String Theory 1

Lecture #9

Chapter 3 Interactions

3.1 Generalities

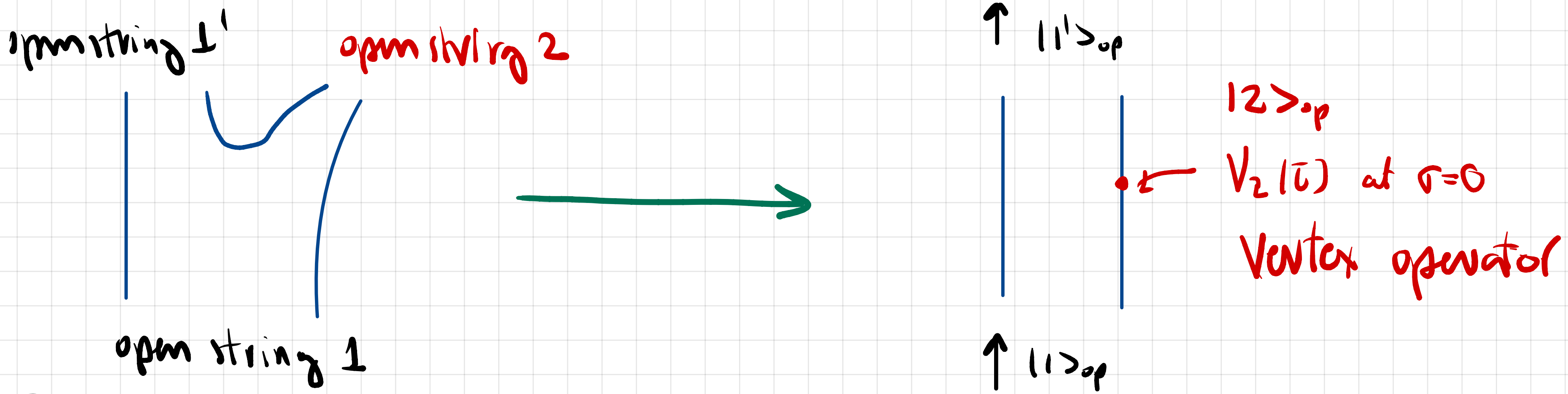
3.2 Vertex operators : introduction

3.3 Construction of vertex operators for the open string

3.4 State-Vertex correspondence (open string

3.5 Closed string

3.2 Introduction of vertex operators (summary)



not known how to do this

we instead describe emission/absorption of a quantum physical state from a fixed string world sheet by the action of a local operator or vertex operator V s.t.

$$\mathcal{H}_{phys} \xrightarrow{V} \mathcal{H}_{phys} \qquad \mathcal{H}_{null} \xrightarrow{V} \mathcal{H}_{null}$$

For the open string a **vertex operator** is a primary operator of <u>weight $h=1$</u>.

This expresses the fact that

$$\mathcal{H}_{phys} \xrightarrow{V} \mathcal{H}_{phys} \qquad\qquad \mathcal{H}_{null} \xrightarrow{V} \mathcal{H}_{null}$$

$$[L_m, V(\tau)] = \partial_\tau \mathcal{O} \quad \forall\, m \geq 0 \qquad\qquad [L_{-m}, V(\tau)] = \partial_\tau \mathcal{O}' \quad \forall\, m > 0$$

$$[L_m, V(\tau)] = \partial_\tau \mathcal{O} \quad \forall\, m \in \mathbb{Z}$$

This, in turn, implies that $\int d\tau\, V(\tau)$ is invariant under conformal transformations

3.3 Vertex operators for the open string

Consider the boundary scalar operator $X^\mu(\tau, 0)$: (on the boundary of Σ)

$$X^\mu(\tau,0) = x^\mu + 2\alpha' p^\mu \tau + i\sqrt{2\alpha'} \sum_{n\neq 0} \frac{\alpha_n^\mu}{n} e^{-in\tau}$$

WL of the end of the string

Conformal transformations:

$$[L_m, X^\mu(\tau,0)] = -i e^{im\tau} \frac{d}{d\tau}\left(X^\mu(\tau,0)\right)$$

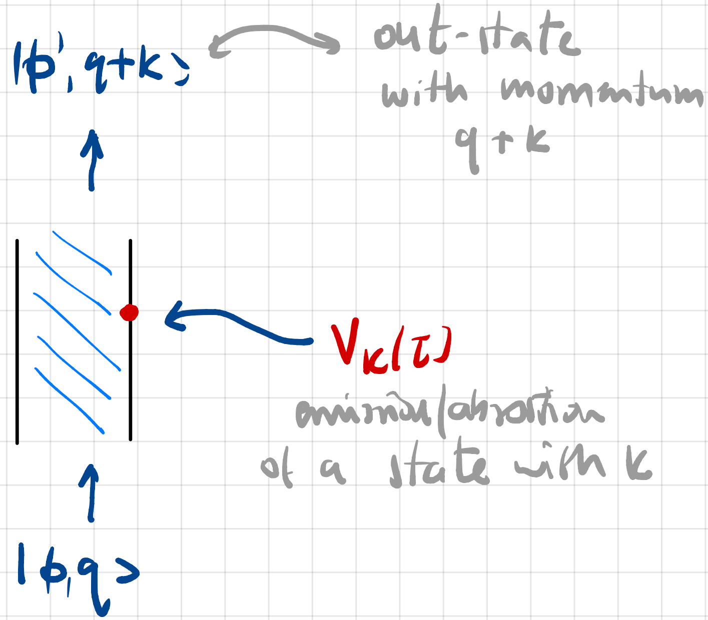
$[L_m, \alpha_n^\mu] = -n\,\alpha_{m+n}^\mu$

$\Rightarrow$ $X^\mu(\tau)$ is an operator with $h=0$

(indeed a WS scalar)

but it is not a VO

Building a VO associated to a state of momentum K



so $\quad V_K(\tau)\,|\phi ; q\rangle = |\phi' ; q+K\rangle$

ie $\quad V_k(\tau)\,|\phi, q\rangle$ = state with momentum "$q+k$"

Thus, we require $\quad [P^\mu, V_K(\tau)] = K^\mu\, V_K(\tau)$ $\quad$ action of the momentum tensor on an operator $V_K(\tau)$

Then the operator must depend on $e^{iK\cdot X(\tau)}$ ← CM coordinate

(QFT: know how to arrange this, this factor shifts momentum with acting on states)

A naive guess is: $V_K(\tau) = e^{iK\cdot X(\tau)}$ $\quad$ (simplest op built from X^μ)

As it stands, it is not well defined: it still needs normal ordering.

Consider the normal ordered expression (convention: annihilation ops on the right!)

$$\boxed{V_k(\tau) \equiv \, : e^{ik\cdot X(\tau)} : \; \equiv e^{\sqrt{2\alpha'}\, k\cdot \sum_{n=1}^{\infty} \frac{\alpha_{-m}}{m} e^{im\tau}} \; e^{ik\cdot x(\tau)} \; e^{-\sqrt{2\alpha'}\, k \sum_{n=1}^{\infty} \frac{\alpha_n}{n} e^{-in\tau}}}$$

$\hat{x}^\mu(\tau) = x^\mu + 2\alpha' \hat{p}^\mu \tau$

$$X^\mu(\tau, 0) = x^\mu + 2\alpha' p^\mu \tau + i\sqrt{2\alpha'} \sum_{n\neq 0} \frac{\alpha_n^\mu}{n} e^{-in\tau}$$

$$= x^\mu(\tau) + i\sqrt{2\alpha'} \sum_{n=1}^{\infty} \frac{1}{n} \left(\alpha_n^\mu e^{-in\tau} - \alpha_{-n}^\mu e^{in\tau} \right)$$

There is still room for reordering eg

$$e^{ik\cdot x} \, e^{i2\alpha' \hat{p}\tau} \quad \text{vs} \quad e^{ik\cdot(x + 2\alpha' \hat{p}\tau)} \qquad \text{etc}$$

we have chosen a particular ordering (the latter)

Baker-Campbell-Hausdorff

and also: $e^{ik\cdot\alpha_m} e^{ik\cdot\alpha_n} = e^{ik\cdot(\alpha_m+\alpha_n)} \, e^{-\frac{1}{2} k\cdot k \, m \, \delta_{m+n,0}}$

$k_\mu k_\nu [\alpha_m^\mu, \alpha_n^\nu] = m\,\delta_{m+n}\eta^{\mu\nu}$

so reordering is free for $k^2=0$ (useful soon!)

The normal ordering in the definition gives the right conformal transformation:

$$[L_m, :e^{ik\cdot X(\tau)}:] = e^{im\tau}\left(-i\partial_\tau + \alpha' m (k\cdot k)\right):e^{ik\cdot X(\tau)}:$$

PS 2

from reordering, gives $h = \alpha'(k\cdot k)$

This is a good vertex operator if $h=1$ ie $\alpha' k^2 = 1$

precisely the tachyon mass-shell condition $(\alpha' k^2 = 1)$!

This is the unique VO that describes emission/absorption of a tachyon.

Things get more interesting for $N=1$ states

Building vertex operators for level one states

For these states: $k^2=0$

so we are looking for the photon emission/absorption vertex operator

$$W(\tau) \sim \underbrace{(\cdot \text{- - -})}_{\text{some appropriate local operator}} \underbrace{:e^{ik\cdot X(\tau)}:}$$

$h = \alpha'(k\cdot k) = 0$

no normal ordering issues as $k^2=0$

Consider

$$[L_m, \partial_\tau X^\mu(\tau)] = \frac{\partial}{\partial \tau}[L_m, X^\mu(\tau)]$$

$$= \frac{\partial}{\partial \tau}\left(-i\left(\partial_\tau X^\mu(\tau)\right) e^{im\tau}\right) = e^{im\tau}\left(-i\,\partial_\tau + m\right) \partial_\tau X^\mu(\tau)$$

Then $\underline{\partial_\tau X^\mu(\tau) \text{ is primary of weight } h=1}$

We could try, for the vertex operator for emission/absorption of a photon with polarisation ς the operator

$$W_{\varsigma,k}(\tau) = \frac{1}{\sqrt{2\alpha'}}\left(\varsigma \cdot \partial_\tau X(\tau)\right) : e^{ik\cdot X(\tau)} :$$

and deal with normal ordering issues.

(Note also that AB does not necessarily have $h_{AB} = h_A + h_B$ ...)
Naively the VO for the tachyon has weight 0 but NO $\Rightarrow$ it has $h=1$

$$\text{In } \underbrace{\xi \cdot \partial_\tau X(\tau)}_{\to \sqrt{2\alpha'} \sum_n \xi \cdot \alpha_n e^{-in\tau}} : \text{ each oscillator operator is contracted with } \xi$$

$$\text{In } e^{iK \cdot X(\tau)} : \text{ each oscillator operator is contracted with } K$$

Then, all potential ambiguities come from commutators of the form:

$$[\alpha_m \cdot \xi, \alpha_n \cdot K] = m \, \delta_{m+n,0} (\xi \cdot K).$$

Precisely when $\xi \cdot K = 0$ (physical polarization)

$$(\xi \cdot \partial_\tau X(\tau)) : e^{iK \cdot X(\tau)} : \text{ is well defined}$$

(i.e. no corrections from normal ordering)

We identify

$$\boxed{W_{s,k}(\tau) = \frac{1}{\sqrt{2\alpha'}} :(\varsigma \cdot \dot{X}(\tau))\, e^{ik\cdot X(\tau)}: \,, \quad k^2=0, \ \varsigma\cdot k = 0 \quad (\hbar = 1)}$$

with the vertex operator for the emission/absorption of a photon.

Note that, for the longitudinal polarization,

$$(k\cdot \dot{x})\, e^{ik\cdot x} = -i\,\partial_\tau (e^{ik\cdot x})$$

which vanishes after integrating over τ.

This means that the longitudinal mode decouples ✓ //

* It is not a coincidence that the rules for constructing vertex operators are closely parallel to the construction of physical states.

In summary: $|\phi\rangle_{\text{phys}} \longmapsto V_\phi(\tau)$ ← conformal primary operator of weight $h=1$

For example, recalling that $:e^{ik\cdot X}:$ has $h=\frac{1}{2}(k\cdot k)\alpha'$

level 0 : $|0;k\rangle \longmapsto :e^{ik\cdot X(\tau)}:$ $h=1$ for $\alpha' k^2 = 1$
tachyon

level 1 : $|\xi;k\rangle \longmapsto :\xi\cdot\dot{X}(\tau)\, e^{ik\cdot X(\tau)}:$ $h=1$ if $k^2=0$ & $\xi\cdot k=0$,
photon of polarisation ξ transverse polarisation

(longitudinal mode decouples $k\cdot\dot{X}\, e^{ik\cdot X(\tau)} = -\partial_\tau\left(e^{ik\cdot X}\right)$)

level 2 : $\gamma_{\mu\nu}\alpha_{-1}^{\mu}\alpha_{-1}^{\nu}|0;k\rangle \longmapsto \underbrace{\gamma_{\mu\nu}\dot{X}^{\mu}\dot{X}^{\nu}}_{h=2}\underbrace{:e^{ik\cdot X}:}_{h=-1}$ $h=1$ if $k^2=-2\alpha'$ & $k\cdot\gamma=0$, γ traceless
massive spin 2 state
(symmetric traceless tensor)

One can show for $\forall N \geq 0$ that for every physical state in $\mathcal{H}_{phys}$ one can construct a corresponding VO (by careful consideration of the normal ordering)

The converse is also true: every VO corresponds to a physical state!

↳ Next: The state-operator correspondence (special feature of CFTs)

3.4 State-Vertex correspondence (open strings)

We want to make this correspondence more explicit:

$|\phi\rangle$ = the phys $\xleftrightarrow{1-1}$ V_ϕ primary operator of weight $h=1$ (ie Vertex operators)

We begin recalling that the Hamiltonian

$$H = L_0 = \alpha' p^2 + N$$

is a time evolution operator. So, if $V(\tau)$ is a vertex operator at the string end point $\sigma=0$, then we can write

$$V(\tau) = e^{i\tau L_0} V(0) e^{-i\tau L_0}$$

& $V(\tau)$ describes the emission/absorption of a physical state from the end of the open string ($\sigma=0$) at τ if $h=1$.

For the tachyon: $V_T(k,\tau) = :e^{ik\cdot X(\tau)}: = e^{i\tau L_0} \underbrace{:e^{ik\cdot X(0)}:}_{V_T(k,0)} e^{-i\tau L_0}$

The point now is that starting from this, one can recover the tachyon state $|k;0\rangle$.

Consider the action of this operator on "zero momentum" string vacuum state $|0;0\rangle$

$L_0 = \frac{1}{2}\alpha_0^2 + N$

$$V_T(k,\tau)|0;0\rangle = e^{i\tau L_0}\left(:e^{ik\cdot X(0)}: \underbrace{e^{-i\tau L_0}\right)|0;0\rangle}_{|0;0\rangle \text{ as } L_0|0;0\rangle = 0}$$

$ik\cdot X(0,0) = ik\cdot x - \sqrt{2\alpha'}\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{1}{n} k\cdot\alpha_n^\mu$

$$V_T(k,\tau)|0;0\rangle = e^{i\tau L_0}\left(e^{\sqrt{2\alpha'}\sum_{n\geq1}\frac{1}{n}k\cdot\alpha_{-n}} : \underbrace{e^{ik\cdot x(0)} : \underbrace{e^{-\sqrt{2\alpha'}\sum_{n\geq1}\frac{1}{n}k\cdot\alpha_n}\right)|0,0\rangle}_{|0,0\rangle \text{ as } \alpha_n|0,0\rangle = 0 \ \forall n\geq 1}}_{|0;k\rangle}$$

So $V_T(K,\tau)|0;0\rangle = e^{i\tau L_0}\, e^{\sqrt{2\alpha'}\sum_{n\geq1}\frac{1}{n}K\cdot\alpha_{-n}}|0;K\rangle$

Wick rotation: pass to imaginary or Euclidean time on the WS

Define $z = e^{i\tau} = e^{t}$ (where $\tau = -it$)

so t is Euclidean WS time

Then:

$L_0 = \alpha' p^2 + \hat{N}$

$$V_T(K,\tau)|0;0\rangle = z^{L_0}\, e^{\sqrt{2\alpha'}\sum_{n\geq1}\frac{1}{n}K\cdot\alpha_{-n}}|0;K\rangle$$

$$= \underset{\alpha'k^2=1}{z^{1+\hat{N}}}\Big(1+\underbrace{\sqrt{2\alpha'}\,K\cdot\alpha_{-1}}_{N=1}+\frac{1}{2}\sqrt{2\alpha'}\,\underbrace{\big(K\cdot\alpha_{-2}+\sqrt{2\alpha'}\,(K\cdot\alpha_{-1})^2}_{N=2}\big)+\cdots\Big)|0;K\rangle$$

$$= z\Big(1+\sqrt{2\alpha'}\,z\cdot\alpha_{-1}+2\,\frac{1}{2}\sqrt{2\alpha'}\,z^2\big(K\cdot\alpha_{-2}+\sqrt{2\alpha'}\,(K\cdot\alpha_{-1})^2\big)+\cdots\Big)|0;K\rangle$$

We can recover the tachyon state $|0;K\rangle$ from V_T in the limit:

$$\lim_{z\to0}\frac{1}{z}V_T(K,\underbrace{-i\log z}_{\tau})|0;0\rangle = \lim_{t\to-\infty} e^{-t}\,V_T(K;it)|0;0\rangle = |0;K\rangle$$

past infinity

For the photon

$$L_0 = \alpha' p^2 + N$$

$$V_S(k,\tau)|0;0\rangle = \frac{1}{\sqrt{2\alpha'}} e^{i\tau L_0} \underbrace{V_S(k,0)\, e^{-i\tau L_0} |0;0\rangle}$$

$$= \frac{1}{\sqrt{2\alpha'}} e^{i\tau(\alpha' p^2 + N)} \zeta\cdot\dot{X}(0)\, \underbrace{e^{ik\cdot X(0)} \underbrace{e^{-i\tau L_0} |0;0\rangle}_{|0;0\rangle}}_{|0;k\rangle}$$

$$\underset{k^2=0}{\text{as}} = \frac{1}{\sqrt{2\alpha'}} z^N \left(\zeta\cdot\dot{X}(\tau=0)\right)|0;k\rangle = z^N \sum_{m>0} (\zeta\cdot\alpha_{-m})\, e^{\sum_{n>0}\frac{1}{n} k\cdot\alpha_{-n}} |0;k\rangle$$

$$= \Big(\underbrace{z(\zeta\cdot\alpha_{-1})}_{N=1} + \underbrace{z^2\left(\zeta\cdot\alpha_{-2} + (\zeta\cdot\alpha_{-1})(\zeta\cdot\alpha_{-1})\right)}_{N=2} + \cdots \Big)|0,k\rangle$$

We then recover the state $|\zeta;k\rangle$:

$$|\zeta;k\rangle = \lim_{z\to 0} \frac{1}{z} W_{\zeta,k}(-i\log z)|0;0\rangle$$

$$= \lim_{t\to -\infty} e^{-t}\, W_{\zeta,k}(it)|0;0\rangle$$

More generally,

$$|\psi\rangle \in \mathcal{H}_{phys} \xleftrightarrow{1-1} V_\psi(\tau) \quad \text{with}$$

$$|\psi\rangle = \lim_{z\to 0} \frac{1}{z} V_\psi(-i\log z)|0;0\rangle$$

$$= \lim_{t\to -\infty} e^{-t} V_\psi(it)|0;0\rangle$$

We can treat annalogously the "out going" states.
For the tachyon, for example:

$$\langle 0;0| V_T(k, i\tau) = \langle 0;k| \, e^{\sqrt{2\alpha'} \sum_{n>0} \frac{1}{n} k\cdot\alpha_n} e^{-i\tau L_0} \quad z^{-L_0}$$

$$= \frac{1}{z} \langle 0;k| \left(1 + \sqrt{2\alpha'}\, z^{-1} (\alpha_1 \cdot k) + 2\sqrt{\alpha'}\, \frac{1}{2} z^{-2} (\alpha_2 \cdot k + (k\cdot\alpha_1)(k\cdot\alpha_1)) + \cdots \right)$$

$$\Rightarrow \langle 0;k| = \lim_{z\to\infty} z \langle 0;0| V_T(k, -i\ln z) = \lim_{t\to\infty} e^{t} \langle 0;0| V_T(k, -i\tau)$$

In summary:

incomming state $$|\psi\rangle = \lim_{z\to 0} \frac{1}{z} V_\psi(-i\log z)\, |0;0\rangle$$

(ie recover an incomming state $|\psi\rangle$ by acting with a VO on the vacuum state Euclidean past infinite limit)

$$\langle\phi| = \lim_{z\to\infty} z\, \langle 0;0|\, V_\psi(-i\log z)$$

(ie recover an outgoing state $\langle\phi|$ by acting in infinite Euclidean future with a VO)

Remark: In 2dim CFT we have we have more generally a state – operator correspondence where

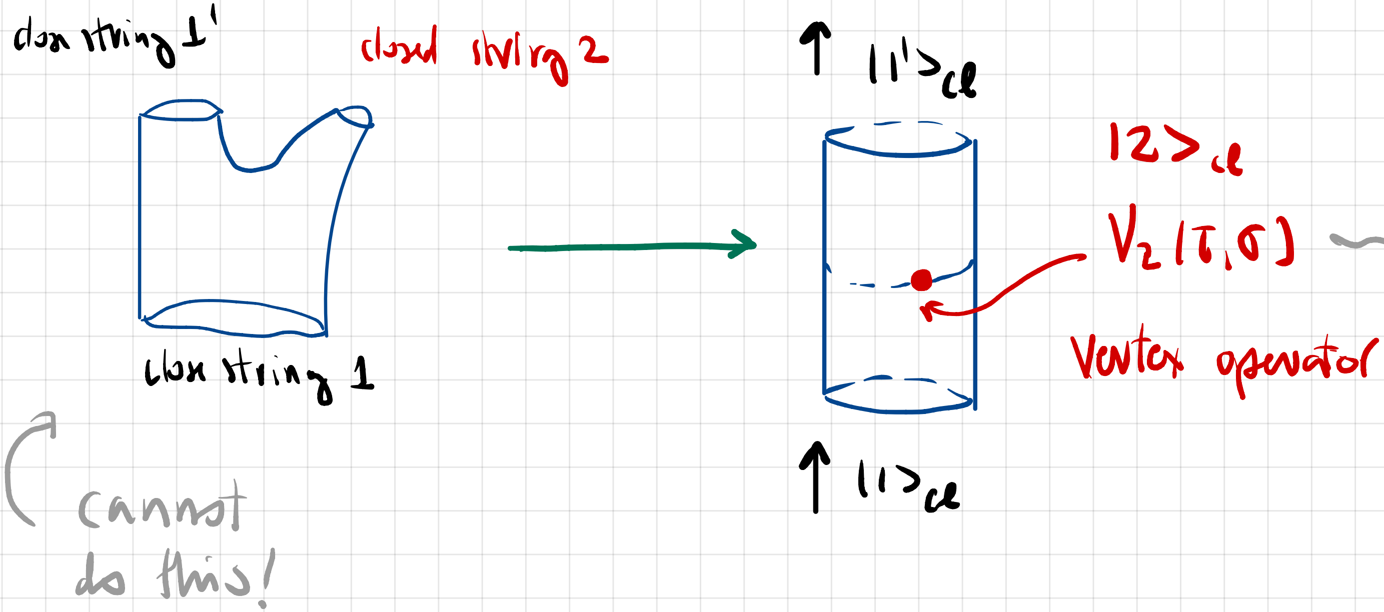
$$|\psi_A\rangle \xleftrightarrow{1-1} A(\tau)$$

"highest weight" state of $\mathcal{H}$ — primary operator of weight h_A

$$|\psi_A\rangle = \lim_{t \to -\infty} z^{h_A} A(it)\,|0,0\rangle$$

$$\langle\psi_A| = \langle 0,0| \lim_{t \to \infty} z^{-h_A} A(it)$$

3.5 Vertex operators for closed strings



$$\int d\tau\, d\sigma\, \underbrace{V_2(\tau,\sigma)}_{\text{inserted on the interior of the WS}}$$

we describe emission/absorption of a quantum physical state from a fixed string world sheet by the action of a local operator or vertex operator V st

$\mathcal{H}_{phys} \xrightarrow{V} \mathcal{H}_{phys}$ $\qquad$ $\mathcal{H}_{null} \xrightarrow{V} \mathcal{H}_{null}$

A primary operator of dimension $(h,\tilde{h})$ is an operator transforming under conformal transfs of the WS according to

$$\phi(\xi^+,\xi^-) \longrightarrow \tilde{\phi}(\tilde{\xi}^+,\tilde{\xi}^-) = \left(\frac{d\xi^+}{d\tilde{\xi}^+}\right)^h \left(\frac{d\xi^-}{d\tilde{\xi}^-}\right)^{\tilde{h}} \phi(\xi^+,\xi^-)$$

The corresponding infinitesimal transformations are

$$\delta\phi(\tau,\sigma) = -\partial_+(\tilde{\epsilon}\phi) - (\tilde{h}-1)(\partial_+\tilde{\epsilon})\phi - \partial_-(\epsilon\phi) - (h-1)(\partial_-\epsilon)\phi$$

This is a total derivative if $h = \tilde{h} = 1$

For $\epsilon_n^{\pm} = \frac{i}{2} e^{2im\xi^{\pm}}$ this gives the action of L_m:

$$[L_m, \phi(\xi^{\pm})] = \frac{1}{2} e^{2im\xi^+}\left(-i\partial_+ + 2mh\right)\phi(\xi^{\pm})$$

$$[\tilde{L}_m, \phi(\xi^{\pm})] = \frac{1}{2} e^{2im\xi^-}\left(-i\partial_- + 2m\tilde{h}\right)\phi(\xi^{\pm})$$

Vertex operators : $\underline{h = \tilde{h} = 1}$

The closed string operator $\quad :e^{ik\cdot X(\xi^{\pm})}:$

is primary with $h=\tilde{h}=\frac{\alpha'}{4}k\cdot k$ so

$$V_T(k;\xi^{\pm}) = :e^{ik\cdot X(\xi^{\pm})}:$$

VO $(h=\tilde{h}=1)$ corresponding to the tachyon $\alpha' k^2 = 4$

For higher levels

$$V(k;\xi^{\pm}) = (\cdots) :e^{ik\cdot X(\xi^{\pm})}:$$

$L_0 = \frac{1}{2}\alpha_0^2 + N \qquad \alpha_0^{\mu} = \tilde{\alpha}_0^{\mu} = \sqrt{\frac{\alpha'}{2}}\,p$

$\boxed{\alpha' M^2 = 2(N+\tilde{N}-2a)}$

As for the open string: with the appropriate normal ordering one can construct a vertex operators $(h=\tilde{h}=1)$ for each $|\varphi\rangle \in \mathcal{H}_{phys}$

For the converse: given a Vertex operator $(h=\tilde{h}=1)$ one can recover a physical state.

If $V[\tau,\sigma]$ is a VO then we can write

translation in $\xi^{\pm}$ $<$ $L_0+\tilde{L}_0 \sim \partial_\tau$, $L_0-\tilde{L}_0 \sim \partial_\sigma$

$$V(\xi^{\pm}] = e^{2i\xi^{-}L_0+2i\xi^{+}\tilde{L}_0}\, V(0,0;k]\, e^{-2i\xi^{-}L_0-2i\xi^{+}\tilde{L}_0}$$

insertion of vertex at the origin $\xi^{\pm}=0$

Wick rotation to Euclidean time $\tau=-it$:

light-cone coordinates: $\xi^{\pm}=-i(t\pm i\sigma)$

and let $z=e^{+i\xi^{+}}$, $\bar{z}=e^{i\xi^{-}}$ $\qquad \bar{\xi}^{+}=-\xi^{-}$

The state vertex correspondence is given by

$$|\psi\rangle = \lim_{t\to-\infty} e^{-4t} \underline{V_\psi(-it,\sigma)} |0,\tilde{0};0\rangle$$

$$= \lim_{|z|\to 0} (z\bar{z})^{-2} \underline{V_\psi}\left(-\frac{i}{2}\log(z\bar{z}), -\frac{i}{2}\log\left(\frac{z}{\bar{z}}\right)\right) |0,\tilde{0};0\rangle$$

$$z = e^{i\xi^+}, \quad \bar{z} = e^{i\xi^-}$$

$$z\bar{z} = e^{2t} \qquad \frac{z}{\bar{z}} = e^{2i\sigma}$$

so $|\psi\rangle$ recovered by the insertion of VO at euclidean past-infinity

↳ 3 point interactions (tree level)