String Theory 1

Lecture #7

Chapter 2 Old covariant quantisation

2.1 Introduction

2.2 Hilbert space (without constraints)

2.3 Constraints, Normal ordering and the Virasoro algebra

2.4 Constraints and physical states

2.5 Mass-shell and level matching conditions

2.6 Level 1 states & dealing with ghosts

2.7 Level 2 states & dealing with ghosts

2.8 Null states and $D=26$

2.9 Physical states of the closed string

In the last few lectures we have been discussing the OCQ of the relativistic string.

All physs constructed by imposing half the constraints

$$L_m|\phi\rangle = 0 \quad \forall\, m>0, \qquad (L_0 - a)|\phi\rangle = 0$$

a from normal order

What is then the role of L_{-m} $m \geq 1$?

The remaining L_{-m} $(m \geq 1)$ lead to redundancies in the spectrum corresponding to the associated gauge symmetries.

↳ next: discuss these redundancies

Summary: Physical states: $(L_0 - a)|\psi\rangle = 0$

$L_m|\psi\rangle = 0\,, \quad m \geq 1$

(only need to check $m = 1, 2$)

open strings

- Ground state

$|k, 0\rangle$

$a < 0$	massive	ground state
$a = 0$	massless	ground state
$a > 0$	tachyonic	ground state (!)

- $N = 1$

$\zeta \cdot \alpha_{-1}|0; k\rangle = |\zeta; k\rangle \qquad \zeta \in \mathbb{R}^{1,D-1}$ polarization vector (D-degrees of freedom)

$\alpha' k^2 = a - 1$: so k is $\begin{cases} \text{~~spacelike if } a > 1~~ \\ \text{lightlike if } a = 1 \\ \text{timelike if } a < 1 \end{cases}$

ghosts! so $a \leq 1$

$\zeta \cdot k = 0$: transverse polarization

$\zeta^2 \geq 0$: polarization is Null or spacelike to avoid ghosts

Critical theory: Consider the case $a=1$ (threshold case)

$N=0$: $|k,0\rangle$ tachyon $[\alpha' M^2 = -1 \text{ (OS)}]$

$N=1$: $|\xi;k\rangle = \xi\cdot\alpha_{-1}|0;k\rangle$ is a massless state $(k^2=0)$

photon: $D-2$ degrees of freedom as the longitudinal state $|k;k\rangle$ decouples

[orthogonal to all physical states $|\xi,k'\rangle$
$\langle k,k|\xi,k'\rangle = 0$]

$\hookrightarrow$ Virasoro constraints for $a=1$ precisely restrict the level 1 states to consistently describe space-time photons

Remark: the decoupling of the longitudinal degree of freedom is due to the fact that it corresponds to a state that is "pure gauge"

ie $|k;k\rangle$ is created by the action of L_{-1} (which is a generator of a conformal transformation). In this sense we say that $|k;k\rangle$ is pure gauge state

Indeed:

$$L_{-1}|0;k\rangle = \sqrt{2\alpha'}\, k\cdot\alpha_{-1}|0;k\rangle = \sqrt{2\alpha'}\,|k;k\rangle$$

$$L_{-1}|0;k\rangle = \Big(\frac{1}{2}\sum_{n=-\infty} \alpha_{-n-1}\cdot\alpha_n\Big)|0,k\rangle = \frac{1}{2}\Big(\underset{k=0}{\alpha_{-1}\cdot\alpha_0} + \underset{k=-1}{\alpha_0\cdot\alpha_{-1}} + \underbrace{\sum_{n=2}^{\infty} \alpha_{n-1}\cdot\alpha_{-n}}_{0}\Big)|0;k\rangle$$

$$= \alpha_{-1}\cdot\alpha_0|0;k\rangle = \sqrt{2\alpha'}\, k\cdot\alpha_{-1}|0;k\rangle$$

more on this soon

$\underline{a < 1}$ $\quad K$ is timelike

$\xi \cdot K = 0 \Rightarrow \xi$ spacelike

(D-1 degrees of freedom)

$\xi^2 > 0$ +ve norm states

<u>massive</u> vector boson in D-dims.

2.7 level 2 states & dealing with ghosts (OS)

Level 2 state for $a=1$ (threshold value as for $a>1$ there are ghosts in the phys.)

Consider the level 2 state $|\phi, K\rangle$ with momentum K

$$|\phi,K\rangle = [c_1\, \alpha_{-1}\cdot\alpha_{-1} + c_2\, \alpha_{-2}\cdot K + c_3\, (\alpha_{-1}\cdot K)(\alpha_{-1}\cdot K)]|0;K\rangle$$

onshell mass condition - $\alpha' K^2 = 1$ ← $\alpha' M^2 = N - a$

$L_1|\phi,K\rangle = 0$ iff $c_1 + c_2 - 2c_3 = 0$

$L_2|\phi,K\rangle = 0$ iff $D c_1 - 4c_2 - 2c_3 = 0$

so $|\phi,K\rangle = c_1[\alpha_{-1}\cdot\alpha_{-1} + \frac{1}{5}(D-1)\,\alpha_{-2}\cdot K + \frac{1}{10}(D+4)(\alpha_{-1}\cdot K)(\alpha_{-1}\cdot K)]|0,K\rangle$

is a physical state for any D

Norm $\langle \phi; K | \phi; K' \rangle = -\frac{2}{25} |c_1|^2 (D-1)(D-26) \; \delta(K-K')$

$\boxed{\text{no}}$ ghosts $\Rightarrow$ $1 \le D \le 26$ ($\exists$ ghosts for $D > 26$)

"null" (physical zero norm) states when $D=26$ (or $D=1$)

so when $a=1$ & $D=26$ there are more "null" states
(proliferation of null states @ $D=26$ sign of large gauge (conf.) symmetry)

1972 Brower ; Goddard, Thorn No ghost theorem

$\hookrightarrow$ $\boxed{\text{For } a=1 \text{ and } D=26 \text{ the phys has no ghosts}}$

Note: there are $\boxed{\text{no}}$ ghosts for $a \le 1$ & $1 \le D \le 25$
(this is a corollary of the no ghost theorem for the critical string)

BUT these theories are inconsistent at the level of string loops (need to look at one loop interactions)

2.8 Null states and $D=26$

<u>Definition</u>:

A state $|\psi\rangle$ is called spurious if

- it is orthogonal to **all** physical states and
- obeys $(L_0 - a)|\psi\rangle = 0$

A null state is a spurious state which is also physical.

A null state has <u>zero norm</u> as it is orthogonal to itself.

An example of a null state is $|k; k\rangle$ at level 1 & $a=1$, and this state is also <u>pure gauge</u>.

(this is expected in gauge theories to find such states as in the Gupta-Bleuler formulation of QED)

Null states are physical states that decouple from the dynamics.

These states are "quotiented out", that is

two physical states are equivalent if they differ by a null state

$$|\psi\rangle_{phys} \sim |\psi\rangle_{phys} + |\varphi\rangle_{null}$$

so we define $\mathcal{H}_{red} = \mathcal{H}_{phys} / \mathcal{H}_{null}$

physically $\boxed{\text{distinct}}$ states

We expect: physical states of the form $L_{-m}|\varphi\rangle$ $\forall m>0$ are null!

spurious states generated by the residual symmetries (conformal transformations)

Consider the state

$$|\psi\rangle = \sum_{m>0} L_{-m} |\chi_{-m}\rangle \quad \text{st} \quad L_0 |\chi_m\rangle = (a-m) |\chi_{-m}\rangle$$

Then $\langle\varphi|\psi\rangle = 0 \quad \forall \; |\varphi\rangle \in \mathcal{H}_{\text{phys}}$

$$(L_0 - a)|\psi\rangle = 0$$

$$\sum_{m>0} (L_0 - a) L_{-m} |\chi_m\rangle = \sum_{m>0} \big(\underbrace{[L_0, L_{-m}]}_{m L_{-m}} + L_{-m} L_0 - a L_{-m}\big)|\chi_{-m}\rangle = \sum_{m>0} L_{-m} \left(m - a + L_0 \right) |\chi_m\rangle$$

So $|\psi\rangle$ is spurious.

One can in fact prove that any spurious state is of this form, ie all spurious states are "pure gauge". (GSW p83)

[pure gauge states ~ states generated by residual symmetries (conformal symmetries!)]

this is expected in gauge theories

One can prove moreover that all spurious states are of the form

$$|\psi\rangle = L_{-1}|\chi_{-1}\rangle + L_{-2}|\chi_{-2}\rangle$$

with $L_0|\chi_{-1}\rangle = (a-1)|\chi_{-1}\rangle$, $L_0|\tilde{\chi}_{-2}\rangle = (a-2)|\chi_{-2}\rangle$

[for $m \geq 3$: can replace L_{-m} by commutators

$$[L_{-p}, L_{-q}] = (-p+q)L_{-p-q}, \quad 1 < p,q < m$$ (no central term as $-p-q \neq 0$)

eg $$[L_{-1}, L_{-2}] = L_{-3}$$

$$L_{-3}|\chi_{-3}\rangle = [L_{-1}, L_{-2}]|\chi_{-3}\rangle = L_{-1}(L_{-2}|\chi_{-3}\rangle) - L_{-2}(L_{-1}|\chi_{-3}\rangle)$$
$$= L_{-1}|\chi_1'\rangle + L_{-2}|\chi_2'\rangle$$

etc. ...]

see GSW p 83

$\boxed{\text{Null states}}$: (ie spurious states which are physical)

clearly spurious

► Consider first: $|\psi\rangle = L_{-1}|\chi\rangle$, $L_0|\chi\rangle = (a-1)|\chi\rangle$

with $L_m|\chi\rangle = 0 \quad \forall\ m>0$.

- $(L_0 - a)|\psi\rangle = 0$ ✓ by construction

- $L_1|\psi\rangle = [L_1, L_{-1}]|\chi\rangle = 2L_0|\chi\rangle = 2(a-1)|\chi\rangle$

  $= 0$ iff $\underline{a=1}$

- $L_2|\psi\rangle = [L_2, L_{-1}]|\chi\rangle = 3L_1|\chi\rangle = 0$

So, for $\underline{a=1}$ we get an infinite set of null states

$|\psi\rangle = L_{-1}|\chi\rangle$, $L_m|\chi\rangle = 0 \quad \forall m>0$, $L_0|\chi\rangle = 0$

(generalising the case $\sqrt{2\alpha'}\ |k;k\rangle = L_{-1}|0;k\rangle$).

► Now consider another example

$$|\psi\rangle = (L_{-2} + \gamma L_{-1}^2)|\chi_2\rangle, \quad L_m|\chi_2\rangle = 0 \quad \forall\, m>0, \quad L_0|\chi_2\rangle = (a-2)|\chi_2\rangle$$

$$L_1|\psi\rangle = [L_1, L_{-2} + \gamma L_{-1}^2]|\chi_2\rangle = \Big(3L_{-1} + \gamma \cdot \underbrace{2(L_0 L_{-1} + L_{-1}L_0)}_{[L_1, L_{-1}^2] = \underbrace{[L_1, L_{-1}]}_{2L_0} L_{-1} + L_{-1}\underbrace{[L_1, L_{-1}]}_{2L_0}}\Big)|\chi_2\rangle$$

$$= \Big(3L_{-1} + 2\gamma \underbrace{[L_0, L_{-1}]}_{L_{-1}} + 4\gamma L_{-1}(a-2)\Big)|\chi_2\rangle$$

$$= \big(3 + 2\gamma + 4\gamma(a-2)\big) L_{-1}|\chi_2\rangle$$

$$L_1|\psi\rangle = 0 \iff \gamma = \frac{3}{2(3-2a)} \qquad \left(\text{for } a=1 \text{ we have } \gamma = \frac{3}{2}\right)$$

$$L_{+2}|\psi\rangle = L_{+2}(L_{-2} + \gamma L_{-1}^2)|\chi\rangle = [L_2, L_{-2} + \gamma L_{-1}^2]|\chi\rangle$$

$$= \Big(4L_0 + \frac{D}{12}\cdot 6 + \gamma\big(\underbrace{[L_2, L_{-1}]}_{3L_1}L_{-1} + \cancel{L_{-1}[L_2, L_{-1}]}\big)\Big)|\chi\rangle$$

$$= \Big(4L_0 + \frac{1}{2}D + 3\gamma[L_1, L_{-1}]\Big)|\chi\rangle = \Big(4L_0 + \frac{1}{2}D + 3\gamma\cdot 2L_0\Big)|\chi\rangle$$

$$= \frac{1}{2}\big(4(2+3\gamma)(a-2) + D\big)|\chi\rangle$$

$$L_2|\psi\rangle = 0 \iff D = 4(2+3\gamma)(2-a)$$

So the spurious state $|\psi\rangle$ is null iff

$$\gamma = \frac{3}{2(3-2a)} \qquad D = 4(2+3\gamma)(2-a)$$

critical bosonic string:

$\underline{a=1}$: $|\psi\rangle$ null if $\gamma = \frac{3}{2}$ & $\underline{D=26}$

critical dimension

- all spurious states are null at higher levels as long as $D=26$
  ie states associated to residual gauge (conformal) symmetries generated by L_{-m} $(m>0)$ $\underline{\text{decouple}}$

Summary

<u>Open strings</u>: by studying the low level spectrum we found

- $a > 1$ $D > 26$: there are ghosts in the phys
- $a = 1$ $D = 26$ critical string

  we found infinite families of null states

- $a \leq 1$ $D \leq 25$ (subcritical case)

no inconsistencies at tree-level (no ghosts)
but inconsistent at the level of string loops (need to look at one loop interactions)

OCQ : $a=1$ $D=26$ needs 1-loop interactions to prove (no proof at tree level)

(no ghosts, many null states)

LCQ : . $a=1$ $D=26$ follows by requiring Lorentz spacetime invariance

. manifestly ghost free

BRST quantization : $a=1$, $D=26$ required for quantum gauge (conformal) invariance.

Too bad we have no time to go over the modern (BRST) quant.!

$$\boxed{\text{From now on} \qquad a=1 \quad D=26}$$

Next • closed string spectrum

2.9 Physical states of the closed string

(at least low level $N=0,1$)

Recall

states are of the form $\prod_{i=1}^{K} \alpha_{-n_i}^{\mu_i} \prod_{j=1}^{K} \tilde{\alpha}_{-m_j}^{\nu_j} |0,\tilde{0};K\rangle$, $n_i, m_j \geq 1$)

$\hookrightarrow$ two oscillator vacua

Physical state conditions

$$\begin{cases} (L_0 - \tilde{L}_0)|\phi\rangle = 0 \iff N = \tilde{N} \\ (L_0 + \tilde{L}_0 - 2a)|\phi\rangle = 0 \iff -\alpha' K^2 = -4a + 2(N+\tilde{N}) \overset{a=1}{\iff} -\alpha' K^2 = 4(N-1) \\ L_m|\phi\rangle = 0 \ \& \ \tilde{L}_m|\phi\rangle \ \forall m \geq 1 \end{cases}$$

(sufficient to prove this for $m=1,2$)

ground state $(N=\tilde{N}=0)$: $|0,\tilde{0};K\rangle$ with $-\alpha' K^2 = -4$ (tachyon)

Level 1 states: $N = \tilde{N} = 1$

General state $|\Omega, K\rangle \equiv \underbrace{\Omega_{\mu\nu}}_{\text{spacetime two-tensor}} \alpha_{-1}^{\mu} \tilde{\alpha}_{-1}^{\nu} |0, \tilde{0}; K\rangle$

mass shell condition: $-\alpha' K^2 = 4(1-a) = 0 \Rightarrow$ massless state for the critical string

Impose Virasoro constraints: only need to impose $L_{+1}|\Omega, K\rangle = 0$
$\tilde{L}_{+1}|\Omega, K\rangle = 0$

$[L_m, \alpha_n^{\mu}] = -n\, \alpha_{m+n}^{\mu}$

$$L_{+1}|\Omega, K\rangle = \Omega_{\mu\nu} L_1 \alpha_{-1}^{\mu} \tilde{\alpha}_{-1}^{\nu} |0; K\rangle = \Omega_{\mu\nu} (\alpha_0^{\mu} \tilde{\alpha}_{-1}^{\nu}) |0; K\rangle$$

$$= \sqrt{\frac{\alpha'}{2}} K^{\mu} \Omega_{\mu\nu} \tilde{\alpha}_{-1}^{\nu} |0; K\rangle = 0 \iff \underline{K^{\mu} \Omega_{\mu\nu} = 0}$$

$\alpha_0^{\mu} = \sqrt{\frac{\alpha'}{2}} \hat{p}^{\mu}$

similarly

$$\tilde{L}_{+1}|\Omega, K\rangle = \sqrt{\frac{\alpha'}{2}} K^{\nu} \Omega_{\mu\nu} \alpha_{-1}^{\mu} |0; K\rangle = 0 \iff \underline{K^{\nu} \Omega_{\mu\nu} = 0}$$

So far $|\Omega, K\rangle = \Omega_{\mu\nu}\, \alpha_{-1}^{\mu}\, \tilde{\alpha}_{-1}^{\nu} |0; K\rangle$ massless state

$\hookrightarrow$ with $K^{\mu}\Omega_{\mu\nu} = 0$, $K^{\nu}\Omega_{\mu\nu} = 0$

Null states: which degrees of freedom of $|\Omega; K\rangle$ are null?

$L_{-1}|\chi\rangle$, $L_0|\chi\rangle = 0$: $|\chi\rangle = \xi \cdot \tilde{\alpha}_{-1}|0, K\rangle$ $(N_{\chi} = 0,\ \tilde{N}_{\chi} = 1)$

$\tilde{L}_{-1}|\tilde{\chi}\rangle$, $\tilde{L}_0|\tilde{\chi}\rangle = 0$: $|\tilde{\chi}\rangle = \xi' \cdot \alpha_{-1}|0, K\rangle$ $(N_{\tilde{\chi}} = 1,\ \tilde{N}_{\tilde{\chi}} = 0)$

$L_{-1}|\chi\rangle, \tilde{L}_{-1}|\tilde{\chi}\rangle$ are spurious by construction

$$L_{-1}|\chi\rangle = \xi_{\mu}\, \underline{L_{-1}}\, \tilde{\alpha}_{-1}^{\mu}|0, K\rangle = \xi \cdot \tilde{\alpha}_{-1}\, \frac{1}{2}(2\, \alpha_{-1} \cdot \alpha_0)|0; K\rangle = \sqrt{\frac{\alpha'}{2}}\, (K \cdot \alpha_{-1})(\xi \cdot \tilde{\alpha}_{-1})|0; K\rangle$$

$L_{-1} = \frac{1}{2}\sum_{k=-\infty}^{\infty} \alpha_{-1-k} \cdot \alpha_k$,

so $\Omega_{\mu\nu} = \sqrt{\frac{\alpha'}{2}}\, K_{\mu}\xi_{\nu}$

Similarly $\tilde{L}_{-1}|\tilde{\chi}\rangle = \sqrt{\frac{\alpha'}{2}}\, (\xi' \cdot \alpha_{-1})(K \cdot \tilde{\alpha}_{-1})|0; K\rangle$ so $\Omega_{\mu\nu} = \sqrt{\frac{\alpha'}{2}}\, \xi'_{\mu} K_{\nu}$

These are also physical when $K^{\mu}\Omega_{\mu\nu} = 0$ & $K^{\nu}\Omega_{\mu\nu} = 0$

ie $K \cdot \xi = 0$ & $K \cdot \xi' = 0$

Thm $|\Omega, K\rangle \equiv \Omega_{\mu\nu}\, \alpha^{\mu}_{-1}\, \tilde{\alpha}^{\nu}_{-1} |0; K\rangle$ massless state

↳ with $K^{\mu}\Omega_{\mu\nu} = 0$, $K^{\nu}\Omega_{\mu\nu} = 0$

and two states $|\Omega, K\rangle$ & $|\hat{\Omega}, K\rangle$ are equivalent if

$|\hat{\Omega}; K\rangle = |\Omega; K\rangle + |\Omega_{null}; K\rangle$; $(\Omega_{null})_{\mu\nu} = \underline{K_{\mu}\xi_{\nu} + K_{\nu}\xi'_{\mu}}$

$K\cdot\xi = 0$, $K\cdot\xi' = 0$

To understand this better decompose the state into space-time Lorentz irreps

$$\Omega_{\mu\nu} = \underset{\text{traceless symmetric}}{\gamma_{\mu\nu}} + \underset{\text{trace}}{\varphi\,\eta_{\mu\nu}} + \underset{\text{antisymmetric}}{b_{\mu\nu}}$$

$$D^2 = \tfrac{1}{2}D(D+1) - 1 + 1 + \tfrac{1}{2}D(D-1)$$

Physical states associated to $\gamma_{\mu\nu}$

$k^\mu \gamma_{\mu\nu} = 0$ ($k^\nu \gamma_{\mu\nu} = 0$) $\quad$ $\gamma_{\mu\nu}$ transverse, symmetric, traceless

$\left(\frac{1}{2}D(D+1)-1\right) - D = \frac{1}{2}D(D-1) - 1$

with gauge invariance: $\left\{ \gamma_{\mu\nu} \longrightarrow \gamma_{\mu\nu} + \xi_\mu k_\nu + \xi_\nu k_\mu \right.$ $\quad$ $\xi' = \xi$ for symmetry

with $\xi \cdot k = 0$ $\quad$ ($\Rightarrow \hat{\gamma}_{\mu\nu}$ is traceless)

degrees of freedom: $\underbrace{\frac{1}{2}D(D+1)}_{\text{sym 2-tensor}} \underbrace{-1}_{\text{trace}} - \underbrace{D}_{k^\mu \gamma_{\mu\nu}=0} - \underbrace{(D-1)}_{\xi \text{ st } \xi\cdot k=0} = \frac{1}{2}(D-2)(D-1)-1$

dim of irrep of $SO(D-2)$ for massless transverse-polarised spin 2 particle

Spacetime interpretation: right degrees of freedom expected of a graviton in the traceless harmonic gauge

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \gamma_{\mu\nu}(x)$$

$$\gamma_{\mu\nu}(x) \sim \gamma_{\mu\nu} + \boxed{\partial_\mu \xi_\nu(x) + \partial_\nu \xi_\mu(x)}$$

$\hookrightarrow$ infinitesimal diffeomorphisms parametrised by ξ which in momentum space become

$\gamma_{\mu\nu}(k) \to \gamma_{\mu\nu}(k) + \xi_\mu k_\nu + \xi_\nu k_\mu$ $\quad$ with $\xi \cdot k = 0$

Physical states associated to $b_{\mu\nu}$ Ramond-Kalb field

$$k^\mu b_{\mu\nu} = 0 \quad \& \quad k^\nu b_{\mu\nu} = 0$$

$b_{\mu\nu}$ transverse, antisymmetric

$S' = -S$ for antisymmetry

with gauge invariance: $b_{\mu\nu} \to b_{\mu\nu} + \xi_\mu k_\nu - \xi_\nu k_\mu$, $\quad \xi \cdot k = 0$

Note that ξ has redundancy $\quad \xi_\mu \sim \xi_\mu + k_\mu \lambda$

degrees of freedom
$$\underbrace{\tfrac{1}{2}D(D-1)}_{\text{antisymmetric 2tensor}} - \underbrace{(D-1)}_{k^\mu b_{\mu\nu}=0} - \underbrace{(D-2)}_{\substack{\xi \text{ s.t. } \xi\cdot k = 0, \\ \xi \sim \xi + k\phi}} = \underbrace{\tfrac{1}{2}(D-2)(D-3)}_{\substack{\text{dim of } SO(D-2) \text{ irrep corresponding to} \\ \text{a massless 2-form}}}$$

In spacetime this is interpreted as a 2-form gauge field

$$b = \tfrac{1}{2} b_{\mu\nu}(x)\, dx^\mu \wedge dx^\nu \sim b + d\xi \;, \quad \underbrace{\xi}_{\text{one form}} \sim \xi + d\lambda$$

↳ next lecture ⇝ · scalar state

· § 3 Interactions