String Theory 1

Lecture #8

<u>In summary</u>: critical strings at low levels

- ground state (both open & closed strings): tachyon
- open string at level 1:

NN

$|\xi;k\rangle = \xi\cdot\alpha_{-1}|0;k\rangle$ is a massless state $\Big\}$ photon

$\xi\cdot k = 0 \qquad \xi_\mu \sim \xi_\mu + \partial_\mu\lambda$

- closed string level $N = \tilde{N} = 1$
  - ▶ symmetric traceless tensor $\boxed{\gamma_{\mu\nu}}$ — metric perturbations (dynamical gravity)

    $\gamma_{\mu\nu}(x) \sim \gamma_{\mu\nu}(x) + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu$
  - ▶ antisymmetric tensor $\boxed{b_{\mu\nu}}$ — Ramond-Kalb 2-form gauge field

    $b_{\mu\nu}(x) \sim b_{\mu\nu}(x) + \partial_\mu\xi_\nu - \partial_\nu\xi_\mu\,,\quad \xi_\mu \to \xi_\mu + \partial_\mu\lambda$

today → ▶ a scalar $\boxed{\varphi}$ — dilaton

(ST II: superstrings retain these features and ∄ tachyon.)

Physical state associated to the scalar   subtle !

We have the state $\frac{1}{2}\varphi\, \alpha_{-1}\cdot\alpha_{-1}|0,K\rangle$ $\Omega_{\mu\nu}=\frac{1}{2}\varphi\,\eta_{\mu\nu}$

Virasoro constraints: $K^{\mu}\Omega_{\mu\nu}=0 \Rightarrow \varphi K_{\nu}=0$

$\Rightarrow K^{\mu}=0$ (eliminates all non-zero momentum modes !)

which implies that the field is a constant so no degrees of freedom.

This is not right: $L_1(\alpha_{-1}\cdot\tilde{\alpha}_{-1}|0;K\rangle)\neq 0$ (and similarly for $\tilde{L}_1$)

so this is $\boxed{\text{not}}$ a physical state!

$\boxed{\text{We can}}$ however construct a level 1 physical state which is a spacetime scalar.

Given to vectors ζ & $\tilde{\zeta}$, define the level 1 state

$$|\varphi_{\zeta,\tilde{\zeta}},K\rangle=\varphi\left[(\zeta\cdot\alpha_{-1})(\alpha_0\cdot\tilde{\alpha}_{-1})+(\alpha_0\cdot\alpha_{-1})(\tilde{\zeta}\cdot\tilde{\alpha}_{-1})+\alpha_{-1}\cdot\tilde{\alpha}_{-1}\right]|0;K\rangle$$

Now impose the Virasoro constraints.

$[L_m, \alpha_n^\mu] = -n\,\alpha^\mu_{m+n}$

$$L_1|\varphi_{s,\tilde{s}};K\rangle = \varphi\, L_1\, \alpha^\mu_{-1}\left[\zeta_\mu(\alpha_0\cdot\tilde{\alpha}_{-1}) + \alpha_{0\mu}(\tilde{\zeta}\cdot\tilde{\alpha}_{-1}) + \alpha_{-1\mu}\right]|0;K\rangle$$

$$= \varphi\left(\alpha_0^\mu + \cancel{\alpha^\mu_{-1} L_1}\right)\left[\zeta_\mu(\alpha_0\cdot\tilde{\alpha}_{-1}) + \alpha_{0\mu}(\tilde{\zeta}\cdot\tilde{\alpha}_{-1}) + \tilde{\alpha}_{-1\mu}\right]|0;K\rangle$$

$$= \varphi\left[\zeta\cdot\alpha_0\,(\alpha_0\cdot\tilde{\alpha}_{-1}) + \frac{\alpha'}{2}\cancelto{0}{K^2}(\tilde{\zeta}\cdot\tilde{\alpha}_{-1}) + \alpha_0\cdot\tilde{\alpha}_{-1\mu}\right]|0;K\rangle$$

$$= \varphi\left[\underline{\zeta\cdot\alpha_0 + 1}\right]\alpha_0\cdot\tilde{\alpha}_{-1}|0;K\rangle$$

$= 0$ iff $\underline{\zeta\cdot K = -\sqrt{\frac{2}{\alpha'}}}$

Similarly: $\tilde{L}_1|\varphi_{s,\tilde{s}};K\rangle = \varphi[\tilde{\zeta}\cdot\alpha_0 + 1]\,\alpha_0\cdot\alpha_{-1}|0;K\rangle$

$= 0$ iff $\tilde{\zeta}\cdot K = -\sqrt{\frac{2}{\alpha'}}$

So $|\varphi_{s,\tilde{s}};K\rangle$ is physical iff $\zeta\cdot K = \tilde{\zeta}\cdot K = -\sqrt{\frac{2}{\alpha'}}$

Despite the fact that it seems to depend on ζ & $\tilde{\zeta}$ this state corresponds to a scalar.

To see this consider the dependence on ζ (and $\tilde{\zeta}$)

$$|\varphi_{s,\tilde{\zeta}};K\rangle - |\varphi_{s',\tilde{\zeta}};K\rangle = \varphi\,(\zeta-\zeta')\cdot\alpha_{-1}\,(\alpha_0\cdot\tilde{\alpha}_{-1})\,|0;K\rangle$$

OTOH $\quad \tilde{L}_{-1}\left(\underbrace{(\zeta-\zeta')\cdot\alpha_{-1}|0;K\rangle}\right) = \left((\zeta-\zeta')\cdot\alpha_{-1}\right)(\tilde{\alpha}_{-1}\cdot\alpha_0)\,|0;K\rangle$

$\tilde{L}_{-1} = \frac{1}{2}\sum_k \tilde{\alpha}_{-1-k}\cdot\tilde{\alpha}_k$

$$|\varphi_{s,\tilde{\zeta}};K\rangle - |\varphi_{s',\tilde{\zeta}};K\rangle = \varphi\,\tilde{L}_{-1}\left((\zeta-\zeta')\cdot\alpha_{-1}|0;K\rangle\right)$$ pure gauge

and similarly for $\tilde{\zeta}$ & $\tilde{\zeta}'$. Then

$$\left\{\begin{array}{l} |\varphi_{s,\tilde{\zeta}};K\rangle - |\varphi_{s',\tilde{\zeta}};K\rangle \\ |\varphi_{s,\tilde{\zeta}};K\rangle - |\varphi_{s,\tilde{\zeta}'};K\rangle \end{array}\right.$$ are spurious

They are also null $\Big\langle$ L_1 : $(\zeta-\zeta')\cdot K = 0$ (as $\zeta\cdot K = -\sqrt{\frac{2}{K}} = \zeta'\cdot K$)

$\tilde{L}_1$: $(\tilde{\zeta}-\tilde{\zeta}')\cdot K = 0$ (as $\tilde{\zeta}\cdot K = -\sqrt{\frac{2}{\alpha'}}\ \tilde{\zeta}'\cdot K$)

Hence we identify a state with ξ ($\bar{\xi}$) and ξ' ($\bar{\xi}'$)

$$|\varphi_{\xi,\tilde{\xi}}; k\rangle \sim |\varphi_{\xi',\tilde{\xi}}; k\rangle + \varphi \tilde{L}_{-1} ((\xi-\xi')\cdot\alpha_{-1}) |0; k\rangle$$

$$|\varphi_{\xi,\tilde{\xi}}; k\rangle \sim |\varphi_{\xi,\tilde{\xi}'}; k\rangle + \varphi L_{-1} ((\tilde{\xi}-\tilde{\xi}')\cdot\tilde{\alpha}_{-1}) |0; k\rangle$$

ie indep of choice of ξ, ξ' up to gauge equivalence

This scalar φ is called the dilaton:

it plays a very important role in the context of string interactions.

Next

- interactions & vertex operators

<u>Final remarks</u>: We have gone over the OCQ of the Polyakov action which describes a free field theory in 1+1 dims

In this scheme we discussed how to construct the Hilbert space of physical states. The consistency of the quantum spectrum (no ghosts) requires

$$a = 1 \quad \& \quad D = 26$$

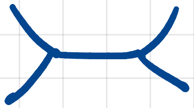
(together with some input from the interacting string to rule out $0 \leq a \leq 1$ & $1 \leq D \leq 25$)

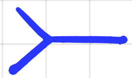
Chapter 3 Interactions

3.1 Generalities

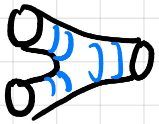
QFT:

- to understand interactions one adds non-linear terms to the action
  - ↳ · doesn't work for the string because anything you try to add breaks gauge invariance.
- scattering amplitudes → Feynman diagrams

  eg  etc.

  interactions encoded at vertices say  3 point int.

  ↳ in string theory this is replaced by, for instance

   or  : no such vertices!

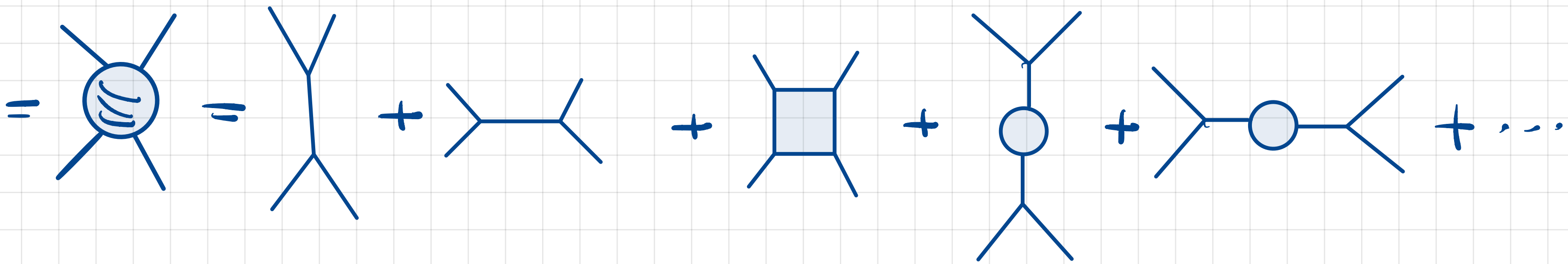
QFT:

- to understand interactions one adds non-linear terms to the action
  - ↳ · doesn't work for the string because anything you try to add breaks gauge invariance.

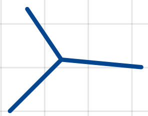
(asymptotic) series for scattering amplitudes: sum over all possible Feynman diagrams with fixed external legs (& integrate over the positions of all vertices)

QFT Feynman diagrams

$\mathcal{A}$ =



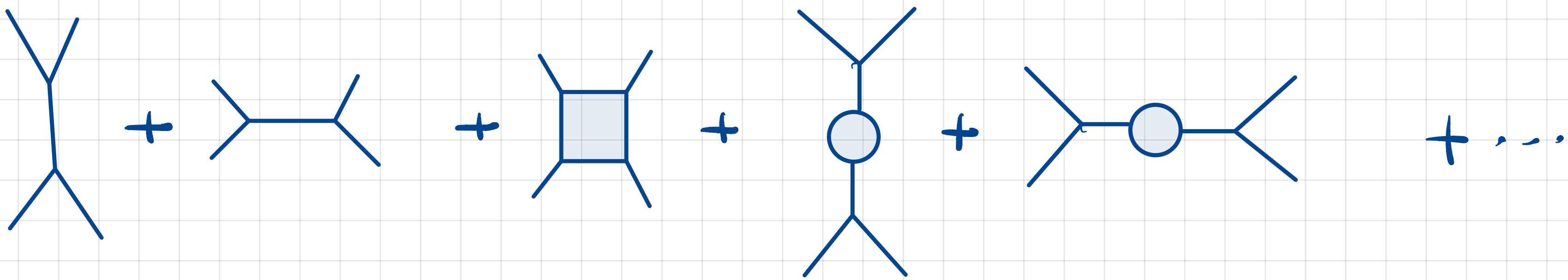
· interaction encoded at vertices



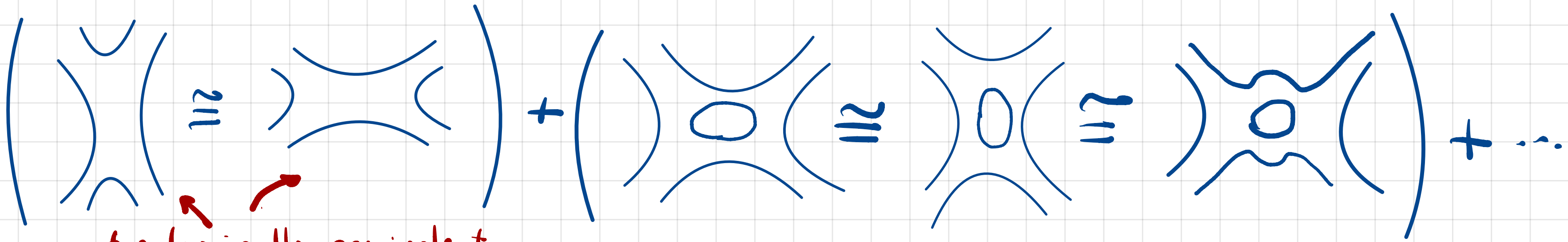
3-point function

(read this off a Lagrangian)

QFT
Feynman diagrams



string theory



topologically equivalent

sums over diagrams of strings branching and joining

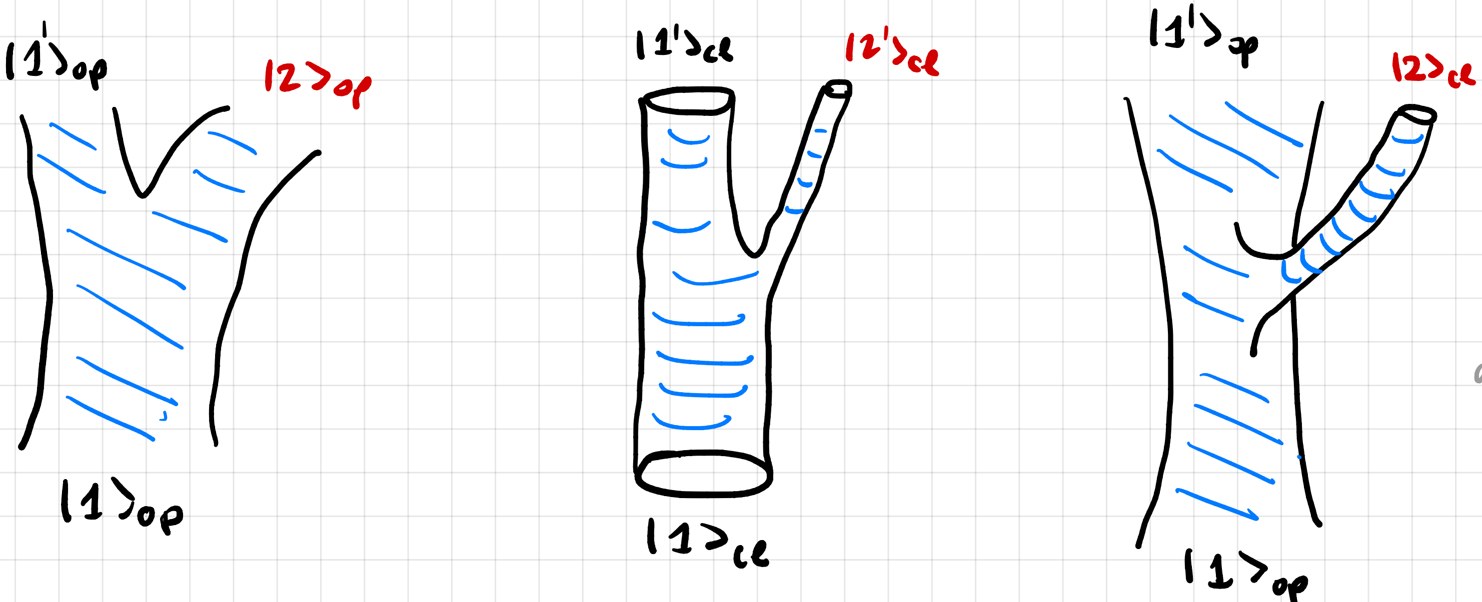
- each diagram locally looks like a 2dim manifold
- different particle diagrams → topologically equiv string diagrams
- solid black lines
  - OS: boundaries of WS
  - CS: only ends of the WS are boundaries

In string theory :

We want to compute for example the amplitude of a given configuration of quantised strings at an initial time to evolve into a new configuration at a later time

$\mathcal{A}$: conf of q-strings $(\Sigma_i) \longrightarrow$ conf of q-strings (Σ_f)

This means we need to describe the branching and joining of quantised strings



open string $|1\rangle_{op}$ which splits into two $|1'\rangle_{op}$ & $|2\rangle_{op}$

open string branches into an OS & a CS

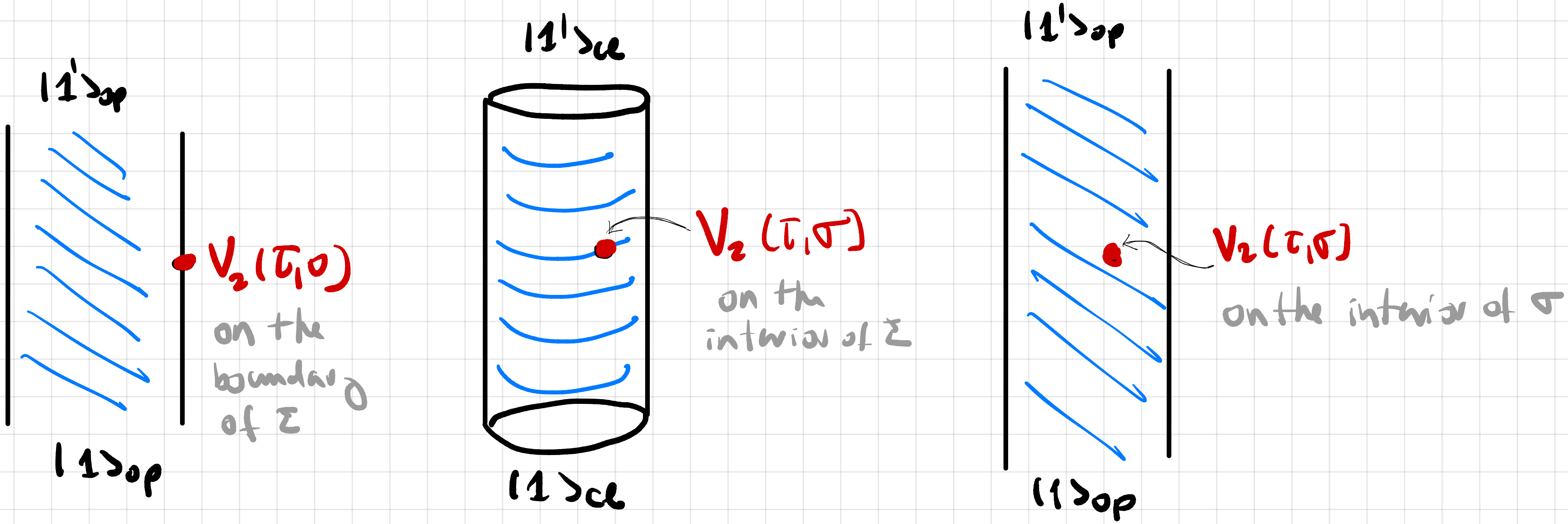
Problem : it is not known how to do this .... !

all we have at hand is the phys/the null physical states of a string

So, we need to work with the quantised formalism we developed.

Suppose $|2\rangle$ is a physical state which is emitted/absorbed.

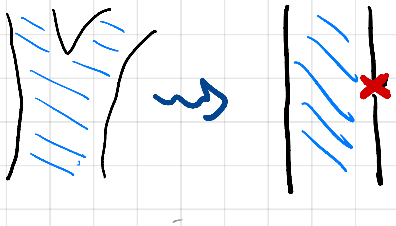
We describe the emission\absorption of a quantum state (say $|2\rangle$) from a fixed string WS by the action of a local operator or Vertex operator.



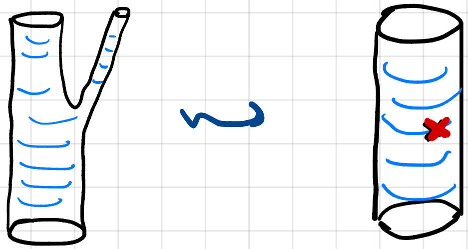
How?

Tools : we will describe this process by

- Wick rotation of the world sheet
  (Lorentzian signature → Euclidean signature)
- conformal transformation



or

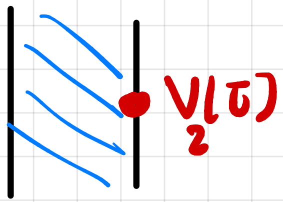


3.2 Vertex operators : Introduction

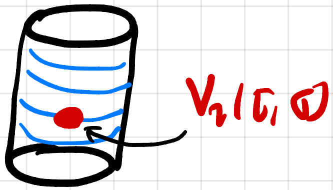
Require: $\boxed{\text{two}}$ key requirements

① time evolution on the worldsheet is a gauge transformation ⇒ position of the vertex operator should not be meaningful.

open string vertex operator $\xrightarrow{\text{what matters is}}$ $\int d\tau \underbrace{V_2(\tau)}_{\text{inserted on the boundary}}$



closed string vertex operator $\rightarrow$ $\int d\tau\, d\sigma \underbrace{V_2(\tau,\sigma)}_{\text{inserted on the interior}}$



② absorption & emission vertex operators to map

$$\mathcal{H}_{\text{phys}} \longrightarrow \mathcal{H}_{\text{phys}}$$

and

$$\mathcal{H}_{\text{null}} \longrightarrow \mathcal{H}_{\text{null}}$$

Let's see how this works for $\boxed{\text{the open strings first}}$

Consider the action of a vertex operator on a physical state $|\phi_{\text{phys}}\rangle$, that is

(local) boundary operator at (say) $\sigma = 0$

$$\int d\tau \; V(\tau, 0) \; |\phi_{\text{phys}}\rangle$$

$\mathcal{H}_{phys} \xrightarrow{V} \mathcal{H}_{phys}$ Applying the physical state conditions:

$$m \geq 0 : \quad (L_m - a\,\delta_{m,0}) \left[\int d\tau\, V(\tau,0) \,|\phi_{phys}\rangle \right] = \int d\tau\, [L_m, V(\tau,0)]\, |\phi_{phys}\rangle$$

$$= 0 \quad \text{if} \quad \boxed{[L_m, V(\tau,0)] = \partial_\tau(\text{local operator}) \qquad m \geq 0}$$

$\int d\tau\, \partial_\tau(\cdots) = (\ )_f - (\ )_i$

local operator at past & future infinity vanish asymptotically

$\mathcal{H}_{null} \xrightarrow{V} \mathcal{H}_{null}$

$$m \geq 1 : \underbrace{\int d\tau\, V(\tau,0)\, \overbrace{[L_{-m} |\phi\rangle]}^{\text{null}}} = \int d\tau \Big\{ \underbrace{[V(\tau,0), L_{-m}]|\phi\rangle}_{\substack{\text{need this to vanish up to a}\\ \text{total derivative}}} + \underbrace{L_{-m} V(\tau,0)|\phi\rangle}_{\substack{\text{null}\\ \langle\psi_{phys}|L_{-m}V(\tau,0)|\phi\rangle = 0}} \Big\}$$

This is null if $\boxed{[V(\tau,0), L_{-m}] = \partial_\tau(\text{local op}), \quad m \geq 1}$

We have seen that if $[L_m, V(\tau, 0)] = \partial_\tau(\text{local op}) \quad \forall m$

then $\quad \mathcal{H}_{\text{phys}} \overset{V}{\longleftrightarrow} \mathcal{H}_{\text{phys}}, \quad \mathcal{H}_{\text{null}} \overset{V}{\longleftrightarrow} \mathcal{H}_{\text{null}}$

The converse is also true: suppose $V(\tau)$ is an operator st $\quad \mathcal{H}_{\text{phys}} \xrightarrow{V} \mathcal{H}_{\text{phys}}, \quad \mathcal{H}_{\text{null}} \xrightarrow{V} \mathcal{H}_{\text{null}}$.

Let $|\phi\rangle$ be a physical (or null) state. Then as $V(\tau)|\phi\rangle$ is physical (or null), we have

$$0 = \int d\tau \, (L_m - \delta_{m,0}) V(\tau) |\phi\rangle = \int d\tau \, [L_m - \delta_{m,0}, V(\tau)] |\phi\rangle$$

$$= \int d\tau \, [L_m, V(\tau)] |\phi\rangle \qquad \forall m$$

Therefore $\quad [L_m, V(\tau)] = \partial_\tau(\text{local operator})$ //

What happens to VOs under conformal transformation?

conformal transformations of the open string are of the form $\tau \to \tilde{\tau}(\tau)$. We want

$\int V(\tau,0)\, d\tau$ to be invariant under $\tau \to \tilde{\tau}$ (as emitted / absorbed state must be inv under $\tau \to \tilde{\tau}$)

$$\int V(\tau,0)\, d\tau \longrightarrow \int \tilde{V}(\tilde{\tau},0)\, d\tilde{\tau} = \int V(\tau,0)\, d\tau$$

$$= \int V(\tau,0)\, \frac{d\tau}{d\tilde{\tau}}\, d\tilde{\tau}$$

so require $V(\tau,0) \longrightarrow \tilde{V}(\tilde{\tau},0) = V(\tau,0)\, \dfrac{d\tau}{d\tilde{\tau}}$ under c transfs

Definition: an operator $A(\tau)$ is a primary operator of weight h if under the transformation $\tau \to \tilde{\tau}(\tau)$ it transforms as

$$\boxed{A(\tau) \longrightarrow \tilde{A}(\tilde{\tau}) = A(\tau)\left(\frac{d\tau}{d\tilde{\tau}}\right)^{h}}$$

For an operator with $h=1$

$$\int \tilde{A}(\tilde{\tau})\, d\tilde{\tau} = \int A(\tau) \frac{d\tau}{d\tilde{\tau}} d\tilde{\tau} = \int A(\tau)\, d\tau$$

ie the integrated operator is conformally invariant.

For infinitesimal transformations $\tau \longrightarrow \tilde{\tau} = \tau + \epsilon(\tau)$

we have

$$A(\tau) \longrightarrow \tilde{A}(\tilde{\tau}) = A(\tau)\left(1 - h\frac{d\epsilon}{d\tau}\right) \qquad \left(\frac{d\tau}{d\tilde{\tau}}\right)^h$$

OTOH (LHS)

$$\tilde{A}(\tilde{\tau}) = \tilde{A}(\tau+\epsilon) \underset{\text{taylor}}{=} \tilde{A}(\tau) + \epsilon\,\partial_\tau A(\tau) + O(\epsilon^2)$$

Then we find the variation of A at τ

$$\delta A(\tau) = \tilde{A}(\tau) - A(\tau) = -\epsilon\,\partial_\tau A - h(\partial_\tau \epsilon) A$$

$$= -\partial_\tau(\epsilon A) - (h-1)\,\partial_\tau \epsilon\, A$$

This is a total derivative when $h = 1$

A conformal transformation (from now on set $\ell = \begin{cases} 2\pi & CS \\ \pi & OS \end{cases}$)

$$\tau \longrightarrow \tilde{\tau} = \tau + \epsilon(\tau) \quad \text{with} \quad \epsilon = -e^{im\tau}$$

corresponds to

$$\delta\phi(\tau) = e^{im\tau}\left(-i\partial_\tau \phi + hm\phi\right)$$

Recall
$\{L_m, \phi(\tau)\}_{PB} = \delta_{\epsilon_m}\phi(\tau)$
$\{\cdot,\cdot\}_{PB} \longrightarrow i[\cdot,\cdot]$
PS 1

So the action of the Virasoro operators on the ops ϕ is

$$-i\,\delta\phi(\tau) = \boxed{[L_m, \phi(\tau)] = e^{im\tau}(-i\partial_\tau + mh)\,\phi(\tau)}$$

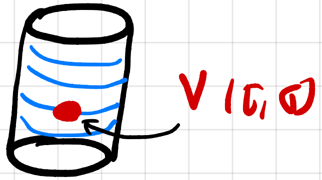
Equivalently, this is the condition for ϕ to have conformal weight h.

For $h=1$: $[L_m, \phi(\tau)] = \partial_\tau\left(-i e^{im\tau}\phi(\tau)\right)$ ✓ TD

The problem is to identify primaries of weight 1 which correspond to the emission/absorption of physical states in the string Hilbert space. Then we use this to compute string amplitudes!

$\boxed{\text{closed strings}}$: analogous

closed string vertex operator $\rightarrow$ $\displaystyle\int d\tau\, d\sigma \underbrace{V(\tau,\sigma)}_{\text{inserted on the interior}}$



A primary operator of dimension $(h, \tilde h)$ is an operator transforming under conformal transfs of the WS according to

$$\phi(\xi^+,\xi^-) \longrightarrow \tilde\phi(\tilde\xi^+,\tilde\xi^-) = \left(\frac{d\xi^+}{d\tilde\xi^+}\right)^{h} \left(\frac{d\xi^-}{d\tilde\xi^-}\right)^{\tilde h} \phi(\xi^+,\xi^-)$$

The corresponding infinitesimal transformations are

$$\delta\phi(\tau,\sigma) = -\partial_+(\tilde\epsilon\phi) - (\tilde h-1)(\partial_+\tilde\epsilon)\phi - \partial_-(\epsilon\phi) - (h-1)(\partial_-\epsilon)\phi$$

This is a total derivative if $h = \tilde h = 1$

For $\epsilon_n^\pm = \frac{i}{2} e^{2im\sigma^\pm}$ this gives the action of L_m:

$$[L_m, \phi(\xi^\pm)] = \frac{1}{2} e^{2im\xi^+}\left(-i\partial_+ + 2mh\right)\phi(\xi^\pm)$$

$$[\tilde L_m, \phi(\xi^\pm)] = \frac{1}{2} e^{2im\xi^-}\left(-i\partial_- + 2m\tilde h\right)\phi(\xi^\pm)$$

↳ Next

- state ↔ vertex correspondence

3.3 Vertex operators for the open string

on the WS

on the boundary of Σ

Consider the boundary scalar operator $X^\mu(\tau, 0)$:

$$X^\mu(\tau,0) = x^\mu + 2\alpha' p^\mu + i\sqrt{2\alpha'} \sum_{n\neq 0} \frac{\alpha_n^\mu}{n} e^{-in\tau}$$

Conformal transformations:

$$[L_m, X^\mu(\tau, 0)] = -i e^{im\tau} \frac{d}{d\tau}\left(X^\mu(\tau,0)\right)$$

$$[L_m, \alpha_n^\mu] = -n\,\alpha^\mu_{m+n}$$

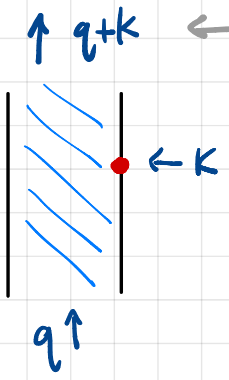
$\Longrightarrow$ $h=0$ (indeed a WS scalar)

so $X^\mu(\tau)$ is not a VO

$$\left([L_m, A(\tau)] = e^{im\tau}(-i\partial_\tau + m h)\, A(\tau) \right)$$

Building a VO associated to a state of momentum K

Construct a vertex operator for the emission / absorption of a state of momentum K



outgoing state. changes the momentum of initial state by K

$V_K(\tau)$ emission/absorption of a state with K

incoming state with q

new out state

so $$V_K(\tau)\,|\phi\,;q\rangle = |\phi'\,;q+K\rangle$$

incoming

momentum of out state $q+K$

Thus, we should have

$$[P^\mu, V_K(\tau)] = K^\mu V_K(\tau)$$

action of the momentum tensor on an operator $V_K(\tau)$

$[P^\mu, V_K(\tau)]\,|\phi, q\rangle = P^\mu V_K(\tau)|\phi, q\rangle - q^\mu V_K(\tau)|\phi, q\rangle = K^\mu V_K(\tau)$

so $P^\mu V_K(\tau)|\phi, q\rangle = (q^\mu + K^\mu) V_K(\tau)|\phi, q\rangle$ ie $V_K(\tau)|\phi, q\rangle$ = state with momentum $q^\mu + K^\mu$

Then the operator $V_k(\tau)$ must include a factor $e^{ik\cdot x(\tau)}$ $\leftarrow$ $x(\tau) = x^\mu + 2\alpha' p^\mu \tau$ CM coord.

Then a naive guess is: $V_k(\tau) = e^{ik\cdot X(\tau)}$ (simplest op built from X^μ)

As it stands, it is not well defined: it still needs normal ordering.

(convention: annihilation ops on the right!)

Consider the normal ordered expression

$$V_k(\tau) =: e^{ik\cdot X(\tau)} : \equiv e^{\sqrt{2\alpha'} k\cdot \sum_{n=1}^{\infty} \frac{\alpha_{-m}}{m} e^{im\tau}} e^{ik\cdot x(\tau)} e^{-\sqrt{2\alpha'} k \sum_{n=1}^{\infty} \frac{\alpha_n}{n} e^{-in\tau}}$$

$\hat{x}(\tau) = x^\mu + 2\alpha' \hat{p}^\mu \tau$

$\ell_s = \sqrt{2\alpha'}$ $\quad X^\mu(\tau) = \underbrace{x^\mu + \ell_s^2 p^\mu \tau}_{x^\mu} + i\ell_s \sum_{\substack{n\in\mathbb{Z} \\ n\neq 0}} \frac{1}{n}\alpha_n^\mu e^{-in\tau}$

There is still room for reordering eg $e^{ik\cdot x}$ vs $e^{ik\cdot(x+2\alpha'\hat{p}\tau)}$ etc

Remark:

$k_\mu k_\nu [\alpha_m^\mu, \alpha_n^\nu] = m\,\delta_{m+n}\,\eta^{\mu\nu}$

$$e^{ik\cdot\alpha_m} e^{ik\cdot\alpha_n} = e^{ik\cdot(\alpha_m+\alpha_n)} e^{-\frac{1}{2}\boxed{k\cdot k\, m\, \delta_{m+n,0}}}$$

so reordering is free for $k^2=0$ (useful soon!)