Analytic Topology: Problem sheet 5

This entire sheet is **optional**.

Any extra material in this part of the question sheet is off the syllabus. These questions, some of which are hard, are intended for further study, or entertainment.

1. One statement of the *Baire Category Theorem* is that any intersection of countably many dense open subsets of $\mathbb{R}$ is non-empty.

(i) Deduce from the above statement that any intersection of countably many dense open subsets of $\mathbb{R}$ is dense.

(ii) We say that a subset of $\mathbb{R}$ is of *first category* if it is contained in the complement of an intersection of countably many dense sets. Intuitively, one can regard first category sets as being small, and the Baire Category Theorem says that $\mathbb{R}$ is not of first category. Prove that any countable union of first category sets is of first category and so is not small.

(iii) Prove the Baire Category Theorem.

[*Hint: Remember that any intersection of a strictly decreasing sequence of closed bounded intervals is non-empty.*]

(iv) (For those who know some measure theory.) Find a subset Q of $\mathbb{R}$ such that $\mathbb{R} \setminus Q$ is of first category, and Q is null.

2. (i) Let A be a subset of $\mathbb{R}$ having the property that for all points x of $\mathbb{R}$ except at most one, there exists an open neighbourhood U of x such that $A \cap U$ is countable. Prove that A is countable.

(ii) Deduce that if A is an uncountable subset of $\mathbb{R}$, there exist (at least) two distinct points x and y such that for every open neighbourhood U of either x or y, $U \cap A$ is uncountable.

(iii) Prove that every uncountable closed subset of $\mathbb{R}$ contains a homeomorphic copy of the Cantor set.

(iv) (Harder) Prove that any intersection of countably many dense open subsets of $\mathbb{R}$ contains a homeomorphic copy of the Cantor set.

3. (i) Let X be a compact zero-dimensional metric space. Note that X is second countable, by Urysohn's Metrisation Theorem. Prove that every clopen subset of X is a finite union of basic open sets. Prove that $\mathscr{B}X$ is countable.

(ii) Let $\mathbb{P}$ be $\wp\mathbb{N}$, considered as a Boolean algebra. Define $h : \mathbb{N} \to \mathscr{S}\mathbb{P}$ so that $h(n) = \{A \subseteq \mathbb{N} : n \in A\}$. Then h is continuous. (Why?) Let $f : \mathbb{N} \to [0, 1]$ be any function. Define a continuous function $g : \mathscr{S}\mathbb{P} \to [0, 1]$ so that for all $n \in \mathbb{N}$, $g(h(n)) = f(n)$, and deduce that $\mathscr{S}\mathbb{P}$ is homeomorpic to $\beta\mathbb{N}$.

[*Hint: Suppose that $p \in \mathscr{S}\mathbb{P}$. What is p? What value might you choose for $g(p)$?*]

(iii) (Much easier.) Now deduce that $\beta\mathbb{N}$ is not metrisable.

4. (Quite hard.) The *Sorgenfrey Line* $\mathbb{S}$ is the real line with the topology generated by sets of the form $(a, b]$, for real numbers a and b. The *Michael Line* $\mathbb{M}$ is the real line equipped with basic open sets of the form $\{r\}$ for irrational r and $(r - \frac{1}{n}, r + \frac{1}{n})$ for r rational and n a natural number.

(i) Prove that the Michael Line is T_4.

(ii) In the product $(\mathbb{R}\setminus\mathbb{Q})\times\mathbb{M}$, let C be the set of points (x, y) such that y is rational, and let D be the set of points (x, x), for x irrational. Prove that C and D are disjoint and closed, but that if U and V are any open sets such that $C \subseteq U$ and $D \subseteq V$, then $U \cap V \neq \varnothing$. Deduce that this product is not normal.

[*Intuitively, this is because, letting $V_{(x,x)}$ be a neighbourhood of the point (x, x) in D such that $V_{(x,x)}$ is contained in V, then too many of the sets $V_{(x,x)}$ are too big and crowd too close together so as to force U and V to intersect. To make this argument formal, you will need the Baire Category Theorem, from further up the sheet.*]

(iii) Prove that the Sorgenfrey Line is T_4.

(iv) Prove that in $\mathbb{S}\times\mathbb{S}$, any subset of the antidiagonal $\{(x, -x) : x \in \mathbb{R}\}$ is closed.

(v) In $\mathbb{S}\times\mathbb{S}$, let C be the set of points of the antidiagonal with rational coordinates, and let D be the set of points of the antidiagonal with irrational coordinates. Prove that if U and V are open sets such that $C \subseteq U$ and $D \subseteq V$, then $U \cap V \neq \varnothing$, and deduce that $\mathbb{S}\times\mathbb{S}$ is not normal.

5. (For those who know some set theory.) Assume ZFC.

(i) Prove that every uncountable closed subset of $\mathbb{R}$ has cardinality $2^{\aleph_0}$.

(ii) Prove that there are $2^{\aleph_0}$ closed subsets of $\mathbb{R}$. [*Hint: how many open subsets are there?*]

(iii) (Hard, and requiring some form of Choice.) A *Bernstein Set* is a subset B of $\mathbb{R}$ such that both B and its complement meet every uncountable closed set. Prove that a Bernstein set exists.