String Theory 1

Lecture #6

Chapter 2 Old covariant quantisation

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2.1 Introduction

Classical theory

$$S_p = -\frac{T}{2}\int_\Sigma d\tau\, d\sigma\, \left(-\partial_\tau X\cdot\partial_\tau X + \partial_\sigma X\cdot\partial_\sigma X\right)$$

in the conformal unit gauge $\gamma_{ab} = \eta_{ab}$.

This is supplemented by the constraints

$$T_{++} = 0 \quad \& \quad T_{--} = 0$$

The OCQ approach consists on promoting

the canonical fields X^μ

& their conjugate momenta $\Pi^\mu = T \partial_\tau X^\mu$ $\rightsquigarrow$ operators

and

the Poisson brackets $\{\cdot, \cdot\}_{PB}$ $\rightsquigarrow$ commutators of operators $i[\cdot, \cdot]$

We get the canonical equal time commutation relations

$$[\Pi^\mu(\tau,\sigma), X^\nu(\tau,\sigma')] = -i\,\delta(\sigma-\sigma')\,\eta^{\mu\nu}$$

(with $[X^\mu(\sigma), X^\nu(\sigma')] = 0$, $[P^\mu(\sigma), P^\nu(\sigma')] = 0$)

The operators X^μ & Π^μ are hermitian

$$X^\mu = (X^\mu)^\dagger, \qquad \Pi^\mu = (\Pi^\mu)^\dagger$$

this replaces the reality conditions of the classical fields

$\{x^\mu, p^\mu, \alpha^\mu_n, (\tilde{\alpha}^\mu_n)\} \rightsquigarrow$ are now operators

The commutation relations for the oscillator modes follow immediately from this:

$$[\hat{p}^\mu, \hat{x}^\nu] = -i\eta^{\mu\nu}$$ $\hat{p}^\mu, \hat{x}^\nu$ are Hermitian

(Heisenberg algebra)

$$[\alpha^\mu_m, \alpha^\nu_n] = m\,\delta_{m+n,0}\,\eta^{\mu\nu}$$ $(\alpha^\mu_n)^\dagger = \alpha^\mu_{-n}$

$$\left([\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n] = m\,\delta_{m+n,0}\,\eta^{\mu\nu} \right.$$ $\left. (\tilde{\alpha}^\mu_n)^\dagger = \tilde{\alpha}^\mu_{-n} \right)$

This forms an infinite set of harmonic oscillators together with the Heisenberg pair $[x^\mu, p^\mu]$ $\left(\alpha_m \longrightarrow \frac{1}{\sqrt{m}}\alpha_m\right)$

Now we construct the Hilbert space in the usual way

2.2 Hilbert space (without constraints)

Identify oscillators
$$\begin{cases} \alpha^\mu_{-n} \ (\tilde{\alpha}^\mu_{-n}), & n>0 \quad \text{raising} \\ \alpha^\mu_n \ (\tilde{\alpha}^\mu_n), & n>0 \quad \text{lowering} \end{cases}$$

Define the oscillator vacuum state $|0\rangle_{vac}$

$$\alpha^\mu_m |0\rangle_{vac} = \tilde{\alpha}^\mu_m |0\rangle_{vac} = 0 \quad \forall m>0$$

ie the state which is annihilated by all $\alpha^\mu_m (\tilde{\alpha}^\mu_m)$ $\forall m>0$

On top of $|0\rangle_{vac}$, we build the osillator Fock spaces

ie states constructed by applying creation operators $\alpha^{\mu}_{-n}\ (\tilde{\alpha}^{\mu}_{-n})$, $n \geq 1$

$$\mathcal{H}^{Fock}_{open} = \mathrm{Span}\left\{ \prod_{i=1}^{k} \alpha^{\mu_i}_{-n_i} |0\rangle_{vac} \right\}_{n_i \geq 1}$$

$$\mathcal{H}^{Fock}_{closed} = \mathrm{Span}\left\{ \prod_{i=1}^{k} \alpha^{\mu_i}_{-n_i} \prod_{j=1}^{l} \tilde{\alpha}^{\mu_j}_{-n_j} |0\rangle_{vac} \right\}_{n_{i,j} \geq 1} = \underset{\mathcal{H}^{Fock}_{open}}{\underset{\cong}{\mathcal{H}^{Fock}_{R}}} \otimes \underset{\mathcal{H}^{Fock}_{open}}{\underset{\cong}{\tilde{\mathcal{H}}^{Fock}_{L}}}$$

It is useful to introduce oscillator number operators

$$N \equiv \sum_{k>0} \alpha_{-k} \cdot \alpha_k \qquad \tilde{N} \equiv \sum_{k>0} \tilde{\alpha}_{-k} \cdot \tilde{\alpha}_k$$

These satisfy:

$$[N, \alpha_n^\mu] = -n \, \alpha_n^\mu \qquad [N, \alpha_{-n}^\mu] = [N, (\alpha_n^\mu)^\dagger] = n \, \alpha_{-n}^\mu$$

(and similar for $[\tilde{N}, \tilde{\alpha}_n^\mu] = \cdots$)

N & $\tilde{N}$ are "counting" operators

$$N \left(\prod_{i=1}^{k} \alpha_{-n_i}^{\mu_i} |0\rangle_{vac} \right) = \left(\sum_{i=1}^{k} n_i \right) \left(\prod_{i=1}^{k} \alpha_{-n_i}^{k} |0\rangle_{vac} \right)$$

$$\tilde{N} \left(\prod_{i=1}^{k} \tilde{\alpha}_{-n_i}^{\mu_i} |0\rangle_{vac} \right) = \left(\sum_{i=1}^{k} n_i \right) \left(\prod_{i=1}^{k} \tilde{\alpha}_{-n_i}^{k} |0\rangle_{vac} \right)$$

Help organise oscillator states into "levels" (N & $\tilde{N}$ eigenstates)

Note that $N \geq 0$ (no states of negative level).

For open strings (or only R-movers closed string)

$N=0 \qquad |0\rangle_{vac}$

$N=1 \qquad \alpha^{\mu}_{-1}|0\rangle_{vac}$

$N=2 \qquad \alpha^{\mu}_{-2}|0\rangle_{vac}\,,\quad \alpha^{\mu_1}_{-1}\alpha^{\mu_2}_{-1}|0\rangle_{vac}$

$N=3 \qquad \alpha^{\mu}_{-3}|0\rangle_{vac}\,,\quad \alpha^{\mu_1}_{-2}\alpha^{\mu_2}_{-1}|0\rangle_{vac}\,,\quad \alpha^{\mu_1}_{-1}\alpha^{\mu_2}_{-1}\alpha^{\mu_3}_{-1}|0\rangle_{vac}$

$\vdots$

Hence

$$\mathcal{H}^{Fock}_{open} = \mathrm{span}\Big\{ \underbrace{\prod_{i=1}^{k} \alpha^{\mu_i}_{-n_i}|0\rangle_{vac}}_{\text{eigenstates of } N} \Big\}_{n_i\geq 1} = \bigoplus_{N=1}^{\infty} \mathcal{H}^{Fock}[N]$$

We are not done: we also have the zero modes $\{x^\mu, p^\nu\}$ (which commute with the oscillator modes).

We define the ground state (in momentum space)

$$|k,0\rangle$$

with the property that it is an eigenvector of the momentum operator

$$\hat{p}^\mu |k,0\rangle = k^\mu |k,0\rangle \qquad k^\mu \in \mathbb{R}^{1,D-1}$$

& it is normalised such that $\langle k'|k\rangle = \delta^{(D)}(k-k')$

Then $\mathcal{H}_{\text{zero-mode}} \simeq L^2(\mathbb{R}^{1,D-1})$

$|k,0\rangle$ in momentum space $\Longleftrightarrow$ $\psi(x) \otimes |0\rangle_{\text{vac}}$

wave fund in spacetime

The Hilbert space is then

$$\mathcal{H}_{\text{open}} = L^2(\mathbb{R}^{1,D-1}) \otimes \mathcal{H}^{\text{Fock}}_{\text{open}} = \text{Span}\left\{ \prod_{i=1}^{K} \alpha^{\mu_i}_{-n_i} |k,0\rangle \right\}_{n_i \geq 1}$$

$$\mathcal{H}_{\text{closed}} = L^2(\mathbb{R}^{1,D-1}) \otimes \mathcal{H}^{\text{Fock}}_{L} \otimes \mathcal{H}^{\text{Fock}}_{R}$$

$$= \text{Span}\left\{ \prod_{i=1}^{h} \prod_{j=1}^{\ell} \alpha^{\mu_i}_{-n_i} \tilde{\alpha}^{\nu_i}_{-m_j} |k,0\rangle \right\}_{n_i, m_j \geq 1}$$

States labelled by spacetime momentum k and tensor indices, so they fall into representations of space-time Poincaré group

Problem (or not?): Consider the state

$$|\varphi\rangle = \alpha^0_{-1}|0;K\rangle \qquad \text{(level 1 state)}$$

Then

$$\langle\varphi|\varphi'\rangle = \langle 0;K|\overbrace{\alpha^0_{+1}}^{(\alpha^0_{-1})^\dagger}\alpha^0_{-1}|0;K'\rangle = \eta^{00}\,\delta(K-K')$$

$$[\ ,\ ]$$

$$= -\,\delta(K-K')$$

Wrong sign! There are negative norm states (ghosts)
($\exists$ of negative norm states $\Rightarrow$ negative probabilities ???)

However: we have not imposed the constraints yet!

2.3 Constraints, normal ordering & Virasoro algebra

Recall the Witt-generators

$$L_m = \frac{1}{2}\sum_{k=-\infty}^{\infty} \alpha_{m-k}\cdot\alpha_k \,, \qquad L_m^{\dagger} = L_{-m} \quad (m\neq 0)$$

$$\tilde{L}_m = \frac{1}{2}\sum_{k=-\infty}^{\infty} \tilde{\alpha}_{m-k}\cdot\tilde{\alpha}_k \,, \qquad \tilde{L}_m^{\dagger} = \tilde{L}_{-m} \quad (m\neq 0)$$

quadratic in the oscillator operators: need a prescription for the ordering of the operators

The operators α_{m-k} & α_k commute $\forall k$ unless $m=0$

in which case $[\alpha^{\mu}_{-k}, \alpha^{\nu}_{k}] = 2k\,\eta^{\mu\nu}$.

Then only "problematic" operator is L_0 (& $\tilde{L}_0$)

(ie L_0 & $\tilde{L}_0$ are not determined by the classical expression)

Classically $L_0 = \frac{1}{2}\alpha_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(\alpha_{-n}\cdot\alpha_n + \alpha_n\cdot\alpha_{-n})$, $\tilde{L}_0 = \cdots$

Depending on how we order the raising/lowering operators (for each n) the action of L_0 (& $\tilde{L}_0$) on states can differ by a c-number (recall α's commute except $[\alpha^\mu_n, \alpha^\nu_{-n}] = n\eta^{\mu\nu}$)

To account properly for the normal ordering we $\boxed{\text{define}}$ the quantum operator L_0 as

$$L_0 = \frac{1}{2}\sum_{n\in\mathbb{Z}} \underbrace{:\alpha_{-n}\cdot\alpha_n:}_{\substack{\text{normal ordered product} \\ \text{(lowering operator to the right)}}} \equiv \frac{1}{2}\alpha_0^2 + \sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n = \frac{1}{2}\alpha_0^2 + N$$

where now the quantum constraint is

$$\boxed{(L_0 - a)|\psi\rangle = 0 \quad ((\tilde{L}_0 - \tilde{a})|\psi\rangle = 0)}$$

$\forall$ physical states $|\psi\rangle$

The constants a (and $\tilde{a}$) cannot be specified yet, but set $a = \tilde{a}$.

Virasoro algebra We need to check the commutator algebra of the operators L_m ($\tilde{L}_m$). A direct computation gives

PS 2

$$\boxed{[L_m, L_n] = (m-n)L_{m+n} + \frac{D}{12}(m^3-m)\delta_{m+n,0}}$$

(similar for $[\tilde{L}_m, \tilde{L}_n]$ for closed strings)

This is the Virasoro algebra with central charge D

We now call L_m the Virasoro generators.

The Virasoro algebra is a central extension of the Witt algebra by $c \in \mathbb{C}$

This can be expressed in terms of an exact sequence

central extension of Witt

$$
\begin{tikzcd}
0 \arrow[r] & \mathfrak{h} \arrow[r] & \mathrm{Vir} \arrow[r] & \mathrm{Witt} \arrow[r] & 0 \\
& \hat{c} : [\hat{c}, \hat{c}] = 0 \arrow[u] & \hat{c} = 0 \arrow[u, squiggly] & \mathrm{Vir}/\langle \hat{c} \rangle \arrow[u, squiggly] &
\end{tikzcd}
$$

$[\hat{c}, L_n] = 0$

The central extension is related to an anomaly of the Weyl invariance (more later).

[The global $sl(2)$ algebra generated by $\{L_0, L_1, L_{-1}\}$ (or $\{\tilde{L}_0, \tilde{L}_1, \tilde{L}_{-1}\}$) is not "anomalous".] $\quad m(m^2-1) = 0$

2.4 Constraints and physical states

↳ Imposing the constraints in the quantum theory to identify the physical states $\mathcal{H}_{phys} \subset \mathcal{H}$

Problem: $\boxed{\text{if}}$ we define $|\psi\rangle \in \mathcal{H}_{phys}$ to be states s.t.

$$L_m |\psi\rangle = 0$$

one finds a contradiction.

Consider $[L_m, L_n]|\Psi\rangle$

(a) $[L_m, L_n]|\Psi\rangle = (L_m L_n - L_n L_m)|\Psi\rangle = 0 \quad \forall m,n$

OTOH

(b) $[L_m, L_n]|\Psi\rangle = \left[\cancel{(m-n)L_{m+n}} + \frac{D}{12}(m^3-m)\delta_{m+n,0}\right]|\Psi\rangle$

$\neq 0 \quad m=-n$

So for $n=-m$

$$[L_m, L_{-m}]|\Psi\rangle = \frac{D}{12}(m^3-m)|\Psi\rangle$$

Imposing all constraints leads to a trivial Hilbert space when $D \neq 0$!

Instead we define physical states $\phi, \psi \in \mathcal{H}_{phys}$ by the constraints

matrix elements of L_m vanish

$$\boxed{\langle \psi | L_m | \phi \rangle = 0} \qquad \forall m \neq 0$$

Definition: a state $|\phi\rangle$ is physical if

- $L_m |\phi\rangle = 0 \qquad \forall\, m \geq 1$
- $(L_0 - a)|\phi\rangle = 0 \qquad$ for fixed $a \in \mathbb{R}$

- $\{L_m\}_{m\geq 1}$ form a closed subalgebra $\left([L_m, L_n] = (m-n) L_{m+n},\ m,n \geq 1\right)$
- $\langle \psi | L_m | \phi \rangle = \langle \phi | \underbrace{L_{-m} | \psi \rangle}_{=0}{}^{*} = 0 \qquad \forall \quad m \leq 1$

so $L_m|\phi\rangle = 0 \ \forall m \geq 0$ incorporates all constraints $m \neq 0$

For closed string:
require the same constraints for $\tilde{L}_m$ for a state to be physical

Important remark: the generators of D-dim spacetime Poincaré symmetries have no normal ordering ambiguities

In fact

$$[P^\mu, L_m] = [M^{\mu\nu}, L_m] = 0 \qquad \forall m$$

$\therefore$ P^μ & $M^{\mu\nu}$ preserve the physical state conditions. This means that states in the phys decompose into representations of $SO(1, D-1)$

(covariant quantisation)

In summary: so far we have

$$\mathcal{H}_{phys} = \left[\bigcap_{m=1}^{\infty} \mathrm{Ker}(L_m) \right] \cap \mathrm{Ker}(L_0 - a)$$

normal ordering

$L_m|\psi\rangle = 0$ $\forall m \geq 1$

$(L_0 - a)|\psi\rangle = 0$

$\forall |\phi\rangle, |\psi\rangle \in \mathcal{H}_{phys}$:

$\langle\phi|L_m|\psi\rangle = 0 \quad \forall m \neq 0$

Situation is even simpler: L_{+1} & L_{+2} generate every possible L_m $m>2$ so only need $L_1|\psi\rangle = 0$ & $L_2|\psi\rangle = 0$!

$([L_1, L_2] = -L_3, \ [L_1, L_3] = -2L_4, \ \text{etc} \ldots)$

$$\boxed{\mathcal{H}_{phys} = \mathrm{Ker}(L_2) \cap \mathrm{Ker}(L_1) \cap \mathrm{Ker}(L_0 - a)}$$

2.5 Mass-shell and level matching conditions

The L_0 (& $\tilde{L}_0$) conditions

Closed string

$L_0 = \frac{1}{2}\alpha_0^2 + N \qquad \alpha_0^\mu = \tilde{\alpha}_0^\mu = \sqrt{\frac{\alpha'}{2}}\, p$

mass shell condition

$$(L_0 + \tilde{L}_0 - 2a)|\varphi, k\rangle = 0 \iff \left(\frac{\alpha'}{2}k^2 + N + \tilde{N} - 2a\right)|\varphi, k\rangle = 0$$

so $\iff \boxed{\alpha' M^2 = 2(N + \tilde{N} - 2a)}$

level matching condition

$$(L_0 - \tilde{L}_0)|\varphi, k\rangle = 0 \iff (N - \tilde{N})|\varphi, k\rangle = 0 \quad \text{so} \quad \boxed{N = \tilde{N}}$$

Open string mass shell condition

$$(L_0 - a)|\varphi, k\rangle = 0 \iff (\alpha' p^2 + N - a)|\varphi, k\rangle = 0 \quad \text{so} \quad \boxed{\alpha' M^2 = N - a}$$

$\alpha_0^\mu = \sqrt{2\alpha'}\, p^\mu$

level 0 ie ground state $|0,k\rangle$

As an exercise, you can show that

$([N, \alpha_n^\mu] = -n\, \alpha_n^\mu)$

$$[N, L_m] = -m\, L_m$$

So L_m shifts N-level by $-m$ (similar for $\tilde{N}$ & $\tilde{L}$)

$N(L_m|\psi\rangle) = (n-m)|\psi\rangle$ for $N|\psi\rangle = n|\psi\rangle$

$\Rightarrow$ at level zero we only need to impose the L_0 conditions.

The L_0-conditions at level $N=0$ are

closed strings $\qquad \alpha' M^2 = -4a$

open strings $\qquad \alpha' M^2 = -a$

states with velocity > c !
q-instability to ground state not the true vac; maybe non-pert corrections fix this?
see ST II

$a<0$	massive	ground state
$a=0$	massless	ground state
$a>0$	tachyonic	ground state (!)

2.6 Level 1 states & dealing with ghosts

Recall: earlier we encountered a problem with negative norm states (ghosts) at level one of the open string (eg: $\alpha^0_{-1}|0;K\rangle$).
We want to see if this issue remains after applying the constraints.

Consider a general level one open string state

$$\xi\cdot\alpha_{-1}|0;K\rangle = |\xi;K\rangle \qquad \xi\in\mathbb{R}^{1,D-1}$$

polarization vector
(D-degrees of freedom)

& impose the physical state conditions for L_0 & L_{+1}.

The L_n conditions for $n\geq 2$ are satisfied automatically once the ones for L_0 & L_{+1} are imposed

$$L_m\,\xi\cdot\alpha_{-1}|0;K\rangle = \xi_\mu[L_m,\alpha^\mu_{-1}]|0;K\rangle = \xi\cdot\alpha_{m-1}|0;K\rangle$$

$$= 0 \qquad m\geq 2$$

- mass-shell condition $\quad \underline{-\alpha' k^2 = \alpha' M^2 = 1 - a}$

- L_{+1} condition:

$$L_{+1}(\zeta\cdot\alpha_{-1})|0;k\rangle = \eta_{\mu\nu}\zeta^\mu \underbrace{[L_{+1},\alpha^\nu_{-1}]}|0;k\rangle$$

$$= \eta_{\mu\nu}\zeta^\mu \alpha^\nu_0 |0;k\rangle = \sqrt{2\alpha'}\,\zeta\cdot k\,|0;k\rangle$$

$[L_m, \alpha^\mu_n] = -n\,\alpha^\mu_{m+n}$

$$L_{+1}|\zeta;k\rangle = 0 \iff \boxed{\zeta\cdot k = 0}$$

hence ζ has $D-1$ independent components

- Norm of a general level 1 state

$$\langle\zeta;k|\zeta';k'\rangle = \langle 0;k|(\zeta\cdot\alpha_{+1})(\zeta'\cdot\alpha_{-1})|0;k'\rangle$$

$$= \zeta_\mu\zeta'_\nu\langle 0;k|[\alpha^\mu_{+1},\alpha^\nu_{-1}]|0;k'\rangle = \zeta\cdot\zeta'\,\delta(k-k')$$

For $\zeta = \zeta'$ require $\zeta^2 \geq 0$ to avoid ghosts

$\hookrightarrow$ space like or null

level 1 summary

$\alpha' K^2 = a-1$: so K is $\begin{cases} \text{spacelike} & \text{if } a>1 \\ \text{lightlike} & \text{if } a=1 \\ \text{timelike} & \text{if } a<1 \end{cases}$

$\zeta \cdot K = 0$: transverse polarization

$\zeta^2 \geq 0$: polarization is Null or spacelike

to avoid ghosts

For $a>1$, K is spacelike (L_0 condition)

Then the constraint $\xi \cdot K = 0$ (L_1 condition)

is satisfied by timelike polarisations ξ, $\xi^2 < 0$.

That is, we would get negative norm states (ghosts).

So for $a>1$, the Virasoro constraints are not enough to eliminate them.

Thus we reject $a>1$ and require $a \leq 1$

Critical theory: Consider the case $a=1$ (threshold case)

$$a=1 \quad \Longrightarrow \quad K^2=0$$

$$\Longrightarrow \quad |\xi; K\rangle = \xi\cdot\alpha_{-1}|0;K\rangle$$ is a massless state

Additionally, there is a zero norm state

$(\xi\cdot K=0)$ $$|K;K\rangle = (K\cdot\alpha_{-1})|0;K\rangle$$ w/ longitudinal polarisation

This state is orthogonal to $\boxed{\text{all}}$ physical states $|\xi', K\rangle$

$$\langle K;K|\xi, K'\rangle = (K\cdot\xi)\,\delta(K-K') = 0$$ as $\xi\cdot K'=0$ for physical states

Hence: the longitudinal polarisation decouples leaving

$D-2$ physical polarisations like a photon!

However: for $a=1$ ground state is a tachyon

Remark: the decoupling of the longitudinal degree of freedom is due to the fact that it corresponds to a state that is "pure gauge":

$$L_{-1}|0;K\rangle = \sqrt{2\alpha'}\, K\cdot\alpha_{-1}|0;K\rangle = \sqrt{2\alpha'}\,|K;K\rangle$$

$$L_{-1}|0;K\rangle = \left(\frac{1}{2}\sum_{n=-\infty} \alpha_{-n-1}\cdot\alpha_{n}\right)|0,K\rangle = \frac{1}{2}\Big(\underset{k=0}{\alpha_{-1}\cdot\alpha_0} + \underset{k=-1}{\alpha_0\cdot\alpha_{-1}} + \sum_{n=2}^{\infty}\alpha_{n-1}\cdot\alpha_{-n}\Big)|0;K\rangle$$

$$= \alpha_{-1}\cdot\alpha_0|0;K\rangle = \sqrt{2\alpha'}\,K\cdot\alpha_{-1}|0;K\rangle$$

ie $|K;K\rangle$ is created by the action of L_{-1} which is a generator of a conformal transformation. In this sense we say that $|K;K\rangle$ is pure gauge state

$\hookrightarrow$ Virasoro constraints for $a=1$ precisely restrict the level 1 states to consistently describe space-time photons

$a < 1$ K is timelike

$\xi \cdot K = 0 \Rightarrow \xi$ spacelike

(D-1 degrees of freedom)

$\xi^2 > 0$ +ve norm states

massive vector boson in D-dims.

Next: Photons & $D = 26$

$a =$