String Theory 1

Lecture #2

Chapter 1 Classical relativistic string

↳ study relativistic classical string propagating in a fixed spacetime $\mathcal{M}$

1.1 Classical relativistic point particle ← in a way that is generalisable to strings

1.2 Classical relativistic string: action principle

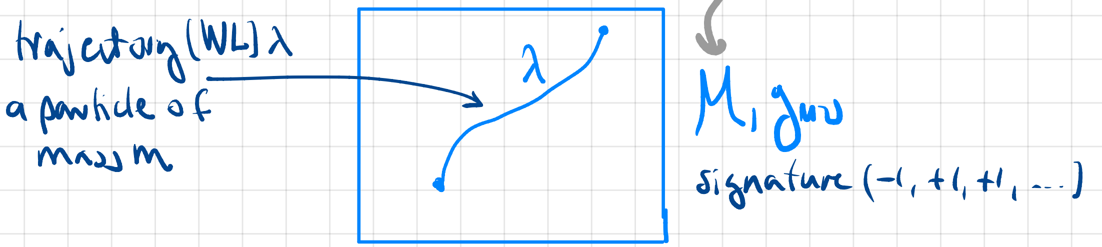
1.3 ...

1.1 Classical relativistic point particle

To motivate the formalism describing the string we first review the motion of a relativistic point-like particle

Consider a particle of mass m traveling along a WL λ in spacetime M with metric $g_{\mu\nu}$.

D dimensional space time



The equations of motion fix the trajectory of the particle to be the one with minimal proper length ($\int_\lambda ds$) ie the particle moves along a geodesic.

Then the action is

$$S[\lambda] = -m \int_\lambda ds$$

dimensionless

$[m] = L^{-1}$

units $c = \hbar = 1$

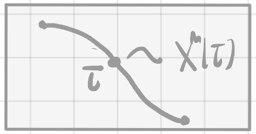
$[L] = [\text{time}] = [\text{mass}]^{-1} = [\text{energy}]^{-1}$

Polchinski exercise (1.1): m indeed the particle's mass (look at non-relativistic limit and see that this gives the appropriate KE & potential energy.

We treat this action as a "0+1" dimensional field theory:

let X^μ $\quad \mu = 0, \dots, D-1$ spacetime coordinates on $\mathcal{M}$

We can introduce a parametrisation of λ, $\tau \in \mathbb{R}$ and associate coordinates $X^\mu(\tau)$ to each point on λ. This is an embedding of λ into $\mathcal{M}$



$$\lambda : \begin{array}{rcl} \mathbb{R} & \longrightarrow & \mathcal{M} \\ \tau & \longmapsto & X^\mu(\tau) \end{array}$$

The point on λ with param τ has coords $X^\mu(\tau)$ on $\mathcal{M}$

<u>GR 1</u>:

use the embedding and the metric on $g_{\mu\nu}$ on $\mathcal{M}$

$$ds = \underbrace{\sqrt{-g_{\mu\nu}\, dX^\mu\, dX^\nu}}_{\text{induced (pullback) metric on } \lambda} = \sqrt{-g_{\mu\nu} \frac{dX^\mu}{d\tau} \cdot \frac{dX^\nu}{d\tau}}\; d\tau$$

line element on λ

infinitesimal proper length of λ

$$S[X] = \int_\tau d\tau \sqrt{-g_{\mu\nu} \frac{dX^\mu}{d\tau} \cdot \frac{dX^\nu}{d\tau}}$$

Symmetries of the action:

▶ 1-dimensional reparametrisation invariance

S is a function of λ CM & we do not care about the choice of parametrisation

$$\tau \longrightarrow \tilde{\tau}(\tau)$$

local diffeomorphisms on λ

to mantain inv. under reparams

$$X^\mu(\tau) \longrightarrow \tilde{X}^\mu(\tilde{\tau}) = X^\mu(\tau)$$

fields X^μ transform as WL scalars

($\frac{dX^\mu}{d\tau}$ WL vectors)

This is a gauge symmetry (redundancy of the description of the motion of the particle)

One can use the reparametrisation invariance to get rid of this redundancy (gauge fix)
For example we can choose $X^0 = t =$ time coord in spacetime. Then

$$S = -m \int dt \sqrt{\underbrace{-\underline{\dot{X}}\cdot\underline{\dot{X}} + 1}_{1-\underline{v}^2}} = S[\underline{X}], \quad \left(\underline{\dot{X}} = \frac{d\underline{X}}{dt}, \quad \underline{X} = (X^1, \ldots, X^{D-1})\right)$$

$-m(1-\underline{v}^2)^{1/2} = -m + \underbrace{\tfrac{1}{2}mv^2}_{KE} + \ldots$

In fact it seems that the motion is described by in terms of the X^μ; that is, seemingly there are D-degrees of freedom. However the motion of the particle should be entirely described by (D-1) spatial coords (plus initial conditions).

▶ The isometry group of $\mathcal{M}$ leaves invariant the line element

If $g_{\mu\nu} = \eta_{\mu\nu}$ (flat Minkowski metric) $\Rightarrow$ spacetime Poincaré invariance

$$X^\mu(\tau) \longrightarrow \Lambda^\mu{}_\nu X^\nu(\tau) + b^\mu, \quad \Lambda \in SO(1, D-1), \quad b \in \mathbb{R}^{1,D-1}$$

isometry group of D-dim Minkowski space

More generally, the space time isometry group is realised as internal (global) symmetries of the WL field theory.

S is a fine classical action but there are two problems:

- it has a square-root $\Rightarrow$ difficult to quantize
  eg not quadratic in the time derivatives!
- what happens if $m = 0$?

To circumvent these problems, look for another "nicer" action which gives the same EOM for the massive particle and describes appropriately massless particles

Consider $\boxed{\text{instead}}$ the action

$$\tilde{S}[e,X] = \frac{1}{2}\int e(\tau)\left(e(\tau)^{-2} g_{\mu\nu}\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau} - m^2\right)d\tau$$

where we introduce a new (auxiliary) field $e(\tau)$ on λ

EOM for e are: $0 = \dfrac{\delta \tilde{S}}{\delta e(\tau)} \iff g_{\mu\nu}\dfrac{dX^\mu}{d\tau}\dfrac{dX^\nu}{d\tau} + e(\tau)^2 m^2 = 0$

For $m \neq 0$ this fixes $e(\tau)$ completely in terms of X^μ

& substituting this back into $\tilde{S}$ we get S

$$\boxed{\tilde{S} \text{ \& } S \text{ are classically equivalent.}}$$

Symmetries of $\tilde{S}$

- spacetime isometries (Poincaré for Minkowski) with $e(\tau)$ invariant
- reparametrization of the world line

$$\tau \longmapsto \tilde{\tau}(\tau)$$

scalars on λ $\quad X^\mu(\tau) \longmapsto \tilde{X}^\mu(\tilde{\tau}) = X^\mu(\tau)$

$$e(\tau) \longmapsto \tilde{e}(\tilde{\tau}) = \frac{d\tau}{d\tilde{\tau}}\, e(\tau)$$

to maintain rep inv.

$(e(\tau)d\tau$ is invariant$)$

We can use reparametrisation invariance to gauge fix $e(\tau)$.

It is convenient to set $e(\tau)$ to be a constant.

let
$$e(\tau) = \begin{cases} 1/m & m \neq 0 \\ 1 & m = 0 \end{cases}$$

$$\tilde{S}_{\text{fixed}} = \begin{cases} \frac{1}{a} m \int \left(g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} - 1 \right) d\tau & m \neq 0 \\ \frac{1}{a} \int g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \, d\tau & m = 0 \end{cases}$$

EOM for X

Euler Lagrange equations $\rightsquigarrow$ geodesic equation (GR1)

$$\frac{d^2X^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta} \frac{dX^\alpha}{d\tau} \frac{dX^\beta}{d\tau} = 0$$

Γ = Christoffel symbols associated to $g_{\mu\nu}$

τ = proper time on WL

EOM for $e(\tau)$: gives an extra constraint.

$m \neq 0$ $\quad g_{\mu\nu} \frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau} + 1 = 0 \Rightarrow \frac{dX^\mu}{d\tau}$ is the TL 4-velocity

(TL geodesic)

$m = 0$ $\quad g_{\mu\nu} \frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau} = 0 \Rightarrow \frac{dX^\mu}{d\tau}$ is the Null 4-velocity

(N geodesic)

Remark: we have fixed the gauge to eliminate redundancies in the description of the system but note that EOM of the extra degrees of freedom are still very important

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Conclusion:

- $\mathcal{S}$ is a good starting point for quantization ← through path integral quant

  (now $\mathcal{S}$ is in fact quadratic in time derivatives)

- "0+1" field theory

Could add interactions, build up Feynman diagrams in a first-quantized theory.

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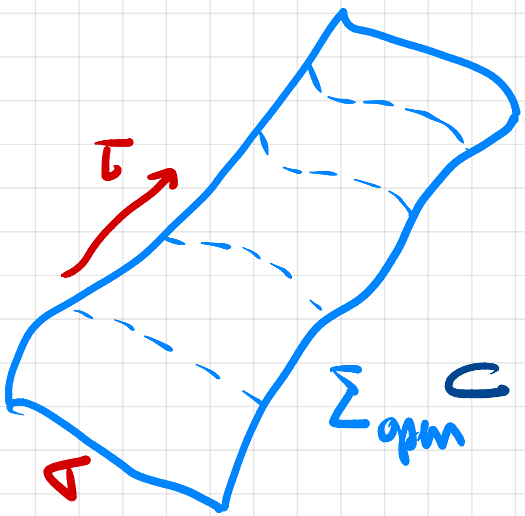
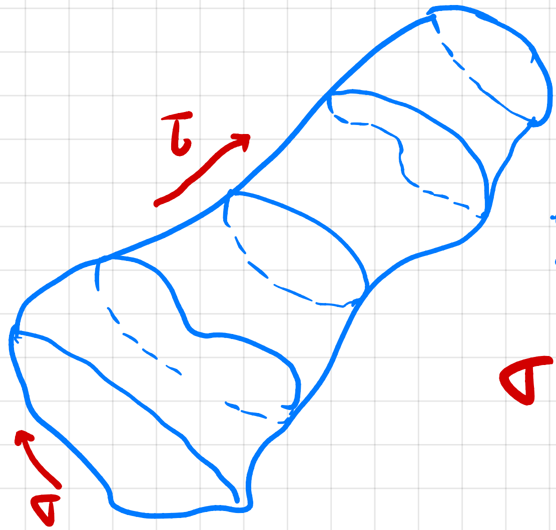
1.2 Classical relativistic string

Generalise to a string

A string sweeps out a 2dim world sheet Σ in M
(1dim object



$\Sigma_{open} \subset M$



$\Sigma_{closed} \subset M$

σ is periodic
$\sigma \equiv \sigma + \ell$

$\sigma \in [0, \ell]$ ℓ = "length" measured in some arbitrary units on Σ

coords on the WS: $\xi^{a} = (\xi^0, \xi^1) = (\tau, \sigma)$, $a = 0, 1$

↳ parametrisation of WS

time coord τ, spatial coordinate σ

Embedding of $\Sigma \hookrightarrow \mathcal{M}$ ← target space

$$(\tau, \sigma) \longmapsto X^\mu(\tau, \sigma)$$

where for closed strings $X^\mu(\tau, \sigma + \ell) = X^\mu(\tau, \sigma)$ same point in space time

$\sim X^\mu(\tau, \sigma) = X^\mu(\tau, \sigma + \ell)$

We now consider a "1+1" dimensional field theory on Σ with fields $X^\mu(\tau, \sigma)$.

Nambu-Goto action: natural generalisation of $S[\lambda]$

(analogue of $S[\lambda] = -m \int_\lambda ds$)

S_{NG} : action which describes the classical motion of a string along minimal area surfaces

$$S_{NG}[\Sigma] = \underbrace{-T}_{} \int_\Sigma dA$$

"tension" ~mass/length $[T]=L^{-2}$

area of Σ

area element (2 dim volume form) $[dA] = L^2$

Euler Lagrange eqs: classical motion of the string along minimal area surfaces.

$$dA = \sqrt{-h}\, d\tau d\sigma = \sqrt{-(\partial_\tau X \cdot \partial_\tau X)(\partial_\sigma X \cdot \partial_\sigma X) + (\partial_\tau X \cdot \partial_\sigma X)^2}\; d\tau d\sigma$$

$$h_{ab} = g_{\mu\nu}(X(\xi)) \frac{\partial X^\mu}{\partial \xi^a} \frac{\partial X^\nu}{\partial \xi^b} = \text{induced worldsheet metric}$$

(pullback of $g_{\mu\nu}$ onto Σ by the embedding $\Sigma \hookrightarrow M$)

$h = \det h_{ab}$

Notation $u \cdot V = g_{\mu\nu} u^\mu V^\nu$ for space-time vectors

What is T? T is interpreted as the string tension

i.e. mass of the string per unit length

ref: 1) D. Tong lecture notes 2) Polchinski problem 1.1
3) Becker + Becker + Schwarz exercise 2.7 with solution

Remark: $T = \frac{1}{2\pi\alpha'}$ α' = Regge slope (historical reasons)

$[T] = L^{-2}$ $[\alpha'] = L^2$

$l_s = 2\pi\sqrt{\alpha'}$ string length scale

so $T = \frac{2\pi}{l_s^2} \Rightarrow$

$M_s = \frac{1}{\sqrt{\alpha'}}$ string mass scale

units $\hbar = c = 1$

$$T = \frac{1}{2\pi} \frac{1}{(l_s / 2\pi)^2} = \frac{2\pi}{l_s^2}$$

$[E] = [m] = [L]^{-1} = [\text{time}]^{-1}$

Symmetries of the NG-action (just as before for $S[\lambda]$)

- 2-dimensional reparametrisation invariance

S is a function of $\Sigma \subset \mathcal{M}$ and we don't care about the parametrization of Σ

can change our choice of parametrization of Σ

$$(\tau,\sigma) \longrightarrow (\tilde{\tau}(\tau,\sigma), \tilde{\sigma}(\tau,\sigma))$$

WS diffeomorphisms

to maintain S invariant

$$X^\mu(\tau,\sigma) \longrightarrow \tilde{X}^\mu(\tilde{\tau},\tilde{\sigma}) = X^\mu(\tau,\sigma)$$

fields X^μ transform like WS scalars

Again: reparametrisations are a gauge symmetry

- The isometry group of $\mathcal{M}$ (the target) leaves invariant the area element

If $g_{\mu\nu} = \eta_{\mu\nu}$ (flat Minkowski metric) $\Rightarrow$ spacetime Poincaré invariance
for example

$$X^\mu(\tau,\sigma) \longrightarrow \Lambda^\mu{}_\nu X^\nu(\tau,\sigma) + b^\mu, \text{ where } \Lambda \in SO(1,D-1),\ b \in \mathbb{R}^{1,D-1}$$

isometry group of Minkowski space

This is an "internal" symmetry ie a global symmetry wrt Σ

S gives a nice classical theory, it describes a 2dim field theory on Σ,

One can compute the Euler-Lagrange eqs (which extremise the area of Σ)

$$\partial_a\left(\sqrt{-h}\, h^{ab}(X)\; g_{\mu\nu}(X(\xi))\, \partial_b X^\nu\right] = 0$$

(where $\delta\sqrt{-h} = \frac{1}{2}\sqrt{-h}\, h^{ab}\,\delta h_{ab}$)

and study the classical dynamics of a string (PS 1)

Even in flat space $g_{\mu\nu} = \eta_{\mu\nu}$ this is hard! it is non linear

So: not clear how to quantise.

The Polyakov action:

Consider

$= h_{ab}$ induced metric on Σ as a subspace of M

$$\boxed{S_P[\gamma_{ab}, X^\mu] = -\frac{T}{2}\int_\Sigma d\tau\, d\sigma \sqrt{-\gamma}\,\gamma^{ab}\,\partial_a X^\mu \partial_b X^\nu g_{\mu\nu}}$$

where we have introduced new fields on Σ

$\boxed{\xi^a = (\tau, \sigma)}$

$\gamma_{ab}(\xi)$ Lorentzian world-sheet metric (auxiliary field)

$\gamma = \det(\gamma_{ab})$

EOM (δS wrt δX): $\partial_a \left(\sqrt{-\gamma}\,\gamma^{ab} g_{\mu\nu}(X)\,\partial_b X^\nu\right) = 0$

looks very similar to that for S_{NG} but here γ is an indep variable !

(In flat space this is linear in X^μ)

Symmetries of the Polyakov action

WS perspective → global symmetries

▸ spacetime isometries
(Poincaré invariance when M = Minkowski)
and γ does not transform

▸ World sheet reparametrisation $\xi^a \longmapsto \tilde{\xi}^a(\xi)$ diffeos of Σ

already there in S_{NG}

$$\gamma_{ab}(\xi) \longmapsto \tilde{\gamma}_{ab}(\tilde{\xi}) = \gamma_{cd}(\xi)\,\frac{\partial \xi^c}{\partial \tilde{\xi}^a}\frac{\partial \xi^d}{\partial \tilde{\xi}^b}$$

symmetric 2 tensor on Σ

&

$$X^\mu(\xi) \longmapsto \tilde{X}^\mu(\tilde{\xi}) = X^\mu(\xi)$$

(WS scalars)

Special to 2 dims

• Weyl invariance ie local scale symmetry acting on the 2dim metric on Σ

function on Σ

$$\gamma_{ab} \longmapsto e^{2\omega(\xi)}\gamma_{ab}, \quad X^\mu \text{ invariant}$$

$$[\sqrt{|\gamma|} \mapsto e^{2\omega}\sqrt{\gamma}\ ;\ \gamma^{ab} \mapsto e^{-2\omega}\gamma^{ab}]$$

Weyl invariance is also a gauge symmetry

Weyl invariance very important in quantisation: anomalous unless $D=26$!

↳ $\boxed{\text{Next}}$

1.2 Classical relativistic string: continued

1.1 General classical solutions