preimage contains $U(f_0.x_1, \ldots, f_0.x_n; B_\epsilon(x_1), \ldots, B_\epsilon(x_n))$, which is a neighbourhood of f_0^{-1}. Hence inversion is continuous.

Suppose $g_0 f_0 \in U(x_1, \ldots, x_n; U_1, \ldots, U_n)$, so that there is $\epsilon > 0$ such that

$$U(x_1, \ldots, x_n; B_\epsilon((g_0 f_0).x_1), \ldots, B_\epsilon((g_0 f_0).x_n)) \subset U(x_1, \ldots, x_n; U_1, \ldots, U_n).$$

Then, if $d(g.(f_0.x_i), (g_0 f_0).x_i) < \epsilon/2$ and $d(f.x_i, f_0.x_i) < \epsilon/2$ for all $1 \leqslant i \leqslant n$, we have

$$\begin{aligned} d((gf).x_i, (g_0 f_0).x_i) &\leqslant d((gf).x_i, (gf_0).x_i) + d((gf_0).x_i, (g_0 f_0).x_i) \\ &= d(f(x_i), f_0(x_i)) + d(g.(f_0.x_i), (g_0 f_0).x_i) < \epsilon, \end{aligned}$$

and so $gf \in U(x_1, \ldots, x_n; U_1, \ldots, U_n)$. Hence the preimage of $U(x_1, \ldots, x_n; U_1, \ldots, U_n)$ under the group operation contains the open neighbourhood

$$\begin{aligned} &U(f_0.x_1, \ldots, f_0.x_n; B_{\epsilon/2}((g_0 f_0).x_1), \ldots, B_{\epsilon/2}((g_0 f_0).x_n)) \\ &\qquad \times U(x_1, \ldots, x_n; B_{\epsilon/2}(f_0.x_1), \ldots, B_{\epsilon/2}(f_0.x_n)) \end{aligned}$$

of (g_0, f_0). We conclude that multiplication is continuous as a map $G \times G \to G$ and G is a topological group. □

Example 1.17 (Isometries of normed spaces)**.** For X a normed space with norm $\|\cdot\|$ the map $d(x,y) := \|x-y\|$ defines a metric on X, and we write $\mathrm{Iso}(X)$ for the group of invertible isometries of X, that is the set of bijections $\phi : X \to X$ such that $\|\phi(x) - \phi(y)\| = \|x - y\|$ for all $x, y \in X$. Proposition 1.16 applied to the evaluation action tells us that $\mathrm{Iso}(X)$ is a topological group when endowed with the topology of pointwise convergence (w.r.t. this action).

If X is a real normed space then the Mazur-Ulam theorem [Väi03] tells us that every invertible isometry is affine linear, but complex conjugation on $\mathbb{C}$ (considered as a complex normed space with norm given by the absolute value) is an invertible isometry that is not affine $\mathbb{C}$-linear.

Example 1.18 (Unitary maps)**.** For V a complex inner product space a **unitary map** is a linear map $\phi : V \to V$ with $\langle \phi(v), \phi(w)\rangle = \langle v, w\rangle$ for all $v, w \in V$. We write $\mathrm{U}(V)$ for the group of invertible unitary maps $V \to V$. The inner product on V defines a norm (by $\|v\| := \langle v, v\rangle^{1/2}$) which in turn defines a metric as in Example 1.17. Proposition 1.16 applied to the evaluation action tells us that $\mathrm{U}(V)$ is a topological group when endowed with the topology of pointwise convergence (w.r.t. this action).

Example 1.19 (Orthogonal groups)**.** The group O_n of orthogonal matrices acts on the metric space $\mathbb{R}^n$ equipped with the Euclidean metric $d(x,y) = \left(\sum_{i=1}^n (x_i - y_i)^2\right)^{1/2}$ via $(M, x) \mapsto Mx$, in such a way that $d(Mx, My) = d(x, y)$ for all $x, y \in \mathbb{R}^n$. Hence by Proposition 1.16 O_n is a topological group when endowed with the topology of pointwise convergence with respect to this action.

Translation invariant Abelian groups

We say that a metric d on a group G (written multiplicatively) is **(left) translation invariant** if

$$d(xy, xz) = d(y, z) \text{ for all } x, y, z \in G.$$

Corollary 1.20. *Suppose that G is an Abelian group and d is a translation invariant metric on G. Then G with the topology induced by d is a Hausdorff topological group.*

Proof. Since G is Abelian we shall write it additively and the group operation is an action of the group on itself. It follows from Proposition 1.16 that G with the topology of pointwise convergence w.r.t. this action is a topological group. On the other hand, writing $B_\epsilon(x) := \{y \in G : d(x, y) < \epsilon\}$ we have

$$U(x_1, \dots, x_n; B_\epsilon(x_1'), \dots, B_\epsilon(x_n')) = B_\epsilon(x_1' - x_1) \cap \dots \cap B_\epsilon(x_n' - x_n)$$

since $d(y + x, x') = d(y, x' - x)$ for all $y, x, x' \in G$ (crucially using the hypothesis that G is Abelian here), and these sets form a base for the topology on G induced by d. In other words the topology of pointwise convergence in this case *is* the topology induced by the metric d, and G with this topology is a topological group. Finally, the topology is Hausdorff since it is induced by a metric. $\square$

Example 1.21 (Normed spaces)**.** For X a normed space with norm $\|\cdot\|$ the map $d(x,y) := \|x-y\|$ defines a translation invariant metric on the (Abelian) additive group of X (*c.f.* Example 1.17). It follows from Corollary 1.20 that X under addition with this topology is a Hausdorff topological group.

If X contains some $x \neq 0$, then the subset $\{\lambda.x : \lambda \in \mathbb{R}\}$ is unbounded in the metric d and hence X is not compact. It follows that the topological group X is compact if and only if $X = \{0\}$; more than this, a classic result of André Weil [Wei74, Corollary 2, p6] tells us that the topological group X is locally compact if and only if the normed space X is finite dimensional.

Write $G^{\mathbb{N}^*}$ for the group of G-valued sequences, *i.e.* sequences $(x_i)_{i\in\mathbb{N}^*}$ with $x_i \in G$, endowed with the group operation $xy := (x_iy_i)_{i\in\mathbb{N}^*}$. This is a group with identity the constant sequence taking the value 1_G, and $x^{-1} = (x_i^{-1})_{i\in\mathbb{N}^*}$.

Example 1.22 (Dyadic Cantor group)**.** Define a metric on $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}^*}$ by

$$d_\pi(x,y) := \inf\{2^{-k} : k \in \mathbb{N}_0, x_1 = y_1, \ldots, x_k = y_k\},$$

which in fact enjoys the stronger triangle inequality

$$d_\pi(x,z) \leqslant \max\{d_\pi(x,y), d_\pi(y,z)\} \text{ for all } x,y,z \in (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}^*}.$$

The metric d_π is also translation invariant and so Corollary 1.20 tells us that $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}^*}$ with this topology is a Hausdorff topological group which we denote $\mathbb{D}$.

If $(x^{(n)})_n$ is Cauchy in d_π then for all k there is N_k such that $x_i^{(n)} = x_i^{(m)}$ for all $i \leqslant k$ and $n,m \geqslant N_k$; letting $y_k := x_k^{(N_k)}$ for all k, we have $x^{(n)} \to y \in \mathbb{D}$. It follows that $\mathbb{D}$ is complete as a metric space. It is also totally bounded, since for every $k \in \mathbb{N}^*$ and $x \in \mathbb{D}$ there is $y \in (\mathbb{Z}/2\mathbb{Z})^k$ such that $d_\pi(x,\tilde{y}) \leqslant 2^{-k}$ where $\tilde{y} := (y_1, \ldots, y_k, 0, \ldots)$. We conclude that $\mathbb{D}$ is compact.

We cannot drop the condition that G is Abelian from Corollary 1.20:

Example 1.23. A homomorphism $f : \mathbb{D} \to S^1$ takes values in $\{1,-1\}$ since every element of $\mathbb{D}$ has order 2; fix such an f.

The map $(x,y)*(x',y') = (x+f(y)x', y+y')$ defines a group operation[2] on the set $\mathbb{R}\times\mathbb{D}$ – this group is denoted $\mathbb{R} \rtimes_f \mathbb{D}$. Equip the group $\mathbb{R} \rtimes_f \mathbb{D}$ with the topology induced by the

[2]This is the semidirect product of the group $\mathbb{R}$ and $\mathbb{D}$ with respect to the homomorphism $\phi : \mathbb{D} \to \mathrm{Aut}(\mathbb{R}); y \mapsto (\mathbb{R} \to \mathbb{R}; x \mapsto f(y)x)$.

metric $d((x,y),(x',y')) := \max\{|x-x'|, d_\pi(y,y')\}$ where d_π is as in Example 1.22. Then

$$\begin{aligned} d((x,y)*(x',y'),(x,y)*(x'',y'')) &= \max\{|x+f(y)x'-(x+f(y)x'')|, d_\pi(y+y',y+y'')\} \\ &= \max\{|x'-x''|, d_\pi(y',y'')\} = d((x',y'),(x'',y'')) \end{aligned}$$

for all $(x,y),(x',y'),(x'',y'') \in \mathbb{R} \rtimes_f \mathbb{D}$ since d_π is translation invariant and $|f(y)| = 1$ for all $y \in \mathbb{D}$. In other words d is left translation invariant.

If f is *not* continuous, then the map

$$\mathbb{R} \rtimes_f \mathbb{D} \to \mathbb{R} \rtimes_f \mathbb{D}; (x,y) \mapsto (x,y)*(1,0) = (x+f(y),y)$$

is not continuous and so certainly $\mathbb{R} \rtimes_f \mathbb{D}$ is not a topological group (*c.f.* Observation 1.31). (It is not immediately obvious that such an f should exist, but we shall see in Exercise IV.3 that it does.)

1.24 A worked example: a square-free topology on the integers.

Although Proposition 1.7 takes the initial topology with respect to just one function, we can define the initial topology for a whole family of functions; we do this here. Write $\mathbb{Z}_{\text{SF}}$ for the additive group of integers endowed with the initial topology with respect to the homomorphisms

$$\gamma_p : \mathbb{Z} \to S^1; z \mapsto \exp(2\pi i z/p) \text{ for } p \text{ prime}.$$

The map γ_p is continuous on $\mathbb{Z}$ endowed with some topology if and only if said topology contains all the sets in $\mathbb{Z}/p\mathbb{Z}$ *i.e.* all sets of the form $x+p\mathbb{Z}$ for $x \in \mathbb{Z}$. If $(x+n\mathbb{Z}) \cap (y+m\mathbb{Z})$ is non-empty – say it contains some z – then it equals $z + \operatorname{lcm}(n,m)\mathbb{Z}$. In particular, the topology on $\mathbb{Z}_{\text{SF}}$ must contain all elements of

$$\mathcal{B} := \bigcup \left\{ \mathbb{Z}/m\mathbb{Z} : m = \prod_{p \in \mathcal{M}} p \text{ for } \mathcal{M} \text{ a finite set of primes} \right\}.$$

Claim. *The set $\mathcal{B}$ is a base for the topology on $\mathbb{Z}_{\text{SF}}$.*

Proof. First $\mathcal{B}$ is a base for a topology on $\mathbb{Z}$ because a) $\mathcal{B}$ is a cover of $\mathbb{Z}$, since $\prod_{p \in \emptyset} p = 1$, and so $\mathbb{Z} \in \mathcal{B}$; and b) if $x+m\mathbb{Z}, y+n\mathbb{Z} \in \mathcal{B}$ then $m = \prod_{p \in \mathcal{M}} p$ and $n = \prod_{p \in \mathcal{N}} p$ for finite sets of primes $\mathcal{M}$ and $\mathcal{N}$, and if $z \in (x+m\mathbb{Z}) \cap (y+n\mathbb{Z})$ then $z+q\mathbb{Z} \subset (x+m\mathbb{Z}) \cap (y+n\mathbb{Z})$ where $q := \prod_{p \in \mathcal{M} \cup \mathcal{N}} p$.

We noted above that $\mathcal{B}$ is certainly a set of open sets in $\mathbb{Z}_{\text{SF}}$, and it remains to note that in any topology containing $\mathcal{B}$ the map γ_p is continuous since $\mathcal{B}$ contains $\mathbb{Z}/p\mathbb{Z}$. □

The base $\mathcal{B}$ explains the subscript in $\mathbb{Z}_{\text{SF}}$: it stands for 'square-free' which is chosen because, by the Fundamental Theorem of Arithmetic, the products $\prod_{p \in \mathcal{M}} p$ are exactly the square-free natural numbers.

Claim. $\mathbb{Z}_{\text{SF}}$ *is a topological group.*

Proof. Negation on $\mathbb{Z}_{\text{SF}}$ is continuous since $-(x + m\mathbb{Z}) = -x + m\mathbb{Z}$ for all $x \in \mathbb{Z}$ and m. Addition is continuous since if (x, y) is in the preimage under addition of $z + m\mathbb{Z}$ then that preimage contains the open neighbourhood $(x + m\mathbb{Z}) \times (y + m\mathbb{Z})$ of (x, y). $\square$

Claim. $\mathbb{Z}_{\text{SF}}$ *is Hausdorff*

Proof. If $x \neq y$ then without loss of generality $(x+1)-y > 1$. Every natural number bigger than 1 has a smallest factor bigger than 1, and this factor will be prime, so there is a prime p with $x + 1 - y \in p\mathbb{Z}$. Then $x + p\mathbb{Z} \cap y + p\mathbb{Z} = (y - 1) + p\mathbb{Z} \cap y + p\mathbb{Z}$. Since $p \nmid 1$ we have $y - 1 + p\mathbb{Z} \neq y + p\mathbb{Z}$ and since the intersection of two cosets of the same subgroup is either equal or empty, the sets $x + p\mathbb{Z}$ and $y + p\mathbb{Z}$ are disjoint open neighbourhoods of x and y respectively, as required. $\square$

Claim. *The compact subsets of* $\mathbb{Z}_{\text{SF}}$ *are exactly the finite sets.*

Proof. Any finite set is compact, but conversely suppose that $S \subset \mathbb{Z}$ is infinite. Let $S_1 := S$, $x_1 = 0$, and $m_1 = 1$. At stage i, let $p_{i+1} > m_i$ be a prime larger than the two smallest elements of S_i, $m_{i+1} := p_{i+1}m_i$ and x_{i+1} be such that $S_{i+1} := S_i \cap (x_{i+1} + m_{i+1}\mathbb{Z})$ is infinite. The choice of p_{i+1} is possible since there are infinitely many primes; and the choice of x_{i+1} is possible since $\mathbb{Z}/m_{i+1}\mathbb{Z}$ is a partition of $\mathbb{Z}$ into finitely many sets and so S_i must have infinite intersection with one of them by the pigeonhole principle.

The fact that $p_{i+1} > m_i$ for each i ensures that the m_is are all products of distinct primes and so $x_i + m_i\mathbb{Z} \in \mathcal{C}$, and the fact that p_{i+1} is larger than the two smallest elements of S_i ensures that $S_{i+1} \neq S_i$. In particular then, we have $\mathbb{Z} = x_1 + m_1\mathbb{Z} \supset x_2 + m_2\mathbb{Z} \supset \dots$ so

$$\mathcal{C} := \{(x_i + m_i\mathbb{Z})\backslash(x_{i+1} + m_{i+1}\mathbb{Z}) : i \in \mathbb{N}^*\}.$$

This is a cover of $\mathbb{Z}$ by disjoint open sets each of which has a non-empty intersection with S. It follows that $\mathcal{C}$ is an open cover with no finite subcover. $\square$

All the non-empty open sets in $\mathbb{Z}_{\text{SF}}$ are infinite and so in particular $\mathbb{Z}_{\text{SF}}$ is *not* locally compact.

Between topologized and topological

To better understand topological groups we shall also look at some weaker structures with some axioms stripped away – centipede mathematics. These structures are also studied in their own right; for a much more detailed development including many examples and open problems see [AT08, Chapters 1 & 2].

Suppose that G is a topologized group written multiplicatively. We say that **left (resp. right) multiplication is continuous** if the maps $G \to G; y \mapsto xy$ (resp. $G \to G; y \mapsto yx$)

are continuous for all $x \in G$. A topologized group in which left (resp. right) multiplication is continuous is said to be a **left-topological** (resp. **right-topological**) **group**. A group which is both a left-topological and a right-topological group is called a **semitopological group**.

Example 1.25 (Example 1.23, revisited)**.** The topologized group $\mathbb{R} \rtimes_f \mathbb{D}$ with operation $*$ is a left-topological group since the maps

$$\mathbb{R} \rtimes_f \mathbb{D} \to \mathbb{R} \rtimes_f \mathbb{D}; (x', y') \mapsto (x, y) * (x', y') = (x + x'f(y), y + y')$$

are continuous for fixed (x, y) whether or not f is continuous. Its topology is the product of two locally compact Hausdorff spaces and so it itself is locally compact and Hausdorff.

Example 1.26 (Opposite and Abelian topologized groups)**.** When G is a topologized group by G^{op} we mean the **opposite group** of G, that is the group with group operation $G \times G \to G; (x, y) \mapsto yx$, endowed with the same topology as G. In this notation G is left-topological if and only if G^{op} is right-topological.

In particular, any Abelian left-topological or right-topological group is semitopological since left (resp. right) multiplication by y is the same as right (resp. left) multiplication by y.

Example 1.27 (The coset topology)**.** For a group G and $H \leqslant G$, equipping G with the topology whose closed sets are unions of left cosets of H makes it into a left-topological group; we call this topology the **(left) coset topology (on G generated by H)**. ⚠ This terminology is not standard.

The open (and closed) sets in G are exactly the unions of left cosets of H, hence if $S \subset G$ then $\overline{S} = SH$. Right multiplication is continuous (if and) only if H is normal in G: Indeed, if right multiplication is continuous then since H is closed, Hy^{-1} is closed for all y, so $Hy^{-1} = SH$ for some $S \subset G$. Let $x \in S$ be such that $y^{-1} \in xH$, whence $y^{-1}H = xH \subset SH = Hy^{-1}$ and so H is normal in G.

Proposition 1.28. *Suppose that X is a topological space and G is a group of homeomorphisms of X. Then G with the topology of pointwise convergence w.r.t. the evaluation action is a semitopological group.*

Proof. For $x_1, \dots, x_n \in X$ and $U_1, \dots, U_n$ open in X we have

$$U(x_1, \dots, x_n; U_1, \dots, U_n)g^{-1} = U(g.x_1, \dots, g.x_n; U_1, \dots, U_n)$$

so right multiplication is continuous. Furthermore,

$$g^{-1}U(x_1, \dots, x_n; U_1, \dots, U_n) = U(x_1, \dots, x_n; g^{-1}.U_1, \dots, g^{-1}.U_n),$$

so left multiplication is continuous since the sets $g^{-1}.U_1, \dots, g^{-1}.U_n$ are open because the action is by continuous functions. $\square$

Example 1.29 (Groups of continuous maps with continuous inverses)**.** For X a normed space we write $\mathrm{GL}(X)$ for the set of continuous linear maps $X \to X$ with continuous linear inverses.

The set $\mathrm{GL}(X)$ is a group of homeomorphisms of X, and hence if $\mathrm{GL}(X)$ is endowed with the topology of pointwise convergence w.r.t. the evaluation action, then $\mathrm{GL}(X)$ becomes a semitopological group by Proposition 1.28.

By contrast with Example 1.18 when $X = \ell_2$, it can be shown that neither composition nor inversion is continuous.

A topologized group in which the group operation is continuous (as a map from the product space $G \times G$) is called a **paratopological group**.

Example 1.30 (The reals with the right order topology)**.** The set[3] $\{(a, \infty) : -\infty \leqslant a \leqslant \infty\}$ is a topology on $\mathbb{R}$ which we call the **right order topology (on $\mathbb{R}$)**; we denote this topologized group $\mathbb{R}_{\mathrm{RO}}$. In particular, if $A \subset \mathbb{R}$ is non-empty and bounded above then $\overline{A} = (-\infty, \sup A]$; and if A is non-empty and not bounded above then $\overline{A} = \mathbb{R}$.

$\mathbb{R}_{\mathrm{RO}}$ is a paratopological group since for $a \in \mathbb{R}$,

$$\{(x, y) : x + y \in (a, \infty)\} = \bigcup_{b \in \mathbb{R}} (a - b, \infty) \times (b, \infty)$$

so that the preimage of the open set (a, ∞) is open in the product topology. Inversion on $\mathbb{R}_{\mathrm{RO}}$ is not continuous since $(-\infty, -a)$ is not open (for any $a \in \mathbb{R}$), and hence $\mathbb{R}_{\mathrm{RO}}$ is not a topological group.

$\mathbb{R}_{\mathrm{RO}}$ is not Hausdorff: Any two non-empty open sets contain all sufficient large reals and hence have non-empty intersection.

The non-empty compact subsets of $\mathbb{R}_{\mathrm{RO}}$ are exactly the sets A that are bounded below (and so have an infimum) and contain their infimum. Indeed, if $\inf A \in A$ then any open cover $\mathcal{U}$ of A contains an open set U containing $\inf A$. But then $A \subset U$, and so $\{U\}$ is a finite subcover of $\mathcal{U}$. Conversely, if A is not bounded below then it is not compact since $\mathcal{U} := \{(a, \infty) : a \in \mathbb{R}\}$ is an open cover of A, but any finite $\mathcal{U}' \subset \mathcal{U}$ has a smallest $a \in \mathbb{R}$ such that $(a, \infty) \in \mathcal{U}'$, and so $\mathcal{U}'$ is not a cover of A; and if A is bounded below but does not contain its infimum then $\{(\inf A + 1/n, \infty) : n \in \mathbb{N}\}$ is a cover of A which does not have a finite subcover by the approximation property.

It follows that $\mathbb{R}_{\mathrm{RO}}$ is locally compact, since for $x \in \mathbb{R}$, $[x - 1, \infty)$ is a compact neighbourhood of x, and also $\mathbb{R}_{\mathrm{RO}}$ is not compact since it is not bounded below.

Observation 1.31. Every paratopological group G is semitopological since the maps $G \to G \times G; x \mapsto (x, y)$ (and $G \to G \times G; x \mapsto (y, x)$) are continuous for all $y \in G$, and the composition of continuous maps is continuous.

[3]For the avoidance of doubt $(-\infty, \infty) := \mathbb{R}$ and $(\infty, \infty) := \varnothing$.

A semitopological group in which inversion is continuous is called a **quasitopological** group.

Example 1.32 (The reals with the cofinite topology)**.** Write $\mathbb{R}_{\mathrm{CF}}$ for the additive group $\mathbb{R}$ equipped with the topology whose proper closed sets are the finite sets. This is a genuine topology and is called the **cofinite** topology on $\mathbb{R}$. In particular if $A \subset \mathbb{R}$ is finite then $\overline{A} = A$ and if A is infinite then $\overline{A} = \mathbb{R}$.

$\mathbb{R}_{\mathrm{CF}}$ is a quasitopological group because $(-x)+U$, $U+(-x)$, and $-U$ are finite whenever U is finite.

If $U, V \subset \mathbb{R}$ are non-empty and open in the cofinite topology, then $U+V = \mathbb{R}$: for $x \in \mathbb{R}$, $x - U$ is infinite and V^c is finite and so $x - U \not\subset V^c$, whence $x \in U + V$ and $U + V = \mathbb{R}$ as claimed. In particular, $\{(x, y) \in \mathbb{R}^2 : x + y \neq 0\}$, which is the preimage under addition of an open set in the cofinite topology, cannot contain a sum of non-empty open sets. It follows that multiplication is not continuous and $\mathbb{R}_{\mathrm{CF}}$ is *not* paratopological.

$\mathbb{R}_{\mathrm{CF}}$ is not Hausdorff: Any two non-empty open sets U and V have finite complements, but $\mathbb{R}$ is infinite and so U is infinite and $U \not\subset V^c$ which is to say that $U \cap V \neq \varnothing$.

$\mathbb{R}_{\mathrm{CF}}$ is compact: Indeed, *any* $A \subset \mathbb{R}$ is compact since if $\mathcal{U}$ is an open cover of A, then (either A is empty and we so compact or we may) let $U \in \mathcal{U}$ be non-empty. U^c is finite and since $\mathcal{U}$ is a cover of A, if $x \in U^c$ has $x \in A$ we may take $U_x \in \mathcal{U}$ such that $x \in U_x$. The set $\{U\} \cup \{U_x : x \in U^c\}$ is a finite subcover of $\mathcal{U}$ and our claim is proved.

Observation 1.33. Every left-topological group G with a continuous inverse is a quasitopological group, since for $y \in G$ the right multiplication map $G \to G; x \mapsto xy = (y^{-1}x^{-1})^{-1}$ is continuous since it is a composition of inversion, left multiplication by y^{-1}, and inversion again.

The following diagram summarises the foregoing. The implications without any text next to them follow *a fortiori* – *i.e.* by simply dropping hypotheses – and the missing implications and non-implications can all be deduced from transitivity of implication and the law of excluded middle.

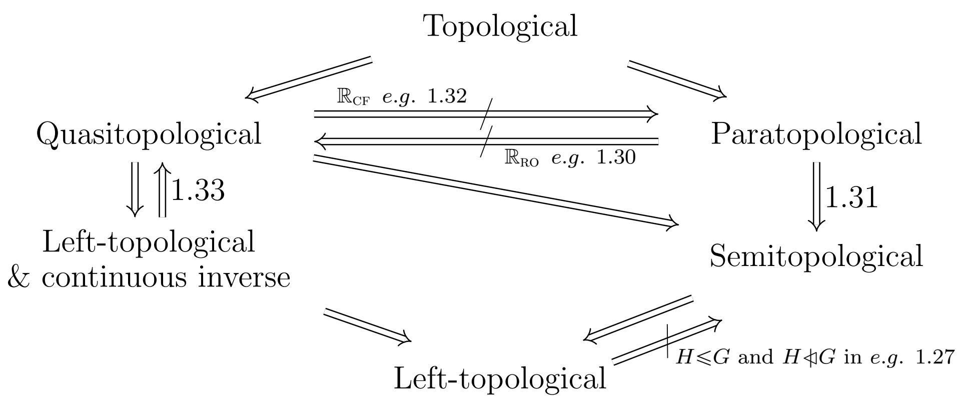


Figure 1: Relationships between types of topologized groups

2 Some basic tools

In this section there are a few key lemmas (Lemmas 2.2, 2.4, 2.13,2.16, 2.18, & 2.21) which we highlight in red because they each capture a crucial technique or idea.

We say $S \subset G$ is **symmetric** if $S = S^{-1}$.

Example 2.1. A group G in which every non-identity element has order 2 has *every* subset symmetric. Moreover, if it is topologized then inversion is just the identity map and so is automatically continuous.

Lemma 2.2 (Key Lemma I)**.** *Suppose that G is a topologized group in which inversion is continuous. If U is a neighbourhood of the identity then there is a symmetric open neighbourhood of the identity $W \subset U$; and if S is symmetric then $\overline{S}$ is symmetric.*

Proof. Since U is a neighbourhood of 1_G, there is an open set $V \subset U$ with $1_G \in V$. Put $W := V \cap V^{-1}$ which is open and contains 1_G, since $1_G^{-1} = 1_G$, and is symmetric. For the second part, since inversion is continuous, the preimage of $\overline{S}$ under inversion is closed, and so $\overline{S} = \overline{S^{-1}} \subset \overline{\overline{S}^{-1}} = \overline{S}^{-1}$. But then $\overline{S}^{-1} \subset (\overline{S}^{-1})^{-1} = \overline{S}$, and we get the result. $\square$

Remark 2.3. Every conclusion of Lemma 2.2 may fail if 'topologized group with continuous inverse' is replaced by 'paratopological group': In $\mathbb{R}_{\mathrm{RO}}$, the only symmetric and open sets are $\varnothing$ and $\mathbb{R}$, hence $(-1, \infty)$ is a neighbourhood of the identity that does not contain a symmetric neighbourhood of the identity; and $\overline{\{0\}} = (-\infty, 0]$ which is not symmetric despite $\{0\}$ being symmetric.

Lemma 2.4 (Key Lemma II)**.** *Suppose that G is a left-topological group, U is open, and V is any subset of G. Then VU is open; U is a neighbourhood of x if and only if $x^{-1}U$ is a neighbourhood of the identity; and $\overline{xV} = x\overline{V}$.*

Proof. First, $VU = \bigcup_{v \in V} vU$, which is a union of open sets since $G \to G; x \mapsto v^{-1}x$ is continuous. Secondly, if U is a neighbourhood of x then there is an open set $U_x \subset U$ containing x. Continuity of $G \to G; z \mapsto xz$ then says that $x^{-1}U_x$ is an open set containing 1_G and contained in $x^{-1}U$, which is to say $x^{-1}U$ is a neighbourhood of the identity. Similarly if $x^{-1}U$ is a neighbourhood of the identity then U is a neighbourhood of x by continuity of $G \to G; z \mapsto x^{-1}z$. Finally, since $G \to G; z \mapsto x^{-1}z$ is continuous, $x\overline{V}$ is closed and contains xV, hence $\overline{xV} \subset x\overline{V}$. Apply this with x replaced by x^{-1} and V replaced by xV to get $\overline{V} \subset x^{-1}\overline{xV}$, whence $\overline{xV} = x\overline{V}$. $\square$

Checking continuity and openness of homomorphisms

Recall a **neighbourhood base** of a point x in a topological space X is a set $\mathcal{B}$ of open neighbourhoods of x such that if N is a neighbourhood of x then there is $B \in \mathcal{B}$ such that $B \subset N$.

Proposition 2.5. *Suppose that G and H are left-topological groups and $\mathcal{B}$ is a neighbourhood base of the identity in H. Then a homomorphism $\theta : G \to H$ is continuous if (and only if) $\theta^{-1}(B)$ is a neighbourhood of the identity for all $B \in \mathcal{B}$; and a homomorphism $\theta : H \to G$ is open if (and only if) $\theta(B)$ is a neighbourhood of the identity for all $B \in \mathcal{B}$.*

Proof. Suppose that $U \subset H$ is open and $\theta(y) \in U$. By Lemma 2.4 there is an open neighbourhood of the identity V_y such that $\theta(y)V_y \subset U$, and hence $B \in \mathcal{B}$ such that $B \subset V_y$. Thus $\theta^{-1}(B) \subset \theta^{-1}(V_y)$ so $y\theta^{-1}(B) \subset \theta^{-1}(U)$ (using that θ is a homomorphism) and hence $\theta^{-1}(U)$ contains a neighbourhood of y *i.e.* $\theta^{-1}(U)$ is open. The parenthetical 'only if' follows since B contains an open neighbourhood of 1_H and $\theta(1_G) = 1_H$. The result for open maps follows similarly. $\square$

Corollary 2.6. *Suppose that G is a semitopological group and $\mathcal{B}$ is a neighbourhood base of the identity such that B^{-1} is a neighbourhood of the identity for all $B \in \mathcal{B}$. Then G is quasitopological.*

Proof. Since G is semitopological the map $G \to G^{\mathrm{op}}; x \mapsto x^{-1}$ is a homomorphism between left-topological groups, and so Proposition 2.5 gives the result. $\square$

Topological closures of subgroups

Extending the definition for subgroups, we say that a subset S of a group G is **normal** if $xS = Sx$ for all $x \in G$.

Corollary 2.7. *Suppose that G is a semitopological group. If $S \subset G$ is normal, then so is $\overline{S}$; if $S \subset G$ is closed under multiplication (*i.e. *$xy \in S$ whenever $x, y \in S$), then so is $\overline{S}$.*

Proof. First, by Lemma 2.4 we have $x\overline{S} = \overline{xS} = \overline{Sx} = \overline{S}x$ for all $x \in G$. Secondly, since G is left-topological by Lemma 2.4 $h\overline{S} = \overline{hS} = \overline{S}$ for all $h \in S$. Hence $Sw \subset \overline{S}$ for all $w \in \overline{S}$. Since G^{op} is left-topological by Lemma 2.4 we have $\overline{S}w = \overline{Sw} \subset \overline{\overline{S}} = \overline{S}$. Hence $\overline{S}^2 \subset \overline{S}$. □

Remark 2.8. We cannot relax 'semitopological' to 'left-topological': First, if G is a group with a non-normal subgroup H then G with the coset topology generated by H has $\overline{\{1_G\}} = H$ which is not normal. Secondly, if G is a group with subgroups H and K such that HK is not closed under multiplication (which, of course, entails that H is not normal). Then G with the coset topology generated by K is left-topological but has $\overline{H} = HK$, so that even though H is closed under multiplication, its topological closure is not.

Proposition 2.9. *Suppose that G is a quasitopological group and $H \leqslant G$. Then $\overline{H}$ is a subgroup of G which is normal if H is normal. In particular, $\overline{\{1_G\}}$ is a normal subgroup of G.*

Proof. By Corollary 2.7 $\overline{H}$ is closed under multiplication and normal if H is normal. By Lemma 2.2, $\overline{H}^{-1} = \overline{H}$. Since H is non-empty it follows that $\overline{H}$ is a group and the result is proved. □

Remark 2.10. We cannot replace 'quasitopological' by 'paratopological' above: $\{0\}$ is a subgroup of $\mathbb{R}_{\mathrm{RO}}$ but $\overline{\{0\}} = (-\infty, 0]$ which is not a subgroup.

If G is a group with a subset S that is not a subgroup, then $\overline{S} = G$ in G_{I} so there is no converse to the above saying if $\overline{H}$ is a subgroup then H is a subgroup; similarly if G has a subgroup H that is not normal, then $\overline{H} = G$ in G_{I} so there is no converse saying that if $\overline{H}$ is normal then H is normal.

In Exercise I.6 there is an example of a compact semitopological group that is not quasitopological, but despite the fact that Proposition 2.9 does not apply we do have the following.

Proposition 2.11. *Suppose that G is a compact semitopological group. Then $\overline{\{1_G\}}$ is a normal subgroup of G.*

Proof. Put $H := \overline{\{1_G\}}$ then by Corollary 2.7, $H^2 \subset H$ and H is normal. Now consider $\mathcal{H} := \{xH : x \in H\}$. This is a set of closed subsets of H by Lemma 2.4, which has the finite intersection property: suppose $x_1H, \dots, x_nH \in \mathcal{H}$. Then $x_iH \supset x_i \cdots x_nH = Hx_i \cdots x_n \supset x_1 \cdots x_{i-1}Hx_i \cdots x_n = x_1 \cdots x_nH$ since $x_1 \cdots x_{i-1}, x_{i+1} \cdots x_n \in H$ and H is closed under multiplication. Since G is compact, $V := \bigcap \mathcal{H}$ is non-empty.

V is closed and non-empty, so there is some $y \in V$. By Lemma 2.4 $yH = \overline{\{y\}} \subset V$, but then $y^2H \in \mathcal{H}$ and so $y^2H \supset V \supset yH$, and since G is a group, $yH \supset H$. Now $H \in \mathcal{H}$, and so $H \supset V \supset yH \supset H$ – in other words $V = H$. But then for all $x \in H$ we have $H \subset xH$, and since $1_G \in H$ we have some $y \in H$ such that $xy = 1_G$ and H is closed under inverses and hence a subgroup. □

Remark 2.12. We cannot relax 'semitopological' to 'left-topological': if G is a finite group with a non-normal subgroup H then G with the coset topology generated by H has $\overline{\{1_G\}} = H$ which is not normal, but it is compact since G is finite. Similarly, we cannot relax the compactness requirement to local compactness in view of the group $\mathbb{R}_{\mathrm{RO}}$ in which the closure of the identity is not even a group (see Remark 2.10).

Lemma 2.13 (Key Lemma III)**.** *Suppose that G is a left-topological group, S is a subset of G and V is an open neighbourhood of the identity. Then $\overline{SV} \subset SVV^{-1}$.*

Proof. Let $A := G\backslash(SVV^{-1})$ and $B := G\backslash(AV)$. B is closed since AV is open by Lemma 2.4. If $x \in AV$ then there is some $v \in V$ such that $xv^{-1} \in A$, so $xv^{-1} \notin SVV^{-1}$. Hence $SV \subset B$ and since B is closed $\overline{SV} \subset B$. Now if $x \in B$ then $x \notin A$ since $1_G \in V$, and hence $x \in SVV^{-1}$ as claimed. □

Corollary 2.14. *Suppose that G is a left-topological group and $H \leqslant G$. If H is a neighbourhood in G then H is open in G; if H is open in G then H is closed in G; if H is closed in G* and *of finite index then H is open in G.*

Proof. First, if H is a neighbourhood then there is a non-empty open set $U \subset H$. But then $H = HU$ is open by Lemma 2.4. For the second part, if H is open then by Lemma 2.13 $\overline{H} \subset HH^{-1} = H$ and so H is closed.

For the last part, since H is closed, every $W \in G/H$ is closed by Lemma 2.4. Since H is of finite index, $\bigcup\{W \in G/H : W \neq H\}$ is a finite union of closed sets and so closed. Finally, since G/H is a partition of G containing H, $H = G\backslash\bigcup\{W \in G/H : W \neq H\}$ is open as required. □

Remark 2.15. $\mathbb{Z}$ is a closed subgroup of the real line $\mathbb{R}$ that is not open, so the hypothesis that H have finite index above cannot be dropped.

Lemma 2.16 (Key Lemma IV)**.** *Suppose that G is a paratopological group and X is a neighbourhood of z. Then there is an open neighbourhood of the identity V such that $zV^2 \subset X$. Moreover, if G is a topological group then V may be taken to be symmetric.*

Proof. Let $U \subset X$ be an open neighbourhood of z. The map $(x, y) \mapsto xy$ is continuous and so $\{(x, y) : xy \in U\}$ is an open subset of $G \times G$. By definition of the product topology there is a set $\mathcal{S}$ of products of open sets in G such that

$$\{(x, y) : xy \in U\} = \bigcup\{S \times T : S \times T \in \mathcal{S}\}.$$

Since $z1_G = z \in U$, there is some $S \times T \in \mathcal{S}$ such that $(z, 1_G) \in S \times T$. Thus S is an open neighbourhood of z and T is an open neighbourhood of the identity, so by Lemma 2.4

$V := (z^{-1}S) \cap T$ is an open neighbourhood of the identity. Now $zV \subset S$ and $V \subset T$ and so $zV^2 \subset U$ as required. Moreover, if G is a topological group inversion is also continuous so by Lemma 2.2 V contains a symmetric open neighbourhood of the identity, and the conclusion follows by nesting. □

Remark 2.17. We cannot replace 'paratopological' by 'quasitopological' above: In $\mathbb{R}_{\text{CF}}$ if X is the complement of some $x \neq z$, then X is open but the sum of any two non-empty open sets is the whole of $\mathbb{R}$ and so cannot be contained in X.

Lemma 2.18 (Key Lemma V). *Suppose that G is a paratopological group and $K_1, \ldots, K_n$ are compact subsets of G. Then $K_1 \cdots K_n$ is compact. In particular, if K is compact then K^n is compact for all*[4] *$n \in \mathbb{N}_0$.*

Proof. The (topological) product of two compact sets is compact so if $K_1 \cdots K_{n-1}$ is compact and K_n is compact then $(K_1 \cdots K_{n-1}) \times K_n$ is compact. But then the continuous image of a compact set is compact and so $K_1 \cdots K_n = (K_1 \cdots K_{n-1})K_n$ is compact and the result follows by induction on n. □

Remark 2.19. Exercise I.5 gives an example of a quasitopological group where the conclusion above does not hold.

A cover $\mathcal{U}$ is a **refinement** of a cover $\mathcal{V}$ of a set X if $\mathcal{U}$ is a cover of X and each set in $\mathcal{U}$ is contained in some set in $\mathcal{V}$.

Observation 2.20. Refinements are transitive meaning that if $\mathcal{W}$ is a refinement of $\mathcal{V}$ and $\mathcal{V}$ is a refinement of $\mathcal{U}$ then $\mathcal{W}$ is a refinement of $\mathcal{U}$.

Lemma 2.21 (Key Lemma VI). *Suppose that G is a paratopological group, $K \subset G$ is compact, and $\mathcal{U}$ is an open cover of K. Then there is an open neighbourhood of the identity $U \subset G$ such that $\{xU : x \in K\}$ is a refinement of $\mathcal{U}$. If G is a topological group then U may be taken to be symmetric.*

Proof. By Lemma 2.16, for every $x \in K$ there is a open neighbourhood of the identity U_x such that xU_x^2 is subset of a set in $\mathcal{U}$. Since K is compact there are elements $x_1, \ldots, x_m \in K$ such that $\mathcal{W} := \{x_1U_{x_1}, \ldots, x_mU_{x_m}\}$ is an open cover of K; let $U := U_{x_1} \cap \cdots \cap U_{x_m}$ which is an open neighbourhood of the identity. Since $\mathcal{W}$ is a cover of K, for each $x \in K$ there is i such that $x \in x_iU_{x_i}$ and hence $xU \subset x_iU_{x_i}^2$. The first part is done and if, additionally, G is assumed to be topological then we can take U_xs to be symmetric and hence U to be symmetric. The result is proved. □

This has a higher dimensional analogue which is related to the tube lemma:

[4]Note that $K^0 = \{1_G\}$ by definition and so is compact since it is finite.

Lemma 2.22. *Suppose that X is a topological space, $K \subset X$ is compact, and $\mathcal{U}$ is an open cover of $K \times K$. Then there is an open cover $\mathcal{W}$ of K such that $\{W \times W' : W, W' \in \mathcal{W}\}$ refines $\mathcal{U}$.*

Proof. For $(x, y) \in K \times K$ there is some $U \in \mathcal{U}$ with $(x, y) \in U$. By definition of the product topology there are U_x and V_y open neighbourhoods of x and y respectively such that $U_x \times V_y \subset U$. Since the topological product of of two compact sets is compact, the cover $\{U_x \times V_y : (x, y) \in K \times K\}$ of $K \times K$ has a finite subcover $\{U_{x_1} \times V_{y_1}, \ldots, U_{x_m} \times V_{y_m}\}$. For every $z \in K$ let

$$W_z := \bigcap\{U_{x_i} : z \in U_{x_i}\} \cap \bigcap\{V_{y_i} : z \in V_{y_i}\}.$$

Write $\mathcal{W}$ for the set of W_zs, which is a (finite) open cover of K such that $\{W \times W' : W, W' \in \mathcal{W}\}$ refines $\mathcal{U}$. □

Where are the compact paratopological groups?

The quasitopological group $\mathbb{R}_{\mathrm{CF}}$ is compact and not a topological group, and Exercise I.6 gives an example of a compact semitopological group that is not quasitopological, but we have not seen an example of a compact paratopological group that is not topological and here is why:

Theorem 2.23. *Suppose that G is a compact paratopological group. Then G is a topological group.*

Proof. Suppose that $K \subset G$ is closed and $x \notin K^{-1}$. For $y \in K$, if $yx \in \overline{\{1_G\}}$ then by Proposition 2.11 $x^{-1}y^{-1} \in \overline{\{1_G\}}$ and so by Lemma 2.4, $x^{-1} \in \overline{\{1_G\}}y = \overline{\{y\}} \subset \overline{K} = K$, a contradiction. Hence $yx \notin \overline{\{1_G\}}$ and again, by Lemma 2.4 there is an open neighbourhood U_y of y such that $U_y x \cap \overline{\{1_G\}} = \varnothing$ and in particular $1_G \notin U_y x$.

Apply Lemma 2.21 to the cover $\{U_y : y \in K\}$ of K to get an open neighbourhood of the identity U such that for all $y \in K$ we have $yU \subset U_z$ for some $z = z(y) \in K$. It follows that $1_G \notin yUx$ for all $y \in K$, so $K^{-1} \cap Ux = \varnothing$. Thus K^{-1} is closed and the result is proved. □

Remark 2.24. We cannot relax 'compact' to 'locally compact' since $\mathbb{R}_{\mathrm{RO}}$ is a locally compact paratopological group that is not a topological group. ⚠In [Rav15] it states that every locally compact paratopological group is a topological group. This does not contradict the above, it is simply using a different definition of local compactness in which every element is contained in a *closed* compact neighbourhood.

Quotient topologies and topological quotient groups

For G a topologized group and $H \leqslant G$, the **quotient topology** on left cosets G/H has $U \subset G/H$ open if and only if $\bigcup U$ is open in G; or, equivalently, $C \subset G/H$ closed if and only if $\bigcup C$ is closed in G.

This topology is the final topology on G/H w.r.t. the quotient map $q : G \to G/H; x \mapsto xH$ – it is the strongest topology (meaning finest topology, or topology with the most open sets) on G/H making q continuous.

Proposition 2.25. *Suppose that G is a topologized group and H is a normal subgroup of G. Then*

(i) if group inversion on G is continuous, then it is continuous on G/H;

(ii) if left (resp. right) multiplication is continuous on G, then it is continuous on G/H;

(iii) and if multiplication is continuous on $G \times G$ then it is also on $(G/H) \times (G/H)$.

In particular, if G is a topological (resp. paratopological, quasitopological, semitopological, or left-topological) group then so is G/H.

Proof. Suppose that $U \subset G/H$ is open. If inversion is continuous on G then

$$\begin{aligned}\bigcup U^{-1} = \bigcup \{(xH)^{-1} : xH \in U\} &= \bigcup \{z^{-1} : z \in xH \in U\} \\ &= \bigcup \{z^{-1} : zH \in U\} = \left\{z^{-1} : z \in \bigcup U\right\} = \left(\bigcup U\right)^{-1}\end{aligned}$$

and so U^{-1} is open in G/H by definition since $\bigcup U$ is open in G. If left multiplication on G is continuous, then for $x \in G$,

$$\bigcup (xH)^{-1}U = \bigcup \{(x^{-1}H)(yH) : yH \in U\} = \bigcup \{x^{-1}yH : yH \in U\} = x^{-1}\bigcup U,$$

and so $(xH)^{-1}U$ is open in G/H and hence left multiplication by xH is continuous. If right multiplication on G is continuous, then for $x \in G$,

$$\bigcup U(xH)^{-1} = \bigcup \{(yH)(xH)^{-1} : yH \in U\} = \bigcup \{yHx^{-1} : yH \in U\} = \left(\bigcup U\right) x^{-1},$$

and so $U(xH)^{-1}$ is open in G/H and hence right multiplication by xH is continuous.

Finally suppose multiplication $G \times G \to G$ is continuous. Define

$$W := \{(zH, wH) \in (G/H) \times (G/H) : (zH)(wH) \in U\},$$
$$\text{and } V := \left\{(z, w) \in G \times G : zw \in \bigcup U\right\}.$$

Suppose that $(xH, yH) \in W$. Then $xy \in (xH)(yH) \subset \bigcup U$ so $(x, y) \in V$ and since V is open there are open sets $S, T \subset G$ such that $x \in S$, $y \in T$, and $S \times T \subset V$. If $s \in S$ and $t \in T$, then $st \in \bigcup U$, and since the latter is a union of cosets of H we have $(st)H \subset \bigcup U$. Since H is normal we have $(sH)(tH) = (st)H \subset \bigcup U$, and so $SH \times TH \subset V$.

By Lemma 2.4, SH and TH are open sets, and so the sets $S' := \{sH : s \in S\}$ and $T' := \{tH : t \in T\}$ are open in G/H; $xH \in S'$ and $yH \in T'$; and $S' \times T' \subset W$. It follows that W is open, and multiplication on $(G/H) \times (G/H) \to G/H$ is continuous. ∎

Example 2.26 (The real line modulo 1)**.** The real line $\mathbb{R}$ (Example 1.3) has a (normal) subgroup $\mathbb{Z}$ and so the group $\mathbb{R}/\mathbb{Z}$ may be given the quotient topology making it into a topological group by Proposition 2.25.

⚠In the literature on topological spaces (though not in these notes) the notation $\mathbb{R}/\mathbb{Z}$ is sometimes used to refer to a different space, called the adjunction space in which all the integers in $\mathbb{R}$ are identified but the rest of $\mathbb{R}$ remains the same. In other language this is a countably infinite bouquet of circles all connected at the point $\mathbb{Z}$.

Example 2.27 (The reals with the circle topology)**.** By Proposition 1.7 $\mathbb{R}$ is a topological group (which we shall denote $\mathbb{R}_C$) when endowed with the initial topology w.r.t. the quotient map $q : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ where $\mathbb{R}/\mathbb{Z}$ is the reals $\pmod 1$ (Example 2.26). We call this the **circle topology** on $\mathbb{R}$. The open sets in the circle topology have the form $U + \mathbb{Z}$ where $U \subset \mathbb{R}$ is open in the real line.

Since $\mathbb{R}_C$ has the initial topology, a set $A \subset \mathbb{R}_C$ is compact if (and only if) $q(A)$ is compact in $\mathbb{R}/\mathbb{Z}$: Indeed, if $\mathcal{U}$ is an open cover of A, we can write $\mathcal{U} = \{q^{-1}(V) : V \in \mathcal{V}\}$ for some set $\mathcal{V}$ of open subsets of $\mathbb{R}/\mathbb{Z}$. Now, if $q(A)$ is compact then there are $V_1, \ldots, V_n \in \mathcal{V}$ such that $q(A) \subset V_1 \cup \cdots \cup V_n$, and hence $A \subset q^{-1}(q(A)) \subset q^{-1}(V_1) \cup \cdots \cup q^{-1}(V_n)$ and so $\{q^{-1}(V_1), \ldots, q^{-1}(V_n)\}$ is a finite subcover of $\mathcal{U}$.

⚠In particular, $A := [0, 1]$ and $B := [0, 1/2) \cup [3/2, 2]$ are compact, but $A \cap B = [0, 1/2)$ is not compact. This phenomenon of the intersection of two compact sets not being compact cannot happen in a Hausdorff space where every compact set is closed, and hence where the intersection of two compact sets is an intersection of a closed set with a compact set which is, therefore, compact.

Regularity in topological groups

A topological space X is said to be **regular** if for all $x \in X$ every neighbourhood of x contains a closed neighbourhood of x. ⚠The literature is inconsistent on the meaning of regular, and for some other authors a regular topological space is required to be Hausdorff.

Remark 2.28. One reason for considering regular topologies without the requirement that they also be Hausdorff is that if X is a regular topological space and $f : Y \to X$ is a function then Y with the initial topology w.r.t. f is also regular. Of course Y need not be Hausdorff even if X is Hausdorff.

Proposition 2.29. *Suppose that G is a topological group. Then G is regular.*

Proof. Let V be a neighbourhood of $x \in G$. By Lemma 2.16 there is a symmetric open neighbourhood of the identity U such that $xU^2 \subset V$, and so by Lemmas 2.4 & 2.13, $x\overline{U} \subset xUU^{-1} = xU^2 \subset V$ as required. □

Remark 2.30. The quasitopological group $\mathbb{R}_{\text{CF}}$ is not regular because the only closed neighbourhood is the whole of $\mathbb{R}$ which cannot be contained in any neighbourhood that is not the whole of $\mathbb{R}$; and the paratopological group $\mathbb{R}_{\text{RO}}$ is not regular for the same reasons.

Theorem 2.23 shows that the topological condition of compactness forces a paratopological group to be topological, so one might wonder if regularity has the same effect; it does not as Exercise I.9 shows.

There are also purely topological conditions that give rise to regularity:

Proposition 2.31. *Suppose that X is a locally compact Hausdorff topological space. Then X is regular.*

Proof. Let V be an open neighbourhood of $x \in X$, which by local compactness we may assume is contained in a compact neighbourhood U. For all $x \neq y \in X$ there is an open set U_y containing y which is disjoint from an open set V_y containing x. $\{U_y : y \in U\backslash V\}$ is an open cover of a closed subset of the compact set U and so has a finite subcover, say $U_{y_1}, \dots, U_{y_m}$. But U is a compact subset of a Hausdorff topological space, so $(U\backslash U_{y_1}) \cap \cdots \cap (U\backslash U_{y_m})$ is closed, contained in V, and contains $V_{y_1} \cap \cdots \cap V_{y_m}$ which is an open set containing x. $\square$

In a Hausdorff topological space a compact set is closed; in a regular topological space we need not have this but we do preserve compactness:

Lemma 2.32. *Suppose that X is a regular topological space and K is compact. Then $\overline{K}$ is compact.*

Proof. Suppose that $\mathcal{U}$ is an open cover of $\overline{K}$. For each $x \in K$ there is $U_x \in \mathcal{U}$ with $x \in U_x$ and by regularity there is an open neighbourhood V_x of x with $\overline{V_x} \subset U_x$. Then $\{V_x : x \in K\}$ is an open cover of K and so has a finite subcover $V_{x_1}, \dots, V_{x_k}$, but then $\overline{K} \subset \overline{V_{x_1}} \cup \cdots \cup \overline{V_{x_k}} \subset U_{x_1} \cup \cdots \cup U_{x_k}$ and so $\mathcal{U}$ has a finite subcover of $\overline{K}$ as required. $\square$

Regularity can be extended from points to compact sets:

Lemma 2.33. *Suppose that X is a regular topological space, and $K \subset B$ with K compact and B open. Then there is an open set C with $\overline{K} \subset C \subset \overline{C} \subset B$.*

Proof. Since B is open, for each $x \in K$ there is an open set U_x containing x and contained in B; and since X is regular, there are open neighbourhoods V_x and W_x of x with $\overline{W_x} \subset V_x$, and $\overline{V_x} \subset U_x$. K is compact and so $K \subset W_{x_1} \cup \cdots \cup W_{x_m}$ for some $x_1, \dots, x_m \in K$. Put $C := V_{x_1} \cup \cdots \cup V_{x_m}$ to get $\overline{K} \subset \overline{W_{x_1}} \cup \cdots \cup \overline{W_{x_k}} \subset C$, and $\overline{C} \subset U_{x_1} \cup \cdots \cup U_{x_m} \subset B$. $\square$

The open mapping theorem

Any bijective group homomorphism is a group isomorphism, but Example 1.6 shows that there are continuous bijective group homomorphisms of topological groups that are not homeomorphic isomorphisms. On the other hand any bijective continuous map from a compact space to a Hausdorff space is a homeomorphism, and the group structure can help to strengthen this:

Theorem 2.34 (Open Mapping Theorem)**.** *Suppose that G is a left-topological group that is a countable union of compact sets, H is a locally compact Hausdorff left-topological group, and $\pi : G \to H$ is a continuous bijective homomorphism. Then π is a homeomorphic isomorphism.*

Proof. Since the inverse of a bijective group homomorphism is a group isomorphism, it suffices to show that $\pi(C)$ is closed whenever C is closed in G. Let K_n be compact in G such that $G = \bigcup_{n\in\mathbb{N}^*} K_n$.

Claim. *There is some $n \in \mathbb{N}^*$ such that $\pi(K_n)$ is a neighbourhood.*

Proof. We use a Baire Category argument, though no familiarity with these is assumed. We construct a nested sequence of closed neighbourhoods inductively: Let U_0 be a compact (and so closed since H is Hausdorff) neighbourhood in H, and for $n \in \mathbb{N}^*$ let $U_n \subset \pi(K_n)^c \cap U_{n-1}$ be a closed neighbourhood.

This is possible since (by the inductive hypothesis) U_{n-1} is a neighbourhood and so contains an open neighbourhood V_{n-1}. But then $\pi(K_n)^c \cap V_{n-1}$ is open, since $\pi(K_n)$ is a continuous image of a compact set and so compact, and therefore closed since H is Hausdorff; and non-empty since otherwise $\pi(K_n)$ contains a neighbourhood. It follows that $\pi(K_n)^c \cap U_{n-1}$ contains an open neighbourhood and so it contains a closed neighbourhood since H is regular by Proposition 2.31.

Now by the finite intersection property of the compact space U_0, the set $\bigcap_n U_n$ is non-empty. This contradicts surjectivity of π since $G = \bigcup_{n\in\mathbb{N}^*} K_n$ and the claim is proved. □

Claim. *If $X \subset H$ is compact then $\pi^{-1}(X)$ is compact.*

Proof. By the previous claim $\pi(K_n)$ contains a neighbourhood (and hence so does $x\pi(K_n)$ by Lemma 2.4) and the set $\{x\pi(K_n) : x \in H\}$ covers X, so by compactness of X there are elements $x_1, \dots, x_m$ such that $X \subset \bigcup_{i=1}^m x_i\pi(K_n)$ and hence $\pi^{-1}(X) \subset \bigcup_{i=1}^m \pi^{-1}(x_i)K_n$ (by injectivity of π). $\pi^{-1}(x_i)K_n$ is compact by Lemma 2.4, and since a finite union of compact sets is compact it follows that $\pi^{-1}(X)$ is contained in a compact set. Finally, X is closed so $\pi^{-1}(X)$ is closed and a closed subset of a compact set is compact as required. □

Finally, suppose that $C \subset G$ is closed, and y is a limit point of $\pi(C)$. H is locally compact so y has a compact neighbourhood X. Now $\pi^{-1}(X)$ is compact and so $\pi^{-1}(X) \cap C$ is compact. But then $X \cap \pi(C)$ is compact since π is continuous, and hence closed since H is Hausdorff. But by design $y \in \overline{X \cap \pi(C)} = X \cap \pi(C) \subset \pi(C)$. □

Corollary 2.35. *Suppose that G is a countable locally compact Hausdorff left-topological group. Then G is discrete. In particular, if G is a compact Hausdorff topological group then G is either finite or uncountable.*

Proof. Since G is countable and finite sets are compact, G_{D} is a topological group that is a countable union of compact sets, and the identity map $G_{\mathrm{D}} \to G$ is a continuous bijective homomorphism. Hence by the Open Mapping Theorem this is a homeomorphism and so G is discrete. Finally, if G is compact and countable then it is compact and discrete and so finite. □

Remark 2.36. None of the hypotheses may be dropped: The real line is an example of an uncountable locally compact Hausdorff topological group that is not discrete (since singletons are not open); the rationals with the subspace topology from the real line are an example of a countable Hausdorff topological group that is not discrete; the rationals with the indiscrete topology, are an example of a countable (locally) compact topological group that is not discrete; and finally, the topological space $\{1/n : n \in \mathbb{N}^*\} \cup \{0\}$ with its subspace topology in $\mathbb{R}$ is a countable compact Hausdorff space (which may be given the group structure of any countably infinite group to make it into a topologized group).

3 Continuous complex-valued functions on topological groups

For a topological space X we write $C(X)$ for the set of continuous functions $X \to \mathbb{C}$. This is closed under pointwise addition and multiplication of functions and contains the constant functions, so it is a $\mathbb{C}$-algebra.

Example 3.1. The set $C(\mathbb{R})$ contains the inclusion $\mathbb{R} \to \mathbb{C}; x \mapsto x$, and since it is a $\mathbb{C}$-algebra it contains all polynomials with complex coefficients.

Example 3.2. For any discrete space X, the space $C(X)$ contains *all* functions $X \to \mathbb{C}$.

Example 3.3. For any indiscrete space X, the space $C(X)$ contains *only* the constant functions.

Example 3.4. For the rationals with the subspace topology inherited from the real line, the function $g : \mathbb{Q} \to \mathbb{C}$ with $g(x) = 0$ if $x^2 < 2$ and $g(x) = 1$ if $x^2 > 2$ is continuous because the preimage of of any subset of $\mathbb{C}$ is either $\varnothing$, $\mathbb{R}$, $(-\infty, \sqrt{2}) \cap \mathbb{Q}$ or $(\sqrt{2}, \infty) \cap \mathbb{Q}$, depending on which elements of $\{0, 1\}$ it contains, and these are all opens sets.

The **support** of a (not necessarily continuous) function $f : X \to \mathbb{C}$ is denoted $\operatorname{supp} f$ and is defined to be the set of $x \in X$ such that $f(x) \neq 0$; f is said to be **compactly supported**[5] if its support is contained in a compact set. We write $C_c(X)$ for the subset of functions in $C(X)$ that are compactly supported.

The set $C_c(X)$ is a subalgebra of $C(X)$ since the union of two compact sets is compact and the support of the sum of two functions is contained in the union of their supports; and the support of the product of two functions is the intersection of their supports which is certainly contained in a compact set if one is. More than this, the function

$$\|f\|_\infty := \sup \{|f(x)| : x \in X\}$$

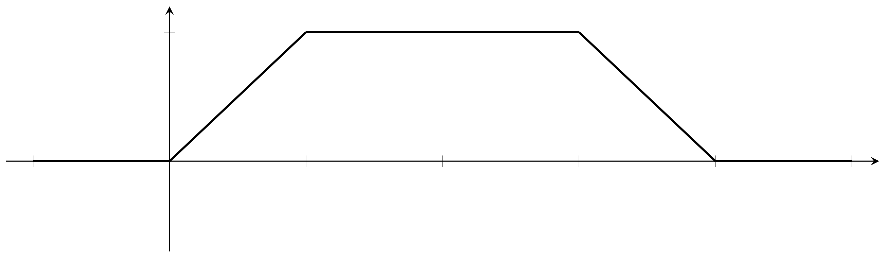
is a norm on $C_c(X)$ called the **uniform norm**. It is well-defined since every continuous (complex-valued) function on a compact set is bounded, and the axioms of a norm are easily checked.

Remark 3.5. ⚠In general $\|\cdot\|_\infty$ is *not* a norm on $C(X)$ since we are not assuming the elements of $C(X)$ are bounded.

⚠In general $C_c(X)$ is *not* complete despite the fact that the uniform limit of continuous functions is continuous since this limit function may not be compactly supported.

[5] ⚠As we have defined it the support of a function that is compactly supported need not actually *be* a compact set; it is simply contained in one.

Example 3.6. The set $C(\mathbb{R})$ contains all polynomials (as we saw in Example 3.1), and in fact all power series of infinite radius of convergence. However, by the Identity Theorem the only one of these functions that is in $C_c(\mathbb{R})$ is the zero function. The sort of function in $C(\mathbb{R})$ that we often have in mind might look like:



Proposition 3.7. *Suppose that G is a left-topological group and $C_c(G)$ contains a function that is not identically zero. Then G is locally compact.*

Proof. Suppose that $f \in C_c(G)$ is not identically zero. Then $\operatorname{supp} f$ is open (since f is continuous), non-empty and contained in a compact set K (since f is compactly supported). It follows that K is a compact neighbourhood of some point $x \in G$, and by Lemma 2.4 $yx^{-1}K$ is then a compact neighbourhood of y for $y \in G$ as required. □

Example 3.8. Since $\mathbb{Q}$ with the subspace topology inherited from the real line is not locally compact we have $C_c(\mathbb{Q}) = \{0\}$.

The regular representation

Given a group G and a function $f : G \to \mathbb{C}$ we write

$$\lambda_x(f)(z) := f(x^{-1}z) \text{ for all } x, z \in G.$$

Proposition 3.9. *Suppose that G is a left-topological group. Then the map*

$$G \to \operatorname{Iso}(C_c(G)); x \mapsto \lambda_x$$

is a well-defined homomorphism. Moreover, if G is a topological group then this map is continuous.

Proof. First recall from Example 1.17 that $\operatorname{Iso}(C_c(G))$ is the set of linear invertible isometries of $C_c(G)$. Since G is left-topological, for $f \in C_c(G)$ the map $z \mapsto \lambda_x(f)(z)$ is continuous. If f has support contained in a compact set K then $\lambda_x(f)$ has support contained in xK, which is itself compact since it is the continuous image of a compact set. Hence $\lambda_x(f) \in C_c(G)$. λ_x is visibly linear and $\|\lambda_x(f)\|_\infty = \|f\|_\infty$ for all $f \in C_c(G)$ and $x \in G$, so by linearity is an isometry. λ_x has $\lambda_{x^{-1}}$ as an inverse and hence the given map maps into $\operatorname{Iso}(C_c(G))$, and so is well-defined. As usual (*c.f.* Cayley's Theorem) we have $\lambda_{xy}(f) = \lambda_x(\lambda_y(f))$ from associativity of the group operation, and the first part is proved.

Now suppose that G is a topological group. The set

$$\{\{\phi \in \mathrm{Iso}(C_c(G)) : \|\phi(f) - f\|_\infty < \epsilon \text{ for all } f \in \mathcal{F}\} \text{ for } \epsilon > 0 \text{ and } \mathcal{F} \subset C_c(G) \text{ finite}\}$$

is a neighbourhood base of the identity in $\mathrm{Iso}(C_c(G))$. Hence by Proposition 2.5 it is enough to show that for $f \in C_c(G)$ the set $\{x \in G : \|\lambda_x(f) - f\|_\infty < \epsilon\}$ contains a neighbourhood of the identity in G.

Let K be a compact set containing the support of f, and let $\mathcal{U}$ be an open cover of G such that $|f(y) - f(y')| < \epsilon$ for all $y, y' \in U \in \mathcal{U}$ (see Observation 3.10 below). By Lemma 2.21 (applied to G^{op}) there is a symmetric open neighbourhood of the identity V such that $\{Vy : y \in K\}$ is a refinement of $\mathcal{U}$ (as a cover of K).

Suppose that $x \in V$ and $y \in G$ is such that $\lambda_x(f)(y) - f(y) \neq 0$. Then either $f(y) \neq 0$ so $y \in K$, but then $V^{-1}y = Vy$ is a subset of an element of $\mathcal{U}$ and so $|\lambda_x(f)(y) - f(y)| < \epsilon$; or $\lambda_x(f)(y) \neq 0$ so $x^{-1}y \in K$, but then $V(x^{-1}y)$ is a subset of an element of $\mathcal{U}$ and so again $|\lambda_x(f)(y) - f(y)| < \epsilon$. Since $\lambda_x(f) - f$ is continuous and compactly supported it attains its bounds so $\|\lambda_x(f) - f\|_\infty < \epsilon$. The result is proved. □

Observation 3.10. For $\Delta := \{z \in \mathbb{C} : |z| < \epsilon/2\}$ and $f \in C(X)$ if $f(x), f(y) \in z + \Delta$ then $|f(x) - f(y)| < \epsilon$ and hence $\mathcal{U} := \{f^{-1}(z + \Delta) : z \in \mathbb{C}\}$ is an open cover of X such that

$$|f(x) - f(y)| < \epsilon \text{ whenever } x, y \in U \in \mathcal{U}.$$

Constructing continuous compactly supported functions

The **interior** of a subset S of a topological space is denoted S°, and is the set of $x \in S$ that are contained in an open set which is itself contained in S. In particular, S° is open.

The **dyadic rationals** in $[0, 1]$ are the set $D := \bigcup_{i=0}^{\infty} D_i$, where

$$D_0 := \{\tfrac{0}{1}, \tfrac{1}{1}\}, D_1 := \{\tfrac{0}{2}, \tfrac{1}{2}, \tfrac{2}{2}\}, D_2 := \{\tfrac{0}{4}, \tfrac{1}{4}, \tfrac{2}{4}, \tfrac{3}{4}, \tfrac{4}{4}\}, \&c.$$

In particular D is dense in $[0, 1]$; we have the nesting $D_0 \subset D_1 \subset \dots$; and every element of $D_{i+1} \backslash D_i$ can be written in the form $\frac{1}{2}(q + q')$ where q and q' are *consecutive* elements of D_i.

Lemma 3.11. *Suppose that G is a paratopological group, V is an open neighbourhood of the identity, and $\overline{K}V \subset (\overline{B})^\circ$ for some set K. Then there are sets $(U_q)_{q \in D}$ with $U_0 = K$, $U_1 = B$, and $\overline{U_q} \subset (\overline{U_{q'}})^\circ$ whenever $q, q' \in D$ have $q < q'$.*

Proof. Set $V_0 := V$ and for $i \in \mathbb{N}_0$ define V_i inductively by Lemma 2.16 to be an open neighbourhood of the identity such that $V_{i+1}^2 \subset V_i$.

We set $U_0 := K$ and $U_1 := B$ and define U_q for $q \in D_{i+1} \backslash D_i$ iteratively for $i \in \mathbb{N}_0$. Suppose that at step i, for all consecutive pairs $q < q'$ in D_i we have $\overline{U_q} V_i \subset \overline{U_{q'}}$ – this is certainly true for $i = 0$. For $q < q'$ consecutive elements of D_i set $U_{\frac{1}{2}(q+q')} := \overline{U_q} V_{i+1}$ so that a) $\overline{U_q} V_{i+1} \subset \overline{U_{\frac{1}{2}(q+q')}}$; and b) $\overline{U_{\frac{1}{2}(q+q')}} V_{i+1} = \overline{\overline{U_q} V_{i+1}} V_{i+1} \subset \overline{\overline{U_q} V_{i+1}^2} \subset \overline{\overline{U_q} V_i} \subset \overline{\overline{U_{q'}}} = \overline{U_{q'}}$, where the first inclusion is by Lemma 2.4. Every element of $D_{i+1} \backslash D_i$ is the average of two consecutive elements of D_i, and the construction is complete.

It remains to note that if $\overline{U_q} V_i \subset \overline{U_{q'}}$ then $\overline{U_q} \subset (\overline{U_{q'}})^\circ$, and the result is proved. □

Nested sets of the type described above can be used to define continuous functions:

Lemma 3.12. *Suppose that X is a topological space, and $(U_q)_{q \in D}$ are such that $\overline{U_q} \subset (\overline{U_{q'}})^\circ$ whenever $q < q'$. Then there is $g \in C(X)$ with $g(x) \in [0, 1]$ for all $x \in X$; $g(x) = 0$ for $x \in \overline{U_0}$; and $g(x) = 1$ for $x \notin (\overline{U_1})^\circ$.*

Proof. For $x \in G$ let $S(x) := \{q \in D : x \in (\overline{U_q})^\circ\}$ and $g(x) := \inf S(x) \cup \{1\}$. Certainly $g(x) \in [0, 1]$. If $x \in \overline{U_0}$ then $q \in S(x)$ for all $q > 0$ by nesting, and hence $g(x) = 0$; if $x \notin (\overline{U_1})^\circ$, then $S(x) = \varnothing$ by nesting, and so $g(x) = 1$. It remains to show $g \in C(X)$.

First, for $\alpha \leqslant 1$ we have $g^{-1}([0, \alpha)) = \bigcup\{(\overline{U_q})^\circ : q < \alpha\}$ is open. Secondly, for $\alpha \geqslant 0$ suppose that $x_0 \in g^{-1}((\alpha, 1])$. Then $g(x_0) > \alpha$ and so there is $q' \in D$ with $q' > \alpha$ such that $x_0 \notin (\overline{U_{q'}})^\circ$ and hence by nesting $x_0 \notin \overline{U_q}$ for any $q \in D$ with $q < q'$; pick $q \in D$ with $\alpha < q < q'$. If $z \notin \overline{U_q}$ then again by nesting $g(z) \geqslant q > \alpha$, and hence $x_0 \in G \backslash \overline{U_q} \subset g^{-1}((\alpha, 1])$. Thus every element of $g^{-1}((\alpha, 1])$ is contained in an open subset of $g^{-1}((\alpha, 1])$, and so $g^{-1}((\alpha, 1])$ itself must be open.

We conclude that $g^{-1}((a, b)) = g^{-1}((a, 1]) \cap g^{-1}([0, b))$ is open for any $a, b \in \mathbb{R}$. The intervals without endpoints in $\mathbb{R}$ form a base for the topology on $\mathbb{R}$, and hence g is continuous as a function into $\mathbb{R}$. Finally, $\mathbb{R}$ is a subspace of $\mathbb{C}$, so $g \in C(X)$ as required. □

Theorem 3.13. *Suppose that G is a regular paratopological group, and $K \subset B$ are compact and open sets respectively. Then there is a continuous function $g \in C(G)$ with $g(x) \in [0, 1]$ for all $x \in G$; $g(x) = 0$ for all $x \in K$; and $g(x) = 1$ for all $x \notin B$.*

Proof. Since the topology is regular and K is compact, by Lemma 2.33, there is an open set C with $\overline{K} \subset C \subset \overline{C} \subset B$. By Lemma 2.32, $\overline{K}$ is compact and so by Lemma 2.21 applied to $\overline{K}$ and the open cover $\{C\}$ there is an open neighbourhood of the identity V such that $\overline{K} V \subset C \subset (\overline{C})^\circ$. Hence by Lemma 3.11 (applied to K and C), and then by Lemma 3.12, we get $g \in C(G)$ with $g(x) \in [0, 1]$ for all $x \in G$; $g(x) = 0$ for all $x \in \overline{K}$; and $g(x) = 1$ for all $x \notin (\overline{C})^\circ$. The result follows since $(\overline{C})^\circ \subset \overline{C} \subset B$. □

Remark 3.14. We know from Proposition 2.29 that every topological group is regular, hence every topological group is a regular paratopological group so in particular the above corollary applies to all topological groups. This consequence is sometimes called the 'complete regularity of topological groups'.

To explain the terminology 'complete regularity', note that if we suppose that G is a topologized group such that the conclusion of Theorem 3.13 holds for all $K \subset B$ with K compact and B open, then G is regular. Indeed, if U is an open neighbourhood of x, then the supposition applied to $K = \{x\}$, which is compact, and $B = U$, gives $g \in C(G)$ with $g(x) = 1$ and $g(y) = 0$ for all $y \notin U$, and so $g^{-1}(\{z \in \mathbb{C} : |z-1| \leqslant 1/2\})$ is a closed neighbourhood of x contained in U.

Surprisingly Theorem 3.13 was only shown relatively recently by Banakh and Ravsky in [Ban17], though the complete regularity of topological groups was known much earlier.

Corollary 3.15. *Suppose that G is a locally compact topological group, and $K \subset B$ are a compact and open set respectively. Then there is $f \in C_c(G)$ with $f(x) \in [0,1]$ for all $x \in G$; $f(x) = 1$ for all $x \in K$; and $f(x) = 0$ for all $x \notin B$.*

In particular, if B is a non-empty open set then there is $f \in C_c(G)$ with $f(x) \geqslant 0$ for all $x \in G$; f is not identically zero; and f has $\operatorname{supp} f \subset B$.

Proof. Since G is locally compact it contains a compact neighbourhood of the identity L; let $H \subset L$ be an open neighbourhood of the identity. KH is open by Lemma 2.4, and so $KH \cap B$ is an open set containing K.

Apply Theorem 3.13 (and Proposition 2.29) to get $g \in C(G)$ with $g(x) \in [0,1]$ for all $x \in G$; $g(x) = 0$ for all $x \in K$; and $g(x) = 1$ for all $x \notin (KH) \cap B$. Let $f := 1 - g$ and note that $\operatorname{supp} f \subset KH \cap B$, which is a subset of KL, which is compact by Lemma 2.18, and also of B. This gives the result. □

Remark 3.16. A topologized group G that is not indiscrete, has a non-empty proper open subset, and so if G satisfies the conclusions of Corollary 3.15, then $C_c(G)$ contains a non-constant function. In particular, by Proposition 3.7 we see that we cannot drop the 'locally compact' hypothesis above.

Exercise II.9 asks for examples of locally compact quasitopological and paratopological groups that are not topological groups, and where there are no non-constant continuous functions into $\mathbb{C}$. Since these are not topological groups, their topologies are not indiscrete, and so it follows that we cannot relax 'topological' to either 'quasitopological' or 'paratopological' above.

Corollary 3.15 can also be bootstrapped to produce continuous partitions of unity for which we first need a technical lemma:

Lemma 3.17. *Suppose that X is a topological space, $K \subset X$ is compact, and $f \in C_c(X), g \in C(X)$ are such that* $\operatorname{supp} f \subset K \subset \operatorname{supp} g$. *Then there is $h \in C_c(X)$ such that $f = gh$.*

Proof. Since K is compact and g is continuous, $|g|$ is continuous it achieves its minimum c on K. Since the support of g contains K we have $c > 0$. The function $\Phi : \mathbb{C} \to \mathbb{C}$ with

$\Phi(z) = \overline{z}$ for $|z| \leqslant 1$ and $\Phi(z) = 1/z$ for $|z| \geqslant 1$ is continuous, and so $h := \frac{f}{c}\Phi\left(\frac{g}{c}\right)$ is continuous because the product and composition of continuous functions is continuous; has compact support since f has compact support; and $f = gh$. The lemma is proved. $\square$

Corollary 3.18. *Suppose that G is a locally compact topological group, K is compact, $\mathcal{U}$ is an open cover of K, and $F \in C(G)$ has $\operatorname{supp} F \subset K$ and $F(x) \in [0,1]$ for all $x \in G$. Then there is some $n \in \mathbb{N}^*$, $U_1, \dots, U_n \in \mathcal{U}$, and $f_1, \dots, f_n \in C_c(G)$ with $\operatorname{supp} f_i \subset U_i$ and $f_i(x) \in [0,1]$ for all $x \in G$ and $1 \leqslant i \leqslant n$, such that $F = f_1 + \cdots + f_n$.*

Proof. Since $\mathcal{U}$ is an open cover of K, for each $x \in K$ there is an open neighbourhood of x, call it $U_x \in \mathcal{U}$, and by Proposition 2.29 there is a closed neighbourhood $V_x \subset U_x$ of x. Since each V_x is a neighbourhood and $\{V_x : x \in K\}$ is a cover of K, compactness tells us that there are elements $x_1, \dots, x_n$ such that $K \subset V_{x_1} \cup \cdots \cup V_{x_n}$. For each i the set $V_{x_i} \cap K$ is a closed subset of a compact set and so compact. Apply Corollary 3.15 to $V_{x_i} \cap K \subset U_{x_i}$ to get $g_i \in C_c(G)$ such that $g_i(x) \in [0,1]$ for all $x \in G$; $g_i(x) = 1$ for all $x \in V_{x_i} \cap K$; and $\operatorname{supp} g_i \subset U_{x_i}$. Since the g_is are non-negative we have

$$\operatorname{supp} F \subset K \subset (V_{x_1} \cap K) \cup \cdots \cup (V_{x_n} \cap K) \subset \operatorname{supp}(g_1 + \cdots + g_n).$$

Thus by Lemma 3.17 there is $h \in C_c(G)$ such that $F = h(g_1 + \cdots + g_n)$ and since F maps into $[0,1]$ and $g_1(x) + \cdots + g_n(x) \geqslant 1$ on the support of F, we conclude that h maps into $[0,1]$; for $1 \leqslant i \leqslant n$ put $f_i = g_i h$.

It remains to check the properties of the f_is. First, $f_i \in C_c(G)$ with $f(x) \in [0,1]$ for all $x \in G$ by design of h and g_i. Secondly, $F = f_1 + \cdots + f_n$ by design. Finally, $\operatorname{supp} f_i \subset \operatorname{supp} g_i \subset U_{x_i} \in \mathcal{U}$. The result is proved. $\square$

4 The Haar integral

We now turn to one of the most beautiful aspects of the basic theory of topological groups. This describes the way the topology and the algebra naturally conspire to produce an integral.

For X a topological space, we say $f \in C_c(X)$ is **non-negative** if $f(x) \geqslant 0$ for all $x \in X$, and write $C_c^+(X)$ for the set of non-negative continuous compactly supported functions on X.

We shall frequently have call to understand elements of $C_c(X)$ as a linear combination of elements of $C_c^+(X)$:

Observation 4.1. The functions $\mathbb{C} \to \mathbb{R}; z \mapsto \operatorname{Re} z$, $\mathbb{C} \to \mathbb{R}; z \mapsto \operatorname{Im} z$, $\mathbb{R} \to \mathbb{R}_{\geqslant 0}; x \mapsto \max\{x, 0\}$ and $\mathbb{R} \to \mathbb{R}_{\geqslant 0}; x \mapsto \max\{-x, 0\}$ are continuous and so any $f \in C_c(X)$ can be written as $f = f_1 - f_2 + if_3 - if_4$ for $f_1, f_2, f_3, f_4 \in C_c^+(X)$, and this decomposition is unique.

We say a linear functional $\int : C_c(X) \to \mathbb{C}$ is **non-negative** if $\int f \geqslant 0$ whenever $f \in C_c^+(X)$. If $f, g \in C_c(X)$ are both real-valued then we write $f \geqslant g$ if $f - g$ is non-negative.

Observation 4.2. If $f, g \in C_c(X)$ are real-valued and $\int$ is a non-negative linear functional $C_c(X) \to \mathbb{C}$ then $\int f \geqslant \int g$ if $f \geqslant g$; and if $f \in C_c(G)$ then $\left|\int f\right| \leqslant \int |f|$ and $\overline{\int f} = \int \overline{f}$.

Example 4.3. The map

$$\int : C_c(\mathbb{R}) \to \mathbb{C}; f \mapsto \int_{-\infty}^{\infty} f(x)\mathrm{d}x,$$

where the integral sign on the right is the Riemann integral, is a non-trivial (meaning not identically zero) non-negative linear map

Remark 4.4. We think of non-negative linear functionals as integrals and in fact the Riesz-Markov-Kakutani Representation Theorem tells us that if X has a sufficiently nice topology then *every* non-negative linear map $C_c(X) \to \mathbb{C}$ arises as an integral against a suitably well-behaved measure on X.

Given a further topological space Y and $F : X \times Y \to \mathbb{C}$ and $x \in X$, we write $\int_y F(x, y)$ for the functional $\int : C_c(Y) \to \mathbb{C}$ applied to the function $Y \to \mathbb{C}; y \mapsto F(x, y)$ (assuming this function is continuous and compactly supported), and similarly for $y \in Y$ and $\int_x F(x, y)$. It will be crucial for us that the order of integration can be interchanged:

Theorem 4.5 (Fubini's Theorem for continuous compactly supported functions)**.** *Suppose that G is a locally compact topological group, $\int$ and $\int'$ are non-negative linear functionals $C_c(G) \to \mathbb{C}$, and $F \in C_c(G \times G)$. Then the map $x \mapsto \int_y' F(x, y)$ is continuous and compactly supported, so that $\int_x \int_y' F(x, y)$ exists. Similarly $y \mapsto \int_x F(x, y)$ is continuous and compactly supported, so that $\int_y' \int_x F(x, y)$ exists and moreover*

$$\int_x \int_y' F(x, y) = \int_y' \int_x F(x, y).$$

Proof. In view of the decomposition in Observation 4.1 and linearity of $\int$ and $\int'$ it is enough to establish the result for F non-negative.

Since $F \in C_c^+(G \times G)$ has support contained in a compact set K, and since the coordinate projection maps $G \times G \to G$ are continuous (and the union of two compact sets is compact) there is a compact set L such that $K \subset L \times L$. It follows that the maps $x \mapsto F(x,y)$ for $y \in G$ and $y \mapsto F(x,y)$ for $x \in G$ are continuous and have support in the compact set L.

We also need an auxiliary 'dominating function' which is a compactly supported continuous function on whose support all of the 'action' happens. For those familiar with the theory of integration, the Dominated Convergence Theorem may come to mind. Concretely, by Corollary 3.15 there is $f \in C_c(G)$ with $f(x) \in [0,1]$ for all $x \in G$; $f(x) = 1$ for all $x \in L$; and $\operatorname{supp} f \subset M$.

For $\epsilon > 0$ (by Observation 3.10) let $\mathcal{U}$ be an open cover of $G \times G$ such that $|F(x,y) - F(x',y')| < \epsilon$ for all $(x,y),(x',y') \in U \in \mathcal{U}$. $M \times M$ is compact and so by Lemma 2.22 there is an open cover $\mathcal{W}$ of M such that $\mathcal{U}' := \{W \times W' : W, W' \in \mathcal{W}\}$ is a refinement of $\mathcal{U}$ (as a cover of $M \times M$ *not* of $G \times G$). First, the support of $\int_y' F(x,y)$ is contained in the (compact) set L and if $x, x' \in W \in \mathcal{W}$ then by design and non-negativity of $\int'$ we have

$$\int_y' F(x',y) = \int_y' F(x',y)f(y) \leqslant \int_y' (F(x,y)+\epsilon)f(y) = \int_y' F(x,y) + \epsilon \int' f,$$

and similarly $\int_y' F(x,y) \leqslant \int_y' F(x',y) + \epsilon \int' f$, whence $|\int_y' F(x',y) - \int_y' F(x,y)| \leqslant \epsilon \int' f$. Since ϵ is arbitrary (and $\int' f$ does not depend on ϵ) it follows that $x \mapsto \int_y' F(x,y)$ is continuous (and compactly supported) and similarly for $y \mapsto \int_x F(x,y)$.

By Corollary 3.18 applied to f supported on the compact set M with the open cover $\mathcal{W}$, there are continuous compactly supported $f_1, \dots, f_n : G \to [0,1]$ such that $f_1 + \cdots + f_n = f$ and $\operatorname{supp} f_i \subset W_i \in \mathcal{W}$. Now, $F(x,y) = F(x,y)f(x)f(y)$ and $f = f_1 + \cdots + f_n$, so

$$F(x,y) = \sum_{i=1}^n \sum_{j=1}^n F(x,y) f_i(x) f_j(y) \text{ for all } x, y \in G.$$

By design of $\mathcal{U}'$ and $\mathcal{U}$, for $1 \leqslant i,j \leqslant n$ there is $\lambda_{i,j} \geqslant 0$ such that $|F(x,y) - \lambda_{i,j}| < \epsilon$ for all $(x,y) \in \operatorname{supp} f_i \times \operatorname{supp} f_j$. We conclude that

$$\sum_{i=1}^n \sum_{j=1}^n \lambda_{i,j} f_i(x) f_j(y) - \epsilon f(x) f(y) \leqslant F(x,y) \leqslant \sum_{i=1}^n \sum_{j=1}^n \lambda_{i,j} f_i(x) f_j(y) + \epsilon f(x) f(y).$$

Since $\int$ and $\int'$ are non-negative linear functionals, we conclude that

$$\left| \int_x \int_y' F(x,y) - \sum_{i=1}^n \sum_{j=1}^n \lambda_{i,j} \int f_i \int' f_j \right| \leqslant \epsilon \int f \int' f$$

and

$$\left| \int_y' \int_x F(x,y) - \sum_{i=1}^n \sum_{j=1}^n \lambda_{i,j} \int f_i \int' f_j \right| \leqslant \epsilon \int f \int' f.$$

The result is proved by the triangle inequality since ϵ is arbitrary (and $\int f$ and $\int' f$ do not depend on ϵ). $\square$

Remark 4.6. ⚠It is *not* enough to assume that $F : G \times G \to \mathbb{C}$ is such that the maps $G \to \mathbb{C}; x \mapsto \int'_y F(x,y)$ and $G \to \mathbb{C}; y \mapsto \int_x F(x,y)$ are well-defined, continuous, and compactly supported. Exercise III.4 asks for an example.

Given a topological group G we say that $\int : C_c(G) \to \mathbb{C}$ is a **(left) Haar integral** on G if $\int$ is a non-trivial (meaning not identically zero) non-negative linear map with

$$\int \lambda_x(f) = \int f \text{ for all } x \in G \text{ and } f \in C_c(G).$$

We sometimes call this last property **(left) translation invariance**.

Remark 4.7. Our definition of Haar integral requires $C_c(G)$ to be non-trivial and hence (Proposition 3.7) for G to support a Haar integral it must be locally compact. It will turn out in Theorem 4.13 that this is enough to guarantee that there is a Haar integral.

Example 4.8. The map $\int$ in Example 4.3 restricted to $C_c(\mathbb{R})$ is a Haar integral, with the only property not already recorded being translation-invariance.

Example 4.9. If G is a discrete group then it supports a left Haar integral:

$$\int : C_c(G) \to \mathbb{C}; f \mapsto \sum_{x \in G} f(x).$$

Exercise III.1 gives a partial converse to this.

The integral of a non-negative continuous function that is not identically 0 is positive, and this already follows from the axioms of a Haar integral. To establish this we begin with a lemma on the comparability of functions:

Lemma 4.10. *Suppose that G is a topological group, $f, g \in C_c^+(G)$ and f is not identically zero. Then there is $n \in \mathbb{N}^*$, $c_1, \dots, c_n \geqslant 0$ and $y_1, \dots, y_n \in G$ such that*

$$g(x) \leqslant \sum_{i=1}^n c_i \lambda_{y_i}(f)(x) \text{ for all } x \in G.$$

Proof. Since f is not identically zero there is some $x_0 \in G$ such that $f(x_0) > 0$ and hence (by Lemma 2.4) an open neighbourhood of the identity U such that $f(x_0 y) > f(x_0)/2$ for all $y \in U$. Let K be compact containing the support of g. Then $\{xU : x \in K\}$ is an open cover of K and so there are elements $x_1, \dots, x_n$ such that $x_1 U, \dots, x_n U$ covers K. But then

$$g(x) \leqslant 2f(x_0)^{-1}\|g\|_\infty \sum_{i=1}^n f(x_0 x_i^{-1} x) = 2f(x_0)^{-1}\|g\|_\infty \sum_{i=1}^n \lambda_{x_i x_0^{-1}}(f)(x) \text{ for all } x \in G,$$

and the result is proved. $\square$

Corollary 4.11. *Suppose that G is a topological group, $\int$ is a left Haar integral on G, and $f \in C_c^+(G)$ has $\int f = 0$. Then f is identically zero.*

Proof. We suppose, for a contradiction, that f is not identically zero. Then by Lemma 4.10 for $g \in C_c^+(G)$ we have $g \leqslant \sum_{i=1}^n c_i \lambda_{y_i}(f)$ for $c_1, \ldots, c_n \geqslant 0$ and $y_1, \ldots, y_n \in G$. By linearity, non-negativity, and translation invariance of the Haar integral

$$\int g \leqslant \sum_{i=1}^n c_i \int \lambda_{y_i}(f) = \sum_{i=1}^n c_i \int f = 0.$$

Since $g \geqslant 0$, non-negativity of the Haar integral implies $\int g \geqslant 0$, and hence $\int g = 0$.

Now, in view of Observation 4.1 we have that $\int h = 0$ for all $h \in C_c(G)$ *i.e.* $\int$ is identically zero contradicting the non-triviality of the Haar integral. The lemma follows. □

Lemma 4.12. *Suppose that G is a topological group, $\int$ is a left Haar integral on G, and $f \in C_c(G)$. Then*

$$\left(\int_x |f(x)|^p\right)^{1/p} \to \|f\|_\infty \text{ as } p \to \infty.$$

Proof. Since f is a continuous function with compact support there is $y \in G$ with $|f(y)| = \|f\|_\infty$, and we may suppose this is non-zero.

For $\epsilon > 0$, the set $U := \{x \in G : |f(x) - f(y)| < \epsilon\}$ is an open neighbourhood of y; since G is regular (Proposition 2.29) U contains a closed neighbourhood of y and intersecting this with a compact neighbourhood of y (which exists since G is locally compact by Proposition 3.7), we have a compact neighbourhood K of y contained in U.

By Corollary 3.15 (applicable since G is locally compact) there is a continuous $h : G \to [0, 1]$ with support contained in K that is not identically 0, and hence by Corollary 4.11 $\int h > 0$. It follows by the triangle inequality that

$$|f(y)|^p \frac{\int |f|}{|f(y)|} \geqslant \int_x |f(x)|^p \geqslant (|f(y)| - \epsilon)^p \int h.$$

Since $r^{1/p} \to 1$ as $p \to \infty$ for any $r > 0$, and $\epsilon > 0$ was arbitrary we get the result. □

Existence of a Haar Integral

Our first main aim is to establish the following.

Theorem 4.13 (Existence of a Haar integral)**.** *Suppose that G is a locally compact topological group. Then there is a left Haar integral on G.*

We begin by defining a sort of approximation: for $f, \phi \in C_c^+(G)$ with ϕ not identically 0 put

$$(f; \phi) := \inf \left\{ \sum_{j=1}^n c_j : n \in \mathbb{N}^*; c_1, \ldots, c_n \geqslant 0; y_1, \ldots, y_n \in G; \text{ and } f \leqslant \sum_{j=1}^n c_j \lambda_{y_j^{-1}}(\phi) \right\}. \quad (4.1)$$

We think of this as a sort of 'covering number' and begin with some basic properties:

Lemma 4.14. *Suppose that G is a topological group, $f, g, \phi, \psi \in C_c^+(G)$ are such that ϕ and ψ are not identically 0. Then*

(i) $(f;\phi)$ *is well-defined;*

(ii) $(\phi;\phi) \leqslant 1$*;*

(iii) $(f;\phi) \leqslant (g;\phi)$ *whenever* $f \leqslant g$*;*

(iv) $(f+g;\phi) \leqslant (f;\phi) + (g;\phi)$*;*

(v) $(\mu f;\phi) = \mu(f;\phi)$ *for* $\mu \geqslant 0$*;*

(vi) $(\lambda_x(f);\phi) = (f;\phi)$ *for all* $x \in G$*;*

(vii) $(f;\psi) \leqslant (f;\phi)(\phi;\psi)$.

Proof. Lemma 4.10 shows that the set on the right of (4.1) is non-empty; it has 0 as a lower bound. (i) follows immediately. For (ii)[6] note that $\phi \leqslant 1.\lambda_{1_G^{-1}}(\phi)$ so that $(\phi;\phi) \leqslant 1$. (iii), (iv), (v), and (vi) are all immediate. Finally, for (vii) suppose $c_1, \dots, c_n \geqslant 0$ are such that $f \leqslant \sum_{j=1}^n c_j \lambda_{y_j^{-1}}(\phi)$, so that by (iii), (iv), (v), and (vi) we have $(f;\psi) \leqslant \sum_{j=1}^n c_j(\phi;\psi)$. The result follows on taking infima. □

To make use of $(\cdot\,;\cdot)$ we need to fix a non-zero reference function $f_0 \in C_c^+(G)$ and for $\phi \in C_c^+(G)$ not identically zero we put

$$I_\phi(f) := \frac{(f;\phi)}{(f_0;\phi)} \leqslant (f;f_0), \tag{4.2}$$

where the inequality follows from Lemma 4.14 (vii).

Many of the properties of Lemma 4.14 translate into properties of I_ϕ. In particular, we have $I_\phi(f_1+f_2) \leqslant I_\phi(f_1) + I_\phi(f_2)$; for suitable ϕ we also have the following converse.

Lemma 4.15. *Suppose that G is a locally compact topological group, $f_1, f_2 \in C_c^+(G)$ and $\epsilon > 0$. Then there is a symmetric open neighbourhood of the identity V such that if $\phi \in C_c^+(G)$ is not identically 0 and has support in V then $I_\phi(f_1) + I_\phi(f_2) \leqslant I_\phi(f_1+f_2) + \epsilon$.*

Proof. Let K be a compact set containing the support of both f_1 and f_2 (possible since the union of two compact sets is compact) and apply Corollary 3.15 to get $F : G \to [0,1]$ continuous, compactly supported, and with $F(x) = 1$ for all $x \in K$.

For $j \in \{1,2\}$ let g_j be continuous such that $(f_1 + f_2 + \epsilon F)g_j = f_j$ (possible by Lemma 3.17 since $\operatorname{supp} f_i \subset K \subset \operatorname{supp} F$). By Observation 3.10 (and the fact that the intersection of two open covers is an open cover) there is an open cover $\mathcal{U}$ of G such that if $x, y \in U \in \mathcal{U}$

[6] As it happens it is easy to prove equality here but we do not need it.

then $|g_j(x) - g_j(y)| < \epsilon$ for $j \in \{1, 2\}$. K is compact; apply Lemma 2.21 to $\mathcal{U}$ to get a symmetric open neighbourhood of the identity V such that $\{yV : y \in K\}$ refines $\mathcal{U}$ as a cover of K.

Now suppose that $\phi \in C_c^+(G)$ is not identically 0 and has support in V, and that $c_1, \ldots, c_n \geqslant 0$ and $y_1, \ldots, y_n \in G$ are such that

$$f_1(x) + f_2(x) + \epsilon F(x) \leqslant \sum_{i=1}^{n} c_i \phi(y_i x) \text{ for all } x \in G.$$

If $\phi(y_i x) g_j(x) \neq 0$ then $x \in K$ and $y_i^{-1} \in xV$ (using $V = V^{-1}$), but xV is a subset of a set in $\mathcal{U}$ so $g_j(x) \leqslant g_j(y_i^{-1}) + \epsilon$ and hence

$$f_j(x) \leqslant \sum_{i=1}^{n} c_i \phi(y_i x) g_j(x) \leqslant \sum_{i=1}^{n} c_i (g_j(y_i^{-1}) + \epsilon) \phi(y_i x) \text{ for all } x \in G, j \in \{1, 2\}.$$

By Lemma 4.14 (ii),(iii), (iv),(v) & (vi) we have

$$(f_j; \phi) \leqslant \sum_{i=1}^{n} c_i (g_j(y_i^{-1}) + \epsilon) \text{ for all } j \in \{1, 2\},$$

but $g_1(y^{-1}) + g_2(y^{-1}) \leqslant 1$ for all $y \in G$, so

$$(f_1; \phi) + (f_2; \phi) \leqslant \sum_{i=1}^{n} c_i (1 + 2\epsilon).$$

Taking infima and then applying Lemma 4.14 (iv) and (v) and the inequality in (4.2) we get

$$\begin{aligned} I_\phi(f_1) + I_\phi(f_2) &\leqslant (1 + 2\epsilon) I_\phi(f_1 + f_2 + \epsilon F) \\ &\leqslant (1 + 2\epsilon)(I_\phi(f_1 + f_2) + \epsilon I_\phi(F)) \\ &\leqslant I_\phi(f_1 + f_2) + (2(f_1 + f_2; f_0) + (F; f_0) + 2\epsilon(F; f_0))\epsilon. \end{aligned}$$

The result follows since $\epsilon > 0$ was arbitrary and F, f_1, f_2 and f_0 do not depend on ϵ. □

With these lemmas we can turn to the main argument.

Proof of Theorem 4.13. By Corollary 3.15 there is $f_0 \in C_c^+(G)$ with f_0 not identically zero. Write F for the set of functions $I : C_c^+(G) \to \mathbb{R}_{\geqslant 0}$ with $I(f) \leqslant (f; f_0)$ for all $f \in C_c^+(G)$ endowed with the product topology *i.e.* the weakest topology such that the maps $F \to [0, (f; f_0)]; I \mapsto I(f)$ are continuous for all $f \in C_c^+(G)$. Since the closed interval $[0, (f; f_0)]$ is compact, F is a (non-empty) product of compact spaces and so compact (this is Tychonoff's Theorem). Let X be the set of $I \in F$ such that

$$I(f_0) = 1 \tag{4.3}$$

$$I(\mu f) = \mu I(f) \text{ for all } \mu \geqslant 0, f \in C_c^+(G), \tag{4.4}$$

and

$$I(\lambda_x(f)) = I(f) \text{ for all } x \in G, f \in C_c^+(G). \tag{4.5}$$

The set X is closed as an intersection of the preimage of closed sets. Moreover, by Lemma 4.14 $I_\phi \in X$ for any $\phi \in C_c^+(G)$ that is not identically zero: the fact that $I(f) \in [0, (f; f_0)]$ follows from the inequality in (4.2); (4.3) by design; (4.4) by (v); and (4.5) by (vi).

This almost gives us a Haar integral (on non-negative functions) except that in general the elements of X are not additive, meaning we do not in general have $I(f+f') = I(f)+I(f')$. To get this we introduce some further sets: for $\epsilon > 0$ and $f, f' \in C_c^+(G)$ define

$$B(f, f'; \epsilon) := \{I \in X : |I(f + f') - I(f) - I(f')| \leqslant \epsilon\}.$$

As with X, the sets $B(f, f'; \epsilon)$ are closed. We shall show that any finite intersection of such sets is non-empty: For any $f_1, f_1', f_2, f_2', \dots, f_n, f_n' \in C_c^+(G)$ and $\epsilon_1, \dots, \epsilon_n > 0$, by Lemma 4.15 there are symmetric open neighbourhoods of the identity $V_1, \dots, V_n$ such that if $\phi \in C_c^+(G)$ is not identically 0 and is supported in V_i then

$$|I_\phi(f_i + f_i') - I_\phi(f_i) - I_\phi(f_i')| \leqslant \epsilon_i. \tag{4.6}$$

The set $V := \bigcap_{i=1}^n V_i$ is a symmetric open neighbourhood of the identity and by Corollary 3.15 there is $\phi \in C_c^+(G)$ that is not identically 0 with support contained in V. I_ϕ enjoys (4.6) for all $1 \leqslant i \leqslant n$, and we noted before that $I_\phi \in X$, hence $I_\phi \in \bigcap_{i=1}^n B(f_i, f_i', \epsilon_i)$. We conclude that $\{B(f, f'; \epsilon) : f, f' \in C_c^+(G), \epsilon > 0\}$ is a set of closed subsets of F with the finite intersection property, but F is compact and so there is some I in all of these sets. Such an I is additive since $|I(f + f') - I(f) - I(f')| \leqslant \epsilon$ for all f, f' and $\epsilon > 0$. It remains to define $\int : C_c(G) \to \mathbb{C}$ by putting

$$\int f := I(f_1) - I(f_2) + iI(f_3) - iI(f_4) \text{ where } f = f_1 - f_2 + if_3 - if_4 \text{ for } f_1, f_2, f_3, f_4 \in C_c^+(G).$$

This decomposition of functions in $C_c(G)$ is unique (noted in Observation 4.1) and so this is well-defined. Moreover, $\int$ is linear since I is additive and enjoys (4.4); it is non-negative since I is non-negative (and $I(0) = 0$); it is translation invariant by (4.5); and it is non-trivial by (4.3). The result is proved. □

Uniqueness of the Haar integral

Our second main aim is to establish the following result.

Theorem 4.16 (Uniqueness of the Haar Integral). *Suppose that G is a locally compact topological group and $\int$ and $\int'$ are left Haar integrals on G. Then there is some $\lambda > 0$ such that $\int = \lambda \int'$.*

For this we introduce a little more notation: Given a topological group G and $f \in C_c(G)$ we write $\widetilde{f}(x) = \overline{f(x^{-1})}$.

Remark 4.17. $\widetilde{\cdot}$ is a conjugate-linear multiplicative involution on $C_c(G)$, since complex conjugation and $x \mapsto x^{-1}$ are both continuous (and continuous images of compact sets are compact).

Proof of Theorem 4.16. Suppose that $f_0, f_1 \in C_c^+(G)$ are not identically 0 and write K for a compact set containing the support of f_0 and f_1 (which exists since finite unions of compact sets are compact). Since G is locally compact there is an open neighbourhood H of 1_G contained in a compact set L.

First, by Corollary 3.15 there is a continuous compactly supported function $F : G \to [0,1]$ with $F(x) = 1$ for all $x \in KL$ (this set is compact by Lemma 2.18, and hence the corollary applies).

Now, suppose $\epsilon > 0$ and apply Observation 3.10 to get an open cover $\mathcal{U}_i$ of G such that if $x, y \in U \in \mathcal{U}_i$ then $|f_i(x) - f_i(y)| < \epsilon$ for $i \in \{0,1\}$; let $\mathcal{U} := \{U_0 \cap U_1 : U_0 \in \mathcal{U}_0, U_1 \in \mathcal{U}_1\}$. By Lemma 2.21 applied to $\mathcal{U}$ and the compact set KL there is a symmetric open neighbourhood of the identity V such that $\{xV : x \in KL\}$ is a refinement of $\mathcal{U}$ as a cover of KL; and by Corollary 3.15 there is $h \in C_c^+(G)$ that is not identically zero and is supported in $V \cap H^{-1}$.

For $x \in G$, translation invariance of $\int'$ (and Observation 4.2) tells us that

$$\int_y' h(y^{-1}x) = \int_y' \overline{\widetilde{h}(x^{-1}y)} = \overline{\int_y' \widetilde{h}(x^{-1}y)} = \overline{\int_y' \widetilde{h}(y)} = \int' \overline{\widetilde{h}}.$$

For $i \in \{0,1\}$, the map $x \mapsto \int_y' f_i(x)h(y^{-1}x) = f_i(x)\int' \overline{\widetilde{h}}$ is continuous and is supported in K and so is compactly supported and $\int_x \int_y' f_i(x)h(y^{-1}x)$ exists and equals $\int f_i \int' \overline{\widetilde{h}}$ (by linearity of $\int$ and $\int'$). On the other hand the map $(x,y) \mapsto f_i(x)h(y^{-1}x)$ is continuous and supported on $K \times L$ and so is compactly supported and hence by Fubini's Theorem (Theorem 4.5), $y \mapsto \int_x f_i(x)h(y^{-1}x)$ exists, and (using translation invariance of $\int$) we have

$$\int f_i \int' \overline{\widetilde{h}} = \int_x \int_y' f_i(x)h(y^{-1}x) = \int_y' \int_x f_i(x)h(y^{-1}x) = \int_y' \int_x f_i(yx)h(x).$$

Since $\{yV : y \in KL\}$ refines $\mathcal{U}$ (as a cover of KL) we have $|f_i(yx) - f_i(y)| < \epsilon$ for $x \in V$ and $y \in KL$; and for $x \in H^{-1}$ and $f_i(yx) \neq 0$ or $f_i(y) \neq 0$ we have $y \in KH$ whence $F(y) = 1$. It follows that

$$f_i(y)h(x) - \epsilon F(y)h(x) \leqslant f_i(yx)h(x) \leqslant f_i(y)h(x) + \epsilon F(y)h(x) \text{ for all } x, y \in G,$$

and so by non-negativity and linearity of $\int$ and $\int'$ we have

$$\int_y' \int_x f_i(y)h(x) - \int_y' \int_x \epsilon F(y)h(x) \leqslant \int_y' \int_x f_i(yx)h(x) \leqslant \int_y' \int_x f_i(y)h(x) + \int_y' \int_x \epsilon F(y)h(x).$$

It follows (using linearity of $\int$) that $|\int' f_i \int h - \int f_i \int' \overline{\tilde{h}}| \leqslant \epsilon \int' F \int h$, and hence by the triangle inequality (and division, which is valid since $\int f_0, \int f_1 \neq 0$ by Corollary 4.11 as f_0 and f_1 are not identically zero) that

$$\left|\frac{\int' f_0}{\int f_0} - \frac{\int' f_1}{\int f_1}\right| \leqslant \left|\frac{\int' f_0}{\int f_0} - \frac{\int' \overline{\tilde{h}}}{\int h}\right| + \left|\frac{\int' \overline{\tilde{h}}}{\int h} - \frac{\int' f_1}{\int f_1}\right| \leqslant \epsilon \int' F \left(\frac{1}{\int f_0} + \frac{1}{\int f_1}\right).$$

Since ϵ was arbitrary (and in particular f_0, f_1, and F do not depend on it) it follows that $\int' f / \int f$ is a constant λ for all $f \in C_c^+(G)$ not identically zero. This constant must be non-zero since $\int'$ is non-trivial, and it must be positive since $\int'$ and $\int$ are non-negative. The result follows from the usual decomposition (Observation 4.1), and the fact that $\int 0, \int' 0 = 0$. $\square$

5 The Peter-Weyl Theorem

Suppose that G is a topological group, and for an inner product space V recall the definition of $\mathrm{U}(V)$ from Example 1.18. A **finite dimensional unitary representation**[7] **of** G is a continuous homomorphism $G \to \mathrm{U}(V)$ for some finite dimensional complex inner product space V.

Example 5.1 (Permutation representation)**.** For $V = \mathbb{C}^n$ with its usual inner product, *i.e.* $\langle x, y\rangle := x_1\overline{y_1} + \cdots + x_n\overline{y_n}$, the map

$$S_n \to \mathrm{U}(\mathbb{C}^n); \sigma \mapsto (\mathbb{C}^n \to \mathbb{C}^n; (x_i)_{i=1}^n \mapsto (x_{\sigma^{-1}(i)})_{i=1}^n)$$

from S_n endowed with the discrete topology, is a finite dimensional unitary representation.

A function $f : G \to \mathbb{C}$ is said to be a **matrix coefficient** if there is a finite dimensional unitary representation $\pi : G \to \mathrm{U}(V)$, and elements $v, w \in V$ such that $f(x) = \langle \pi(x)v, w\rangle$ for all $x \in G$.

Example 5.2. Suppose that $\pi : G \to \mathrm{U}(V)$ is a finite dimensional unitary representation of a topological group G and $e_1, \dots, e_n$ is an orthonormal basis for V. If we write $A_{i,j} := \langle \pi(x)e_i, e_j\rangle$ and suppose that $\lambda \in \mathbb{C}^n$ is the vector for $v \in V$ written w.r.t. the basis $e_1, \dots, e_n$ (*i.e.* $\lambda_i = \langle v, e_i\rangle$), then λA – the matrix A pre-multiplied by the row vector λ – is $\pi(x)v$ written w.r.t. the basis $e_1, \dots, e_n$. This is the reason for the terminology 'matrix coefficient'.

Remark 5.3. All matrix coefficients are continuous, since continuity of $\pi : G \to \mathrm{U}(V)$ and the definition of the topology on $\mathrm{U}(V)$ means that $x \mapsto \pi(x)v$ is continuous for all $v \in V$, and the projections $v \mapsto \langle v, w\rangle$ are continuous for all $w \in V$, so the resulting composition is also continuous.

Lemma 5.4. *Suppose that G is a compact topological group. Then there is a unique left Haar integral $\int$ on G with $\int 1 = 1$ such that*

$$\langle f, g\rangle := \int f\overline{g} \text{ for all } f, g \in C(G)$$

is an inner product on $C(G)$ and for each $x \in G$, λ_x is unitary w.r.t. this inner product. Furthermore, $\|f\|_2 := \langle f, f\rangle^{1/2}$ and $\|f\|_1 := \int |f|$ define norms on $C(G)$ and

$$\|f\|_1 \leqslant \|f\|_2 \leqslant \|f\|_\infty \text{ for all } f \in C(G).$$

[7] ⚠ A unitary representation is usually a continuous group homomorphism $\pi : G \to \mathrm{U}(H)$ for a complex Hilbert space H, not merely a complex inner product space. Every finite dimensional complex inner product space is complete and so a Hilbert space, and so our definition is not at variance with this.

Proof. By Theorem 4.13 there is a left Haar integral $\int'$ on G. Since G is compact the constant function 1 is compactly supported and so by Corollary 4.11, $\int' 1 > 0$. Diving by this positive constant we get a left Haar integral $\int$ with $\int 1 = 1$. Now suppose that $\int'$ is another left Haar integral with $\int' 1 = 1$. By Theorem 4.16 $\int' = \lambda \int$ for some $\lambda > 0$, but since $\int 1 = 1 = \int' 1$ we conclude that $\lambda = 1$ and $\int = \int'$ giving the claimed uniqueness.

Linearity in the first argument and conjugate-symmetry of $\langle \cdot, \cdot \rangle$ follow from linearity of the Haar integral and Observation 4.2 respectively. $\langle f, f \rangle \geqslant 0$ for all $f \in C(G)$ since $\int$ is non-negative and $\langle \cdot, \cdot \rangle$ is then positive definite by Corollary 4.11.

The Haar integral is left-invariant so

$$\langle f, g \rangle = \int f\overline{g} = \int \lambda_x(f\overline{g}) = \int \lambda_x(f)\overline{\lambda_x(g)} \text{ for all } f, g \in C(G),$$

and the first part is proved.

For any inner product $f \mapsto \langle f, f \rangle^{1/2}$ is a norm, so $\|\cdot\|_2$ is a norm. Absolute homogeneity of $\|\cdot\|_1$ follows from the fact that the modulus of a complex number is multiplicative and $\int$ is linear, and the triangle inequality follows from, non-negativity, linearity and the triangle inequality for the modulus of a complex number. $\|f\|_1 \geqslant 0$ by non-negativity of $\int$, and finally $\|\cdot\|_1$ is positive definite by Corollary 4.11.

Since G is compact the constant functions 1 and $\|f\|_\infty^2$ are both in $C(G)$. By the Cauchy-Schwarz inequality (which holds for all inner products) we have

$$\|f\|_1 = \int |f| = \langle 1, |f| \rangle \leqslant \|1\|_2 \||f|\|_2 = \|f\|_2 \text{ for all } f \in C(G);$$

and by non-negativity of $\int$ we have

$$\|f\|_2^2 = \int |f|^2 \leqslant \int \|f\|_\infty^2 = \|f\|_\infty^2 \text{ for all } f \in C(G).$$

The result is proved. □

Remark 5.5. For the remainder of this section we write $\int$ for the unique Haar integral in Lemma 5.4, and use the notation $\langle \cdot, \cdot \rangle$, $\|\cdot\|_2$ and $\|\cdot\|_1$ as in this lemma.

For $f, g \in C(G)$ we define their **convolution** to be the function

$$x \mapsto f * g(x) := \int_y f(y)g(y^{-1}x) = \langle f, \lambda_x(\widetilde{g}) \rangle.$$

Lemma 5.6 (Basic properties of convolution)**.** *Suppose that G is a compact topological group. Then*

(i) $C(G) \to C(G); g \mapsto g * f$ is well-defined and linear for all $f \in C(G)$;

(ii) $h * (g * f) = (h * g) * f$ for all $f, g, h \in C(G)$;

(iii) $\lambda_x(g * f) = \lambda_x(g) * f$ *for all* $x \in G$, $f, g \in C(G)$*;*

(iv) $\langle g * f, h\rangle = \langle g, h * \widetilde{f}\rangle$ *for all* $f, g, h \in C(G)$ *(recall* $\widetilde{f}$ *from just before Remark 4.17);*

(v) $\|h * f\|_\infty \leqslant \min\{\|h\|_1\|f\|_\infty, \|h\|_2\|\widetilde{f}\|_2\}$ *for all* $f, h \in C(G)$*.*

Proof. By the first part of Fubini's Theorem (Theorem 4.5) the function $g * f \in C(G)$ since $(x, y) \mapsto g(x)f(x^{-1}y)$ is continuous and compactly supported. Since $\int_x$ is linear, $g \mapsto g * f$ is well-defined and linear giving (i).

For (ii) we apply λ_y to the integrand $z \mapsto g(z)f(z^{-1}y^{-1}x)$ using that $\int_z$ is a left Haar integral; then Fubini's Theorem (Theorem 4.5) since $(z, y) \mapsto h(y)g(y^{-1}z)f(z^{-1}x)$ is continuous; and finally linearity of $\int_y$ to see that

$$\begin{aligned} h * (g * f)(x) &= \int_y h(y) \int_z g(z)f(z^{-1}y^{-1}x) \\ &= \int_y h(y) \int_z g(y^{-1}z)f(z^{-1}x) = \int_z \left(\int_y h(y)g(y^{-1}z)\right) f(z^{-1}x) = (h * g) * f(x) \end{aligned}$$

as claimed.

For (iii) note that $\lambda_t(g * f)(x) = g * f(t^{-1}x) = \langle g, \lambda_{t^{-1}x}(\widetilde{f})\rangle = \langle g, \lambda_{t^{-1}}(\lambda_x(\widetilde{f}))\rangle = \langle \lambda_t(g), \lambda_x(\widetilde{f})\rangle = \lambda_t(g) * f(x)$ since λ_t is unitary w.r.t. $\langle \cdot, \cdot\rangle$ by Lemma 5.4.

For (iv), since the function $(x, y) \mapsto g(x)f(x^{-1}y)\overline{h(y)}$ is continuous and compactly supported, by Fubini's Theorem (Theorem 4.5) and linearity of $\int_y$; and then Observation 4.2 we have

$$\begin{aligned} \langle g * f, h\rangle &= \int_y \int_x g(x)f(x^{-1}y)\overline{h(y)} \\ &= \int_x g(x) \int_y f(x^{-1}y)\overline{h(y)} = \int_x g(x) \int_y \overline{h(y)\widetilde{f}(y^{-1}x)} = \langle g, h * \widetilde{f}\rangle, \end{aligned}$$

as required.

Finally, (v) follows on the one hand since

$$|h * f(x)| \leqslant \int_y |h(y)||f(y^{-1}x)| \leqslant \int |h|\|f\|_\infty = \|h\|_1\|f\|_\infty,$$

and on the other since $|h * f(x)| = |\langle h, \lambda_x(\widetilde{f})\rangle| \leqslant \|h\|_2\|\lambda_x(\widetilde{f})\|_2 = \|h\|_2\|\widetilde{f}\|_2$. The result is proved. □

Remark 5.7. As usual, in view of the associativity in (ii) there is no ambiguity in omitting parentheses when writing expressions like $h * g * f$.

Proposition 5.8. *Suppose that* G *is a compact topological group* G, $f \in C(G)$ *and* $(g_n)_{n\in\mathbb{N}^*}$ *is a sequence of elements of* $C(G)$ *with* $\|g_n\|_1 \leqslant 1$*. Then there is a subsequence* $(g_{n_i})_{i\in\mathbb{N}^*}$ *such that* $g_{n_i} * f$ *converges uniformly to some element of* $C(G)$ *as* $i \to \infty$*.*

Proof. For each $j \in \mathbb{N}^*$, let $\mathcal{U}_j$ be an open cover of G such that if $x, y \in U \in \mathcal{U}_j$ then $|f(x) - f(y)| < 1/j$. Since G is compact apply Lemma 2.21 to get an open neighbourhood of the identity U_j such that $\{xU_j : x \in G\}$ refines $\mathcal{U}_j$; and by compactness again there is a finite cover $\{x_{1,j}U_j, \dots, x_{k(j),j}U_j\}$ which refines $\{xU_j : x \in G\}$.

By Lemma 5.4 (v) $g_n * f(x) \in [-\|f\|_\infty, \|f\|_\infty]$. Let $n_{0,i} = i$ for all $i \in \mathbb{N}^*$, and suppose that $j \geqslant 1$. By the Heine-Borel theorem (for $\mathbb{R}^{k(j)}$) there is subsequence $(n_{j,i})_i$ of $(n_{j-1,i})_i$ such that $g_{n_{j,i}} * f(x_{k,j})$ converges for all $1 \leqslant k \leqslant k(j)$. Setting $n_i := n_{i,i}$ we have that the tail of $(n_i)_i$ is a subsequence of $(n_{j,i})_i$ for all j and so $g_{n_i} * f(x_{k,j})$ converges, say to $z_{k,j}$, as $i \to \infty$ for all $1 \leqslant k \leqslant k(j)$ and $j \in \mathbb{N}^*$.

Suppose $\epsilon > 0$ and let $j := \lceil 3\epsilon^{-1} \rceil$. For all $1 \leqslant k \leqslant k(j)$ let M_k be such that $|g_{n_i} * f(x_{k,j}) - z_{k,j}| < \epsilon/6$ for all $i \geqslant M_k$; let $M := \max\{M_k : 1 \leqslant k \leqslant k(j)\}$ and suppose that $i, i' \geqslant M$.

For $x \in G$ there is some $1 \leqslant k \leqslant k(j)$ such that $x \in x_{k,j}U_j$ and hence for all $y \in G$ we have $y^{-1}x, y^{-1}x_{k,j} \in y^{-1}x_{k,j}U_j$ which is a subset of an element of $\mathcal{U}_j$, so $|f(y^{-1}x) - f(y^{-1}x_{k,j})| < 1/j$. Thus for $g \in C(G)$ with $\|g\|_1 \leqslant 1$ we have

$$\begin{aligned} |g * f(x) - g * f(x_{k,j})| &= |\langle g, \lambda_x(\widetilde{f}) - \lambda_{x_{k,j}}(\widetilde{f})\rangle| \\ &\leqslant \|g\|_1 \|\lambda_x(\widetilde{f}) - \lambda_{x_{k,j}}(\widetilde{f})\|_\infty \leqslant \sup_{y \in G} |f(y^{-1}x) - f(y^{-1}x_{j,k})| \leqslant \frac{1}{j} \leqslant \epsilon/3. \end{aligned}$$

In particular this holds for $g = g_{n_i}$ and $g = g_{n_{i'}}$, so that

$$\begin{aligned} |g_{n_i} * f(x) - g_{n_{i'}} * f(x)| \leqslant{}& |g_{n_i} * f(x) - g_{n_i} * f(x_{k,j})| + |g_{n_i} * f(x_{k,j}) - z_{k,j}| \\ &+ |z_{k,j} - g_{n_{i'}} * f(x_{k,j})| + |g_{n_{i'}} * f(x_{k,j}) - g_{n_{i'}} * f(x)| < \epsilon. \end{aligned}$$

Since $x \in G$ was arbitrary it follows that the sequence of functions $(g_{n_i} * f)_i$ is uniformly Cauchy and so converges to a continuous function on G. The result is proved. $\square$

We say that $V \leqslant C(G)$ is **invariant** if $\lambda_x(v) \in V$ for all $v \in V$.

Example 5.9. Suppose that $V \leqslant C(G)$ is invariant and finite dimensional. Then $\pi : G \to \mathrm{U}(V); x \mapsto (V \to V; v \mapsto \lambda_x(v))$ is a finite dimensional unitary representation.

For any $V \leqslant C(G)$ write $V^\perp$ for the set of $w \in C(G)$ such that $\langle v, w\rangle = 0$ for all $v \in V$.

Proposition 5.10. *Suppose that G is a compact topological group and $f \in C(G)$. Then there is an invariant space $W \leqslant C(G)$ with $\dim W \leqslant \epsilon^{-2}\|f\|_2^2$ such that if $g \in W^\perp$ then $\|g * f\|_2 \leqslant \epsilon\|g\|_2$.*

Proof. Let V be the set of vectors of the form

$$h_1 + \cdots + h_n \text{ where } n \in \mathbb{N}_0, h_i * \widetilde{f} * f = \lambda_i h_i \text{ and } \lambda_i \geqslant \epsilon^2 \text{ for all } 1 \leqslant i \leqslant n. \tag{5.1}$$

This is an invariant space by Lemma 5.6 (iii). For $v \in V$ we shall write $v = h_1 + \cdots + h_n$ to mean a decomposition as in (5.1) with the additional requirements that h_i is not identically zero (so $\|h_i\|_2^2 \neq 0$ since h_i is continuous), and $\lambda_i \neq \lambda_j$ for $i \neq j$, which is possible since the map $T : C(G) \to C(G); h \mapsto h * \widetilde{f} * f$ is linear. (The zero vector is represented as a sum with no terms.)

In fact T is positive definite and so the h_is, which are eigenvectors with corresponding eigenvalues λ_i, are perpendicular for different eigenvalues. In our language the relevant parts of this follow since if $h_i * \widetilde{f} * f = \lambda_i h_i$ and $h_j * \widetilde{f} * f = \lambda_j h_j$, then

$$\lambda_i\langle h_i, h_j\rangle = \langle \lambda_i h_i, h_j\rangle = \langle h_i * \widetilde{f} * f, h_j\rangle = \langle h_i, h_j * \widetilde{f} * f\rangle = \langle h_i, \lambda_j h_j\rangle = \overline{\lambda_j}\langle h_i, h_j\rangle.$$

Applying this identity with $j = i$ for some $h_j \neq 0$ we see that λ_i is real. Then applying it again with $\lambda_i \neq \lambda_j$ we have $\langle h_i, h_j\rangle = 0$. In particular, if $v = h_1 + \cdots + h_n$ in the way discussed after (5.1) then

$$\|v * \widetilde{f}\|_2^2 = \langle v * \widetilde{f} * f, v\rangle = \sum_{i=1}^{n} \lambda_i \|h_i\|_2^2 \geqslant \epsilon^2 \sum_{i=1}^{n} \|h_i\|_2^2 = \epsilon^2\|v\|_2^2. \tag{5.2}$$

If V contains n linearly independent vectors, then by the Gram-Schmidt process[8] there are orthonormal vectors $v_1, \ldots, v_n \in V$. For $x \in G$, by Bessel's inequality[9]

$$\sum_{i=1}^{n} |\langle v_i, \lambda_x(f)\rangle|^2 \leqslant \|\lambda_x(f)\|_2^2 = \|f\|_2^2.$$

Integrating against x and using (5.2) we have

$$n\epsilon^2 \leqslant \sum_{i=1}^{n} \int_x |v_i * \widetilde{f}(x)|^2 = \int_x \sum_{i=1}^{n} |\langle v_i, \lambda_x(f)\rangle|^2 \leqslant \int_x \|f\|_2^2 = \|f\|_2^2.$$

---

[8]Given $e_1, e_2, \ldots$ linearly independent, the Gram-Schmidt process in an inner product space defines

$$u_i := e_i - \sum_{k=1}^{i-1} \langle e_i, v_k\rangle v_k \text{ and } v_i := u_i/\|u_n\|.$$

It can be shown by induction that $v_1, v_2, \ldots$ is an orthonormal sequence.

[9]Bessel's inequality is the fact that if $v_1, v_2, \ldots$ is an orthonormal sequence in an inner product space then $\sum_{i=1}^{n} |\langle v_i, v\rangle|^2 \leqslant \|v\|^2$ for all v. To prove it note that because the v_is are orthonormal we have

$$\left\|\sum_{i=1}^{n} \langle v_i, v\rangle v_i\right\|^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} \langle v_i, v\rangle\overline{\langle v_j, v\rangle}\langle v_i, v_j\rangle = \sum_{i=1}^{n} |\langle v_i, v\rangle|^2.$$

Hence by the Cauchy-Schwarz inequality

$$\left(\sum_{i=1}^{n} |\langle v_i, v\rangle|^2\right)^2 = \left|\left\langle v, \sum_{i=1}^{n} \langle v_i, v\rangle v_i\right\rangle\right|^2 \leqslant \|v\|^2 \left\|\sum_{i=1}^{n} \langle v_i, v\rangle v_i\right\|^2 = \|v\|^2 \left(\sum_{i=1}^{n} |\langle v_i, v\rangle|^2\right).$$

Cancelling gives the inequality.

It follows that $\dim V \leqslant \epsilon^{-2}\|f\|_2^2$.

Write $W := \{k * \widetilde{f} : k \in V\}$, which is invariant by Lemma 5.6 (iii) and the fact V is invariant. Let $M := \sup\{\|g * f\|_2 : g \in W^{\perp} \text{ and } \|g\|_2 \leqslant 1\}$. We shall be done if we can show that $M^2 \leqslant \epsilon^2$.

Claim. *If $h \in V^{\perp}$ then $\|h * \widetilde{f}\|_2 \leqslant M\|h\|_2$.*

Proof. First, $h * \widetilde{f} \in W^{\perp}$: To see this, for $v \in V$ write $v = h_1 + \cdots + h_n$ to mean a decomposition as in (5.1). Then

$$\langle h * \widetilde{f}, v * \widetilde{f}\rangle = \sum_{i=1}^{n} \langle h, h_i * \widetilde{f} * f\rangle = \sum_{i=1}^{n} \lambda_i \langle h, h_i\rangle = 0.$$

Now let $k \in W^{\perp}$ have $\|k\|_2 = 1$ such that $\|h * \widetilde{f}\|_2 = \langle h * \widetilde{f}, k\rangle = \langle h, k * f\rangle \leqslant \|h\|_2 \|k * f\|_2 \leqslant M\|h\|_2$ as claimed. □

Let $g_n \in W^{\perp}$ have $\|g_n * f\|_2 \to M$ and $\|g_n\|_2 \leqslant 1$. By Cauchy-Schwarz we have $\|g_n\|_1 \leqslant 1$ and we may apply Proposition 5.8 to pass to a subsequence which converges uniformly. Hence by relabelling we may now additionally assume that $g_n * f \to h$ uniformly for some $h \in C(G)$. In particular, $\|g_n * f\|_2 \to \|h\|_2$ and $\langle h, g_n * f\rangle \to \|h\|_2^2$ and hence $\|h\|_2 = M$. Moreover, if $v \in V$ then $\langle g_n * f, v\rangle = \langle g_n, v * \widetilde{f}\rangle = 0$, and the former converges to $\langle h, v\rangle$, whence $h \in V^{\perp}$.

Combining this with the claim above we have

$$\begin{aligned} \|h * \widetilde{f} - M^2 g_n\|_2^2 &= \|h * \widetilde{f}\|_2^2 - 2M^2 \operatorname{Re}\langle h * \widetilde{f}, g_n\rangle + M^4 \|g_n\|_2^2 \\ &\leqslant M^2 \|h\|_2^2 - 2M^2 \operatorname{Re}\langle h, g_n * f\rangle + M^4 \to 0. \end{aligned}$$

Hence $M^2 g_n \to h * \widetilde{f}$ in $\|\cdot\|_2$, and since convergence in $\|\cdot\|_2$ is mapped to uniform convergence by convolution operators we have $M^2 g_n * f \to h * \widetilde{f} * f$. Uniqueness of limits then ensures $M^2 h = h * \widetilde{f} * f$. If $M^2 \geqslant \epsilon^2$ then $h \in V$, but then since $h \in V^{\perp}$ we see that h is identically zero. In this case $M = \|h\|_2 = 0$ and certainly $M^2 \leqslant \epsilon^2$ as required. The result is proved. □

Theorem 5.11 (The Peter-Weyl Theorem, Part I)**.** *Suppose that G is a compact topological group. Then matrix coefficients are dense in $C(G)$ with the uniform norm.*

Proof. Suppose that $f \in C(G)$ and let $\epsilon > 0$. Observation 3.10 gives us an open cover $\mathcal{U}_j$ of G such that if $x, y \in U \in \mathcal{U}_j$ then $|\widetilde{f}(x) - \widetilde{f}(y)| < \epsilon/2$. Since G is compact, by Lemma 2.21 there is an open neighbourhood of the identity U such that $\{xU : x \in G\}$ refines $\mathcal{U}$, and by Lemma 2.16 there is an open set V such that $V^2 \subset U$. By Corollary 3.15, there is $g \in C(G)$ non-negative and not identically 0 such that $\operatorname{supp} g \subset V$. By rescaling g we may assume that $\int g = 1$. The support of $g * g$ is contained in $V^2 \subset U$ and by Fubini's Theorem (Theorem 4.5) we therefore have $\int g * g = 1$. But then

$$|g * g * \overline{f}(x) - \overline{f}(x)| = \left|\int_y g * g(y)\overline{f}(y^{-1}x) - \overline{f}(x)\right| = \left|\int_y g * g(y)(\widetilde{f}(x^{-1}y) - \widetilde{f}(x^{-1}))\right| \leqslant \epsilon,$$

for all $x \in G$ and so $\|\overline{f} - g * g * \overline{f}\|_\infty \leqslant \epsilon/2$.

Let $\delta < \epsilon\|g\|_2^{-1}\|\widetilde{f}\|_2^{-1}/2$ for reasons which will be come clear shortly. By Proposition 5.10 there is a finite dimensional invariant space $W \leqslant C(G)$ such that $\|h * g\|_2 \leqslant \delta\|h\|_2$ for all $h \in W^\perp$. Write $\pi_W : C(G) \to C(G)$ for the map projecting onto W. Then $g - \pi_W(g) \in W^\perp$ and so $\|g * g - \pi_W(g) * g\|_2 \leqslant \delta\|g - \pi_W(g)\|_2 \leqslant \delta\|g\|_2$. By Lemma 5.6 (v) we have

$$\|g * g * \overline{f} - \pi_W(g) * g * \overline{f}\|_\infty \leqslant \delta\|g\|_2\|\widetilde{f}\|_2.$$

By the triangle inequality we have $\|\overline{f} - \pi_W(g) * g * \overline{f}\|_\infty < \epsilon$. Finally, writing $k := (g * \overline{f})^\sim$ we have by definition; since λ_x is unitary; since W is invariant; since π_W is self-adjoint (meaning $\langle \pi_W v, w\rangle = \langle v, \pi_W w\rangle$ for all $v, w \in C(G)$); and again since λ_x is unitary, that

$$\begin{aligned}
\pi_W(g) * g * \overline{f}(x) = \langle \pi_W(g), \lambda_x(k)\rangle &= \langle \lambda_{x^{-1}}(\pi_W(g)), k\rangle \\
&= \langle \pi_W(\lambda_{x^{-1}}(\pi_W(g))), k\rangle \\
&= \langle \lambda_{x^{-1}}(\pi_W(g)), \pi_W(k)\rangle \\
&= \langle \pi_W(g), \lambda_x(\pi_W(k))\rangle = \overline{\langle \lambda_x(\pi_W(k)), \pi_W(g)\rangle}.
\end{aligned}$$

Hence $\overline{\pi_W(g) * g * \overline{f}(x)}$ is a matrix coefficient. Since $\epsilon > 0$ was arbitrary the result is proved. $\square$

Remark 5.12. ⚠ There are other important parts to the Peter-Weyl Theorem which we have not included here.

6 The dual group

Suppose that G is a topologized group. We write $\widehat{G}$ for the set of continuous homomorphisms $G \to S^1$ (where S^1 is as in Example 1.11), and call these **characters**. ⚠ While characters are elements of $C(G)$, they are *not* in $C_c(G)$ unless G is compact.

Example 6.1 (Characters of the circle group)**.** For $n \in \mathbb{Z}$ the maps $S^1 \to S^1; z \mapsto z^n$ are continuous homomorphisms of S^1 and so characters. As it happens these are the only characters but we shall not show this here.

Example 6.2 (Sign of permutations)**.** Suppose that S_n is endowed with a topology making it a left topological group such that A_n is topologically closed. Then the map $S_n \to S^1; \sigma \mapsto \operatorname{sgn}(\sigma)$ is a continuous homomorphism.

Example 6.3 (§1.24, contd.)**.** For m a square-free natural number, and $r \in \mathbb{Z}$ the maps $\mathbb{Z}_{\mathrm{SF}} \to S^1; z \mapsto \exp(2\pi i z r/m)$ are continuous homomorphisms and so characters.

Example 6.4 (Legendre symbol)**.** Given a finite Abelian Hausdorff topological group G, if it has a unique element of order 2, then it has a unique character of order 2. (In fact this is an 'if and only if' which can either be proved directly or by combining what follows with Proposition 6.5, Remark 6.13, and Theorem 6.22.)

The map $G \to G; x \mapsto x^2$ is a homomorphism and so its image, S, is a subgroup. By hypothesis it has kernel of size 2 and so by Lagrange's Theorem S has index 2 in G. Define the map $\chi : G \to S^1$ by $\chi(x) = 1$ if $x \in S$ and $\chi(x) = -1$ is $x \in G\backslash S$. This is a homomorphism, and since G is finite and Hausdorff it is discrete and hence χ is continuous. On the other hand, if $\chi' : G \to S^1$ is a character of order 2 then its kernel must contain S. Since χ' is non-trivial and S has index 2 in G, we have $\ker \chi' = S$ and hence $\chi = \chi'$. In other words, G has a unique character of order 2.

For p prime the multiplicative group $\mathbb{F}_p^*$ is cyclic and if it is odd this group has even order so has a unique element of order 2. The corresponding unique character of order 2 is called the Legendre symbol and features in number theory.

The set $\widehat{G}$ has a natural topology on it and to define this we make some notation: for K a compact subset of G and $\delta > 0$ write

$$U(K, \delta) := \{\gamma \in \widehat{G} : |\gamma(x) - 1| < \delta \text{ for all } x \in K\}.$$

Proposition 6.5. *Suppose that G is a topologized group. Then $\widehat{G}$ is a group with multiplication and inversion defined by*

$$(\gamma, \gamma') \mapsto (x \mapsto \gamma(x)\gamma'(x)) \text{ and } \gamma \mapsto (x \mapsto \overline{\gamma(x)}),$$

and identity, $1_{\widehat{G}}$, the character taking the constant value 1; and there is a topology on $\widehat{G}$ making it into a Hausdorff Abelian topological group with $(U(K,\delta))_{K,\delta}$ as K ranges compact subsets of G and $\delta > 0$, is a neighbourhood base of the identity.

Proof. The fact that $\widehat{G}$ is an Abelian group with the given multiplication, inversion, and identity follows since S^1 is an Abelian group under multiplication and $z^{-1} = \overline{z}$ when $z \in S^1$.

For the topological aspects of the proposition we begin by showing that

$$\tau := \left\{ \bigcup_{\gamma \in \Gamma} \gamma U(K_\gamma, \delta_\gamma) : \Gamma \subset \widehat{G} \text{ and for all } \gamma \in \Gamma \text{ the set } K_\gamma \text{ is compact and } \delta_\gamma > 0 \right\}$$

is a topology. To this end if $\lambda \in \gamma U(K,\delta) \cap \gamma' U(K',\delta')$, then by compactness of K (resp. K') there is $\delta_0 < \delta$ (resp. $\delta_0' < \delta'$) such that $|(\lambda\overline{\gamma})(x) - 1| \leqslant \delta_0$ for all $x \in K$ (resp. $|(\lambda\overline{\gamma'})(x) - 1| \leqslant \delta_0'$ for all $x \in K'$). Let $K_\lambda := K \cup K'$, which is compact, and $\delta_\lambda := \min\{\delta - \delta_0, \delta' - \delta_0'\}$, which is positive, and suppose that $\mu \in \lambda U(K_\lambda, \delta_\lambda)$. By the triangle inequality

$$|(\mu\overline{\gamma})(x) - 1| = |(\mu\overline{\gamma})(x)\overline{\lambda}(x) - \overline{\lambda}(x)| \leqslant |(\mu\overline{\gamma})(x)\overline{\lambda}(x) - 1| + |\overline{\lambda}(x) - 1| < \delta_\lambda + \delta_0 < \delta \text{ for } x \in K,$$

and similarly $|(\mu\overline{\gamma})(x) - 1| < \delta'$ for all $x \in K'$. In other words, $\lambda U(K_\lambda, \delta_\lambda) \subset \gamma U(K,\delta) \cap \gamma' U(K',\delta')$, and hence taking the union over $\lambda \in \gamma U(K,\delta) \cap \gamma' U(K',\delta')$ we have that $\gamma U(K,\delta) \cap \gamma' U(K',\delta') \in \tau$ as required.

Since τ is a topology, to see that $(U(K,\delta))_{K,\delta}$ is a neighbourhood base of the identity it is enough to note that if $1_{\widehat{G}} \in \gamma U(K,\delta)$ then, again by compactness, there is $\delta' < \delta$ such that $|\gamma(x) - 1| \leqslant \delta'$ for all $x \in K$ and hence $U(K, \delta - \delta') \subset \gamma U(K,\delta)$.

Now suppose that $\gamma\lambda \in \mu U(K,\epsilon)$ for some $\mu \in \widehat{G}$. Since $\gamma\lambda\overline{\mu}$ is continuous and K is compact there is some $\delta > 0$ such that $|(\gamma\lambda\overline{\mu})(x) - 1| < \epsilon - \delta$ for all $x \in K$. But then if $\gamma' \in \gamma U(K, \delta/2)$ and $\lambda' \in \lambda U(K, \delta/2)$ we have

$$\begin{aligned} |(\gamma'\lambda'\overline{\mu})(x) - 1| &\leqslant |(\gamma'\lambda'\overline{\mu})(x) - (\gamma\lambda'\overline{\mu})(x)| + |(\gamma\lambda'\overline{\mu})(x) - (\gamma\lambda\overline{\mu})(x)| + |(\gamma\lambda\overline{\mu})(x) - 1| \\ &< \delta/2 + \delta/2 + \epsilon - \delta = \epsilon. \end{aligned}$$

It follows that $\gamma'\lambda' \in \mu U(K,\epsilon)$ and so the preimage of $\gamma\lambda$ contains a neighbourhood of (γ,λ) in $\widehat{G} \times \widehat{G}$ *i.e.* multiplication is continuous.

Since $\widehat{G}$ is paratopological and hence semitopological Corollary 2.6 and the fact that $\overline{U(K,\delta)} = U(K,\delta)$ tells us inversion is continuous; we conclude that $\widehat{G}$ is a topological group. Finally, the topology is Hausdorff since if $\gamma \neq \lambda$ then there is some $x \in G$ such that $\gamma(x) \neq \lambda(x)$; put $\epsilon := |\gamma(x) - \lambda(x)|/2$ and note that $\gamma U(\{x\},\epsilon)$ and $\lambda U(\{x\},\epsilon)$ are disjoint open sets containing γ and λ respectively. □

We call the topology above the **compact-open** topology, the topological group $\widehat{G}$ the **dual group** of G, and its identity the **trivial character**.

Example 6.6 (Dual group of indiscrete topological groups)**.** When G is a group with the indiscrete topology the only continuous functions are constant, and hence there is only one continuous homomorphism, the trivial character. It follows that $\widehat{G_{\mathrm{I}}} = \{1_{\widehat{G_{\mathrm{I}}}}\}$, and since there is only one topology and one group structure on a set of size 1 this completely determines the dual group.

Example 6.6 gave topological reasons for the dual group being trivial, but there can also be algebraic reasons:

Example 6.7 (Dual group of non-Abelian simple groups)**.** Suppose that G is a non-Abelian simple[10] topological group. Since G is non-Abelian there are elements $x, y \in G$ with $xy \neq yx$, but then $xyx^{-1}y^{-1} \neq 1_G$. If $\gamma \in \widehat{G}$ then

$$\gamma(xyx^{-1}y^{-1}) = \gamma(x)\gamma(y)\gamma(x)^{-1}\gamma(y)^{-1} = 1$$

since S^1 *is* Abelian. We conclude that the kernel of γ is non-trivial, but all kernels are normal subgroups and since G is simple it follows that $\ker \gamma = G$ *i.e.* γ is trivial. In other words $\widehat{G} = \{1_{\widehat{G}}\}$.

Proposition 6.8. *Suppose that G is a compact topologized group. Then $\widehat{G}$ is discrete.*

Proof. Suppose that $\gamma \neq 1_{\widehat{G}}$ so there is $x \in G$ such that $\gamma(x) \neq 1$. Let $y \in G$ be such that $|\gamma(y) - 1|$ is maximal (which exists since G is compact and $x \mapsto |\gamma(x) - 1|$ is continuous) and note that by assumption this is positive. If $|\gamma(y) - 1| < 1$ then we have

$$\begin{aligned} |\gamma(y^2) - 1| = |\gamma(y)^2 - 1| &= |(2 + (\gamma(y) - 1))||\gamma(y) - 1| \\ &\geqslant (2 - |\gamma(y) - 1|)|\gamma(y) - 1| > |\gamma(y) - 1|. \end{aligned}$$

This is a contradiction, whence $\gamma \notin U(G, 1)$ and so $\{1_{\widehat{G}}\} = U(G, 1)$ is open so the topology is discrete. $\square$

Example 6.9 (Dual group of discrete finite cyclic groups)**.** Suppose that C is a finite cyclic group endowed with the discrete topology. Since C is cyclic it is generated by some element x, and the map

$$\phi : C \to \widehat{C}; x^r \mapsto (C \to S^1; x^l \mapsto \exp(2\pi i r l/|C|))$$

is a well-defined homeomorphic isomorphism. To see this note that ϕ is well-defined in the sense that different representations of an element in the domain produce the same image: $x^r = x^{r'}$ implies $|C| \mid r - r'$ and hence $\exp(2\pi i r l/|C|) = \exp(2\pi i r' l/|C|)$; and ϕ is well-defined in the sense that $\phi(x^r)$ as defined is genuinely an element of $\widehat{C}$: $x^l = x^{l'}$

[10] A **simple group** is a group whose only normal subgroups are the trivial group and the whole group *e.g.* A_n, the alternating group on n elements, when $n \geqslant 5$ as shown in Part A: Group Theory.

implies $|C| \mid l - l'$ and hence $\exp(2\pi i r l/|C|) = \exp(2\pi i r l'/|C|)$ so that $\phi(x^r)$ is itself a well-defined function; it is continuous since C is discrete; and it is a homomorphism since $\exp(2\pi i r(l+l')/|C|) = \exp(2\pi i r l/|C|)\exp(2\pi i r l'/|C|)$.

ϕ is a homomorphism since $\exp(2\pi i(r+r')l/|C|) = \exp(2\pi i r l/|C|)\exp(2\pi i r' l/|C|)$. ϕ is injective since if $\exp(2\pi i r l/|C|) = 1$ for all l then $|C| \mid r$ so $x^r = 1_C$. ϕ is surjective since if $\gamma : C \to S^1$ is a homomorphism then $\gamma(x)^{|C|} = 1$ so $\gamma(x) = \exp(2\pi i r/|C|)$ for some $r \in \mathbb{Z}$, and $\gamma = \phi(x^r)$.

We conclude that $\phi : C \to \widehat{C}$ is a bijective group homomorphism and hence ϕ^{-1} is a group homomorphism. Since C is finite, C is compact and so $\widehat{C}$ is discrete by Proposition 6.8 and hence ϕ^{-1} is continuous as required.

There is only one infinite cyclic group up to isomorphism so to complement the above we have:

Example 6.10 (The dual of $\mathbb{Z}_{\mathrm{D}}$)**.** The map

$$\phi : S^1 \to \widehat{\mathbb{Z}_{\mathrm{D}}}; z \mapsto (\mathbb{Z}_{\mathrm{D}} \to S^1; n \mapsto z^n)$$

is well-defined because any map from a discrete group is continuous, and it certainly takes $z \in S^1$ to homomorphisms of $\mathbb{Z}$. ϕ is visibly a homomorphism. If $\phi(z) = \phi(w)$ then $z = \phi(z)(1) = \phi(w)(1) = w$, so ϕ is injective; and any homomorphism $\mathbb{Z} \to S^1$ is determined by where it maps 1, and so ϕ is surjective.

Compact subsets of discrete topological spaces are finite, and so if $K \subset \mathbb{Z}_{\mathrm{D}}$ is compact then there if $m \in \mathbb{N}^*$ such that $|n| \leqslant m$ for all $n \in K$. Then for $\delta > 0$, if $|z - w| < \delta/m$ and $n \in K$ we have

$$|z^n - w^n| = |z^{|n|} - w^{|n|}| \leqslant |z - w|(|z|^{|n|-1} + \cdots + |w|^{|n|-1}) \leqslant |z - w||n| < \delta.$$

In other words ϕ is continuous.

The continuous image of a compact set is compact and so $\widehat{\mathbb{Z}_{\mathrm{D}}}$ is compact. It is also Hausdorff by Proposition 6.5, and so by the Open Mapping Theorem (Theorem 2.34), ϕ is a homeomorphic isomorphism. In words the dual of $\mathbb{Z}_{\mathrm{D}}$ is homeomorphically isomorphic to S^1.

Orthogonality of characters

Characters on compact Abelian topological groups are particularly useful because they convert topological information into algebraic information. For this subsection it is useful to use the normalisation of Lemma 5.4, specifically $\int$ denotes the unique left Haar integral on G with $\int 1 = 1$ and we write

$$\langle f, g\rangle := \int_x f(x)\overline{g(x)} \text{ for all } f, g \in C(G).$$

Lemma 6.11. *Suppose that G is a compact Abelian topological group, $\gamma, \gamma' \in \widehat{G}$. Then*

$$\langle \gamma, \gamma' \rangle = \begin{cases} 1 & \text{if } \gamma = \gamma' \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The first case is immediate. If $\gamma \neq \gamma'$ then there is $y \in G$ such that $\gamma(y) \neq \gamma'(y)$. Now

$$\gamma'(y)\langle \gamma, \gamma' \rangle = \langle \gamma, \lambda_y(\gamma') \rangle = \langle \lambda_{y^{-1}}(\gamma), \gamma' \rangle = \gamma(y)\langle \gamma, \gamma' \rangle,$$

and so $\langle \gamma, \gamma' \rangle = 0$. The result is proved. □

Lemma 6.12. *Suppose that G is a compact Abelian topological group $\Gamma \subset \widehat{G}$ is finite, $\omega : \Gamma \to \mathbb{C}$, and $k \in \mathbb{N}^*$. Then*

$$\int_x \left| \sum_{\gamma \in \Gamma} \omega(\gamma)\overline{\gamma(x)} \right|^{2k} = \sum_{\gamma_1 \cdots \gamma_k = \gamma_1' \cdots \gamma_k'} \omega(\gamma_1) \cdots \omega(\gamma_k)\overline{\omega(\gamma_1') \cdots \omega(\gamma_k')}.$$

Proof. We expand out the integrand:

$$\begin{aligned} \left| \sum_{\gamma \in \Gamma} \omega(\gamma)\overline{\gamma(x)} \right|^{2k} &= \left(\sum_{\gamma \in \Gamma} \omega(\gamma)\overline{\gamma(x)} \right)^k \overline{\left(\sum_{\gamma' \in \Gamma} \omega(\gamma')\overline{\gamma'(x)} \right)^k} \\ &= \sum_{\gamma_1, \ldots, \gamma_k, \gamma_1', \ldots, \gamma_k' \in \Gamma} \omega(\gamma_1) \cdots \omega(\gamma_k)\overline{\omega(\gamma_1') \cdots \omega(\gamma_k')} \cdot (\overline{\gamma_1 \cdots \gamma_k}\gamma_1' \cdots \gamma_k')(x). \end{aligned}$$

Apply $\int$ to both sides. By linearity of $\int$ and Lemma 6.11 the contribution to the right hand side is 0 unless $\gamma_1 \cdots \gamma_k = \gamma_1' \cdots \gamma_k'$ in which case it is 1. The result is proved. □

Remark 6.13. The dual group of a countable group can be uncountable as Example 6.10 shows, but the dual group of a finite group must be finite: Indeed, if G is finite then $x^{|G|} = 1$ for all $x \in G$ and so if $\chi : G \to S^1$ is a homomorphism then it must map into the $|G|$th roots of unity. Since there are at most $|G|^{|G|}$ maps from G into the set of $|G|$th roots of unity, it follows that $|\widehat{G}| \leqslant |G|^{|G|}$ and so is finite.

The orthogonality of characters, however, gives a stronger bound: if G is finite, then G is, in particular, compact and so the elements of $\widehat{G}$ are orthonormal and hence linearly independent elements of $C(G)$. The space $C(G)$ is a subspace of the space of functions $G \to \mathbb{C}$ and so has dimension at most $|G|$, whence $|\widehat{G}| \leqslant |G|$.

Local compactness in the dual group

We can make use of the Haar integral we have developed to show that if G is a locally compact topological group then the dual group is also locally compact. To do this we need a lemma.

Lemma 6.14. *Suppose that G is a locally compact topological group supporting a Haar integral $\int$, $f_0 \in C_c^+(G)$ has $\int f_0 \neq 0$, and $\kappa, \delta > 0$. Then there is an open neighbourhood of the identity $L_{\delta,\kappa}$ such that if $\left|\int f_0\gamma\right| \geqslant \kappa$ then $|1-\gamma(y)| < \delta$ for all $y \in L_{\delta,\kappa}$.*

Proof. Write K for a compact set containing the support of f_0 and U for a compact neighbourhood of the identity. UK is compact by Lemma 2.18. Apply Corollary 3.15 to get a continuous compactly supported $F : G \to [0,1]$ such that $F(x) = 1$ for all $x \in UK$.

By Proposition 3.9 there is an open neighbourhood of the identity $L_{\delta,\kappa}$ (which we may assume is contained in U since U is a neighbourhood and so contains an open neighbourhood of the identity) such that $\|\lambda_y(f_0)-f_0\|_\infty < \delta\kappa/\int F$ for all $y \in L_{\delta,\kappa}$. (Note $\int F > 0$ by Corollary 4.11.) For $y \in L_{\delta,\kappa}$, the support of $\lambda_y(f_0) - f_0$ is contained in UK (since $L_{\delta,\kappa} \subset U$) and so

$$\int |\lambda_y(f_0) - f_0| \leqslant \|\lambda_y(f_0) - f_0\|_\infty \int F < \delta\kappa.$$

Now, if $y \in L_{\delta,\kappa}$ then

$$\begin{aligned}|1-\gamma(y)|\kappa \leqslant \left|(\gamma(y)-1)\int f_0\gamma\right| &= \left|\int f_0\lambda_{y^{-1}}(\gamma) - \int f_0\gamma\right| \\ &= \left|\int \lambda_y(f_0)\gamma - \int f_0\gamma\right| \leqslant \int |\lambda_y(f_0) - f_0| < \delta\kappa.\end{aligned}$$

Dividing by κ gives the claim. □

Theorem 6.15. *Suppose that G is a locally compact topological group. Then $\widehat{G}$ is locally compact.*

Proof. Let $\int$ be a left Haar integral on G (which exists by Theorem 4.13). Since $\int$ is non-trivial there is $f_0 \in C_c^+(G)$ such that $\int f_0 \neq 0$ and we may rescale so that $\int f_0 = 1$. Write K for a compact set containing the support of f_0 and define

$$V := \{\gamma \in \widehat{G} : |\gamma(x) - 1| \leqslant 1/4 \text{ for all } x \in K\},$$

so that V certainly contains, $U(K, 1/4)$, an open neighbourhood of the identity.

Write M for the set of maps $G \to S^1$ endowed with the product topology so that M is a (non-empty) product of compact set and so compact. This is Tychonoff's Theorem, and our approach here has parallels with another place we used Tychonoff's Theorem: in the proof of Theorem 4.13. The set $\widehat{G}$ is contained in the set M, but the compact-open topology on $\widehat{G}$ is *not*, in general, the same as that induced on $\widehat{G}$ as a subspace of M. Our aim is to make use of the compactness on M to show that $\widehat{G}$ is locally compact in the compact-open topology.

First we restrict to homomorphisms: write H for the set of homomorphisms $G \to S^1$, which is a closed subset of M since it is the intersection over all pairs $x, y \in G$ of the set of $f \in M$ such that $f(xy) = f(x)f(y)$, which are closed since the maps evaluation maps $f \mapsto f(x)$ are closed in the product topology. With the sets $L_{\delta,3/4}$ as in Lemma 6.14 write

$$C := \bigcap_{\delta>0, x\in L_{\delta,3/4}} \{f \in H : |f(x) - 1| \leqslant \delta\}$$

which is also closed as an intersection of closed sets. By Proposition 2.5 as sets we have $C \subset \widehat{G}$ since the sets $\{z \in S^1 : |1-z| \leqslant \delta\}$ form a neighbourhood base of the identity in S^1, and if $f \in C$ then $f^{-1}(\{z \in S^1 : |1-z| \leqslant \delta\}) \supset L_{\delta,3/4}$ which is a neighbourhood of the identity in G.

If $\gamma \in V$ then $\left|1 - \int f_0\gamma\right| \leqslant \int f_0|1-\gamma| \leqslant 1/4$, so by the triangle inequality $\left|\int f_0\gamma\right| \geqslant 3/4$ and hence Lemma 6.14 tells us that $\gamma \in C$. Thus (as sets) $V \subset C \subset \widehat{G}$ and so

$$V = \bigcap_{x\in K} \{f \in C : |f(x) - 1| \leqslant 1/4\},$$

which is again a closed subset of M.

Our aim is to show that V is compact in the compact-open topology on $\widehat{G}$. This follows if every cover of the form $\mathcal{U} = \{\gamma U(K_\gamma, \delta_\gamma) : \gamma \in V\}$ (where K_γ is compact and $\delta_\gamma > 0$) has a finite subcover. Write $L_\gamma := L_{\delta_\gamma/2,1/2}$ (where these sets are as in Lemma 6.14 applied to f_0) and note that by compactness of K_γ there is a finite set T_γ such that $K_\gamma \subset T_\gamma L_\gamma$. Write

$$U_\gamma := \{f \in M : |f(x) - 1| < \delta_\gamma/2 \text{ for all } x \in T_\gamma\}$$

which is an open set in M since T_γ is finite. Suppose that $\lambda \in (\gamma U_\gamma) \cap V$ for some $\gamma \in V$. Then since $\gamma, \lambda \in V$, the triangle inequality gives

$$\begin{aligned}\left|1 - \int f_0\overline{\gamma}\lambda\right| &\leqslant \int f_0|1-\overline{\gamma}\lambda| = \int f_0|1-\overline{\gamma}+\overline{\gamma}-\overline{\gamma}\lambda| \\ &\leqslant \int f_0|1-\gamma| + \int f_0|1-\lambda| \leqslant 1/2.\end{aligned}$$

Hence $\left|\int f_0\overline{\gamma}\lambda\right| \geqslant 1/2$ by the triangle inequality again. Lemma 6.14 applied towith f_0 gives $|1-\overline{\gamma(y)}\lambda(y)| < \delta_\gamma/2$ for all $y \in L_\gamma$. But $\overline{\gamma}\lambda \in U_\gamma$ so we also have $|1-\overline{\gamma(z)}\lambda(z)| < \delta_\gamma/2$ for all $z \in T_\gamma$. Thus, if $x \in K_\gamma$ then there is $z \in T_\gamma$ and $y \in L_\gamma$ such that $x = zy$ and

$$\begin{aligned}|1-\overline{\gamma(x)}\lambda(x)| &\leqslant |1-\overline{\gamma(z)}\lambda(z)| + |\overline{\gamma(z)}\lambda(z) - \overline{\gamma(zy)}\lambda(zy)| \\ &= |1-\overline{\gamma(z)}\lambda(z)| + |1-\overline{\gamma(y)}\lambda(y)| < \delta_\gamma.\end{aligned}$$

We conclude that $\gamma U_\gamma \cap V \subset \gamma U(K_\gamma, \delta_\gamma) \cap V$. Finally $\{\gamma U_\gamma : \gamma \in V\}$ is a cover of V by sets that are open in M. M is compact and V is closed as a subset of M so V is compact as a subset of M, and hence $\{\gamma U_\gamma : \gamma \in V\}$ has a finite subcover which leads to a finite subcover of our original cover $\mathcal{U}$. The result is proved. □

Reflexive topological groups

For G a topological group write

$$\alpha_G : G \to \widehat{\widehat{G}}; x \mapsto (\gamma \mapsto \gamma(x)).$$

Observation 6.16. Certainly $\alpha_G(xy)(\gamma) = \gamma(xy) = \gamma(x)\gamma(y) = \alpha_G(x)(\gamma)\alpha_G(y)(\gamma)$ for all $x, y \in G$ and $\gamma \in \widehat{G}$, and so α_G *is* a homomorphism.

When α_G is a homeomorphic isomorphism we say that G is **reflexive**.

To analyse the continuity of α_G in the following auxiliary lemma can be useful.

Lemma 6.17. *Suppose that G is a locally compact topological group. Then the map*

$$G \times \widehat{G} \to S^1; (x, \gamma) \mapsto \gamma(x) \tag{6.1}$$

is continuous.

Proof. For $\delta > 0$ and $\gamma \in \widehat{G}$ the set $\{x \in G : |\gamma(x) - 1| < \delta/2\}$ is an open neighbourhood of 1_G and so by regularity of G (Proposition 2.29) there is an open neighbourhood of 1_G, call it L, such that $\overline{L} \subset \{x \in G : |\gamma(x) - 1| < \delta/2\}$. We may take L to be a subset of a compact set since G is locally compact, whence $\overline{L}$ is a subset of the closure of a compact set which is compact by Lemma 2.32.

Suppose further that $x \in G$, and $(x', \gamma') \in xL \times \gamma U(x\overline{L}, \delta/2)$ – an open neighbourhood of (x, γ). Then

$$\begin{aligned} |\gamma'(x') - \gamma(x)| &\leqslant |\gamma'(x') - \gamma(x')| + |\gamma(x') - \gamma(x)| \\ &= |(\gamma'\overline{\gamma})(x') - 1| + |\gamma(x^{-1}x') - 1| < \delta, \end{aligned}$$

and the result follows. □

For $\Lambda \subset \widehat{G}$ compact and $\delta > 0$ define

$$\mathrm{Bohr}(\Lambda, \delta) := \{x \in G : |\gamma(x) - 1| < \delta \text{ for all } \gamma \in \Lambda\}.$$

Proposition 6.18. *Suppose that G is a locally compact topological group. Then α_G is a continuous homomorphism.*

Proof. By Proposition 2.5 it suffices to show that the sets $\mathrm{Bohr}(\Lambda, \delta)$ are open for $\Lambda \subset \widehat{G}$ compact and $\delta > 0$. Fix $x_0 \in \mathrm{Bohr}(\Lambda, \delta)$. For each $\lambda \in \Lambda$, Lemma 6.17 gives us open neighbourhoods of the respective identities $U_\lambda \subset G$ and $\Gamma_\lambda \subset \widehat{G}$ such that $x_0U_\lambda \times \lambda\Gamma_\lambda$ is a subset of $\{(x, \gamma) : |\gamma(x) - 1| < \delta\}$. The sets $\{\lambda\Gamma_\lambda : \lambda \in \Lambda\}$ form an open cover of Λ and so there is a finite subcover $\lambda_1\Gamma_{\lambda_1}, \ldots, \lambda_m\Gamma_{\lambda_m}$ of Λ; let $U' := U_{\lambda_1} \cap \cdots \cap U_{\lambda_m}$. Then $x_0U' \subset \mathrm{Bohr}(\Lambda, \delta)$, and the set is open as required. □

Remark 6.19. In Exercise IV.8 we shall see that there are reflexive topological groups that are not locally compact. On the other hand, in [MP95] it is shown that if G is a reflexive topological group such that the map in (6.1) is continuous then G is locally compact, and so Exercise IV.8 gives an example where the map in (6.1) is not continuous.

To analyse the injectivity of α_G we use the following consequence of the Peter-Weyl Theorem:

Proposition 6.20. *Suppose that G is a compact Hausdorff Abelian topological group and $x \neq 1_G$. Then there is $\gamma \in \widehat{G}$ such that $\gamma(x) \neq 1$.*

Proof. By Corollary 3.15 there is a continuous $f \in C(G)$ such that $f(x) \neq f(1)$. By Theorem 5.11 there is an inner product space V, $v, w \in V$ and a continuous homomorphism $\pi : G \to \mathrm{U}(V)$ such that $|f(z) - \langle \pi(z)v, w\rangle| < \frac{1}{2}|f(x) - f(1)|$ for all $z \in G$. In particular, by the triangle inequality $\pi(x)v \neq \pi(1_G)v = v$.

Let $W := \{u \in V : \pi(x)u = u\}$ and $V_0 := \{u \in V : \langle u, u'\rangle = 0 \text{ for all } u' \in W\}$. Then

(i) $\pi(y)v \in V_0$ for all $y \in G$ and $v \in V_0$, since $\langle \pi(y)v, u'\rangle = \langle v, \pi(y)^*u'\rangle = \langle v, u'\rangle = 0$ for all $u' \in W$;

(ii) and $V_0 \neq \{0\}$ since if $u' \in W$ then $\langle \pi(x)v - v, u'\rangle = \langle v, \pi(x)^*u'\rangle - \langle v, u'\rangle = 0$, so $0 \neq \pi(x)v - v \in V_0$.

Suppose that $i \in \mathbb{N}_0$ and $\pi(y)v \in V_i$ for all $v \in V_i$ and $y \in G$ and $V_i \neq \{0\}$. If there is $y \in G$ such that $\pi(y)$ is not a scalar multiple of the identity on V_i then let V_{i+1} be an eigenspace of $\pi(y)$ restricted to V_i corresponding, say, to some eigenvalue λ_{i+1}. We have $0 < \dim V_{i+1} < \dim V_i$; since G is Abelian, if $v \in V_{i+1}$ and $z \in G$ then

$$\pi(y)(\pi(z)v) = \pi(yz)v = \pi(zy)v = \pi(z)(\pi(y)v) = \pi(z)(\lambda_{i+1}v) = \lambda_{i+1}\pi(z)v$$

and so $\pi(z)v \in V_{i+1}$. By induction we conclude that this terminates with some space $V_j \neq \{0\}$ such that $\pi(z)$ is a scalar multiple of the identity on V_j for every $z \in G$; let $\gamma(z)$ be this scalar. Since π is a continuous homomorphism, so is γ (indeed, γ is a matrix coefficient). Moreover since $V_j \neq \{0\}$ there is $0 \neq v' \in V_j$ and $\gamma(x)v' = \pi(x)v' \neq v'$ since $V_j \leqslant V_0$, whence $\gamma(x) \neq 1$ and the result is proved. $\square$

Remark 6.21. We cannot relax any of the hypotheses in a strong sense: if we drop 'Abelian' then Example 6.7 shows that the dual may be trivial; if we drop 'Hausdorff' then Example 6.6 shows that the dual may be trivial; and if we drop 'compact' then Exercise IV.5 shows that the dual may be trivial.

Theorem 6.22 (Pontryagin duality for compact Hausdorff Abelian topological groups)**.** *Suppose that G is a compact Hausdorff Abelian topological group. Then G is reflexive.*

Proof. By Proposition 6.5 and Theorem 6.15, $\widehat{\widehat{G}}$ is a locally compact Hausdorff Abelian topological group, so by Proposition 6.18 and the Open Mapping Theorem (Theorem 2.34) it is enough to show that α_G is a bijection. First, it is injective by Proposition 6.20. Secondly, since G is compact and α_G is continuous, the set $\alpha_G(G)$ is compact and so closed (since $\widehat{\widehat{G}}$ is Hausdorff). Hence for surjectivity of α_G it is enough to show that the image of α_G is dense.

To show that α_G has dense image, suppose that $\phi \in \widehat{\widehat{G}}$. By Proposition 6.8 the group $\widehat{G}$ is discrete and hence any compact subset is finite and so it is enough to show that for finite $\Gamma \subset \widehat{G}$ there is $y \in G$ such that $\phi(\gamma) = \gamma(y)$ for all $\gamma \in \Gamma$. To begin suppose, as we may, that Γ includes the trivial character, and define

$$f(x) := \sum_{\gamma \in \Gamma} \phi(\gamma)\overline{\gamma(x)} \text{ and } g(x) := \sum_{\gamma \in \Gamma} \overline{\gamma(x)}.$$

By Lemma 6.12 and the fact that ϕ is a homomorphism, for any $k \in \mathbb{N}^*$, we have

$$\begin{aligned}\int_x |f(x)|^{2k} &= \sum_{\substack{\gamma_1,\dots,\gamma_k,\gamma_1,\dots,\gamma_k' \in \Gamma \\ \gamma_1\cdots\gamma_k = \gamma_1'\cdots\gamma_k'}} \phi(\gamma_1)\cdots\phi(\gamma_k)\overline{\phi(\gamma_1')}\cdots\overline{\phi(\gamma_k')} \\ &= \sum_{\substack{\gamma_1,\dots,\gamma_k,\gamma_1,\dots,\gamma_k' \in \Gamma \\ \gamma_1\cdots\gamma_k = \gamma_1'\cdots\gamma_k'}} \phi(\gamma_1\cdots\gamma_k\overline{\gamma_1'}\cdots\overline{\gamma_k'}) = \sum_{\substack{\gamma_1,\dots,\gamma_k,\gamma_1,\dots,\gamma_k' \in \Gamma \\ \gamma_1\cdots\gamma_k = \gamma_1'\cdots\gamma_k'}} 1 = \int_x |g(x)|^{2k}.\end{aligned}$$

Since f and g are continuous, by Lemma 4.12 we have $\|f\|_\infty = \|g\|_\infty$. But $g(0_G) = |\Gamma|$ and $|g(x)| \leqslant |\Gamma|$, whence there is $y \in G$ such that $|f(y)| = |\Gamma|$.

Since $\phi(1_{\widehat{G}}) = 1$, $1_{\widehat{G}}(y) = 1$, and $|\phi(\gamma)\overline{\gamma(y)}| = 1$ for all $\gamma \in \Gamma$ we have $|f(y) - 1| \leqslant |\Gamma| - 1$. Hence $|f(y)|^2 + 1 - 2\operatorname{Re} f(y) \leqslant |\Gamma|^2 - 2|\Gamma| + 1$, but $|f(y)| = \Gamma$ and so $\operatorname{Re} f(y) \geqslant |\Gamma|$ and

$$\sum_{\gamma \in \Gamma} |\phi(\gamma) - \gamma(y)|^2 = 2|\Gamma| - 2\operatorname{Re} f(y) \leqslant 0.$$

It follows that each summand is 0 *i.e.* $\phi(\gamma) = \gamma(y)$ for all $\gamma \in \Gamma$. The result is proved. □

Remark 6.23. In view of Proposition 6.5, if G is a reflexive topological group then G must be Hausdorff and Abelian, and the above can be extended to show that all locally compact Hausdorff Abelian topological groups are reflexive – this is often called Pontryagin duality.

On the other hand, Exercise IV.8 shows that there are reflexive groups that are *not* locally compact.

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