String Theory 1

Lecture # 10

3 Interactions

Last lecture state-vertex correspondence

open string:

incomming state $\quad |\psi\rangle = \lim_{z\to 0} \frac{1}{z} V_\psi(-i\log z)\,|0;0\rangle$

$\tau = -it$

$z = e^{i\tau} = e^{t}$

(ie recover an incomming state $|\psi\rangle$ by acting in infinite Euclidean past with a VO)

$$\langle\phi| = \lim_{z\to\infty} z\,\langle 0;0|\,V_\psi(-i\log z)$$

(ie recover an outgoing state $\langle\phi|$ by acting in infinite Euclidean future with a VO)

closed string $\quad z = e^{i\xi^{+}} \qquad \bar z = e^{i\xi^{-}} \qquad z\bar z = e^{2t} \qquad \frac{z}{\bar z} = e^{2i\sigma}$

$$|\psi\rangle = \lim_{z\to 0} (z\bar z)^{-1} V_\psi(-it\,,\sigma)\,|0;0\rangle$$

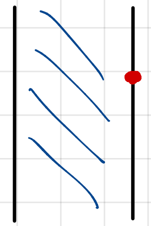
3.6 Three point scattering

tree level

3-point open string interaction (open string)

$\langle 0; -k_3|$

outgoing OS tachyon



absorption of OS tachyon at τ $(\sigma=0)$

$|0; k_1\rangle$

incoming open string tachyon

$$= A_3^{op}(k_1, k_2, k_3)$$

3-tachyon interaction

divide out by an infinite volume of the gauge group

$$= g_o \int_{-\infty}^{\infty} d\tau \, \langle 0; -k_3 | V_T(\tau; k_2) | 0; k_1 \rangle \Big/ \mathrm{Vol(conf.)}$$

ansatz

open string coupling constant

value of τ is immaterial

expect this because τ-translations correspond to a residual gauge invariance

$L_0 |\psi\rangle_{phys} = |\psi\rangle_{phys}$

$$A_3^{\infty}(k_1,k_2,k_3) = g_0 \int d\tau \, \underbrace{\langle 0;-k_3 | e^{i\tau L_0}}_{\langle 0;-k_3| e^{i\tau}} V_T(0;k_2) \underbrace{e^{-i\tau L_0} |0;k_1\rangle}_{e^{-i\tau}|0;k_1\rangle} \Big/ \mathrm{Vol(conf.)}$$

$$= g_0 \, \langle 0;-k_3 | \underbrace{V_T(0;k_2)}_{} |0;k_1\rangle \left(\int d\tau \Big/ \mathrm{Vol(conf.)} \right)$$

$$: \underbrace{e^{k\cdot \sum_{m=1}^{\infty} \frac{\alpha_{-m}}{m}}}_{\underbrace{1+\text{ creation operators}}_{\to 1}} \, e^{ik\cdot x(0)} \, \underbrace{e^{-k \sum_{n=1}^{\infty} \frac{\alpha_n}{n}}}_{\underbrace{1+\text{ annihilation operators}}_{\to 1}} :$$

$$= g_0 \, \langle 0;-k_3 | e^{ik_2\cdot x(0)} |0;k_1\rangle \left(\int d\tau \Big/ \mathrm{Vol(conf.)} \right)$$

$$\mathcal{A}_3^{\varphi}(k_1,k_2,k_3) = g_0 \langle 0;-k_3 | 0; k_1+k_2 \rangle \left(\int d\tau \Big/ \mathrm{Vol(conf.)} \right)$$

$$= g_0\, \delta(k_1+k_2+k_3) \left[\int_{-\infty}^{\infty} d\tau \Big/ \mathrm{Vol(conf.)} \right]$$

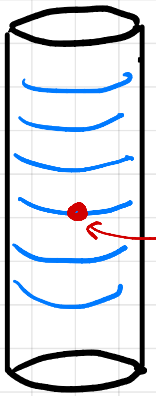
alternatively: gauge fix at $\tau = 0$ (more later: use Fadeev-Popov procedure)

$$\boxed{\mathcal{A}_3^{\varphi}(k_1,k_2,k_3) = g_0\, \delta(k_1+k_2+k_3)}$$

expected result from Feynman rules for a 3-point vertex interaction

three point tree level closed string diagrams
(tachyon absorbing a tachyon)

$\langle 0; -k_3 |$



$|0; k_1\rangle$

$V_T^{cl}(k_2, \xi^{\pm})$ $= A_3^{cl}(k_1, k_2, k_3)$

$$= g_{cl} \int d^2\xi^{\pm} \, \langle 0; -k_3 | V_T^{cl}(k_2, \xi^{\pm}) | 0; k_1 \rangle \Big/ \mathrm{Vol(conf.)}$$

closed string coupling

Recall

$$V_T^{cl}(\xi^{\pm}, k) = e^{2i\xi^- L_0 + 2i\xi^+ \tilde{L}_0}\, V_T^{cl}(0, k)\, e^{-2i\xi^- L_0 - 2i\xi^+ \tilde{L}_0}$$

Hamiltonian: $H \sim (L_0 + \tilde{L}_0) \sim \partial_\tau$

$L_0 - \tilde{L}_0 \sim \partial_\sigma$

$$A_3^{cl}(k_1, k_2, k_3)$$

$(L_0 - 1)\,|\text{phys}\rangle = 0$

$$= g_{cl} \int d^2\xi^{\pm}\, \underbrace{\langle 0; -k_3 |\, e^{2i\xi^- L_0 + 2i\xi^+ \tilde{L}_0}}_{\langle 0; -k_3 |\, e^{2i\xi^- + 2i\xi^+}}\, V_T^{cl}(0, k)\, \underbrace{e^{-2i\xi^- L_0 - 2i\xi^+ \tilde{L}_0}\, |0; k_1\rangle}_{e^{-2i\xi^- - 2i\xi^+}\, |0; k_1\rangle} \Big/ \text{Vol(conf.)}$$

$$= g_{cl} \int d^2\xi^{\pm} \langle 0; -k_3 |\, \underbrace{V_T^{cl}(k_2, 0)}_{(1 + \text{creation}) : e^{ik_2 \cdot x} : (1 + \text{annihilation})}\, |0; k_1\rangle \Big/ \text{Vol(conf.)}$$

$$\mathcal{A}_3^{cl}(k_1,k_2,k_3) = g_{cl} \int d^2\xi^{\pm} \langle 0; -k_3 | 0; k_1+k_2 \rangle \Big/ \mathrm{Vol(conf.)}$$

$$= g_{cl}\, \delta(k_1+k_2+k_3) \int d^2\xi^{\pm} \Big/ \mathrm{Vol(conf.)}$$

$$\boxed{\mathcal{A}_3^{cl}(k_1,k_2,k_3) = g_{cl}\, \delta(k_1+k_2+k_3)}$$

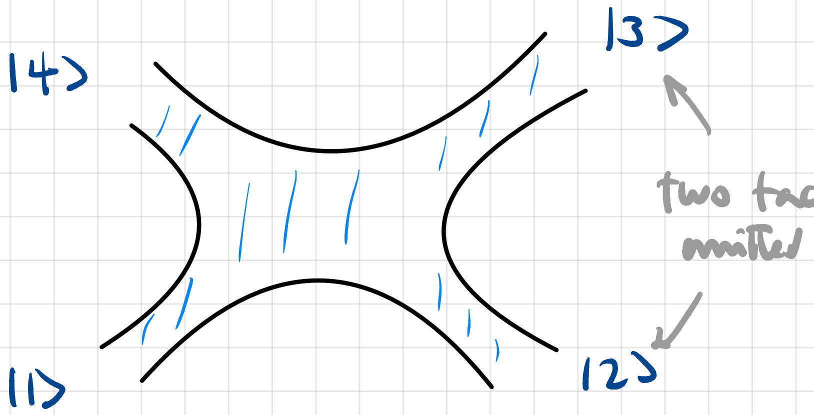
PS 3 3 point amplitudes with tachyons & level-one states

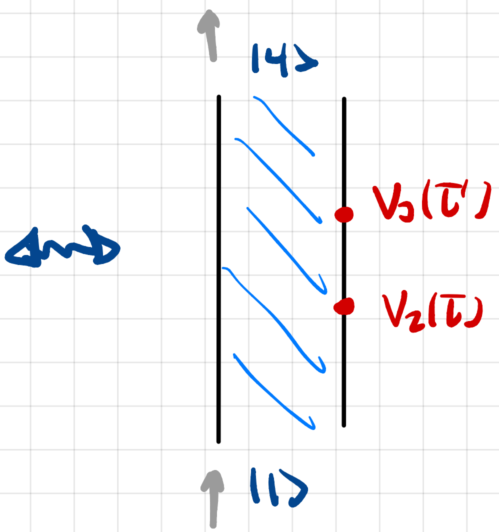
3.7 The Veneziano amplitude

or 4-point tachyon amplitude

[This section: features of amplitudes than can be generalized to n-point amplitudes in open & closed string amplitudes]

outgoing with mom $-k_4$





incoming tachyon with mom k_1

$$A_4(k_1,k_2,k_3,k_4) = g_o^2 \int_{\tau'>\tau} d\tau\, d\tau' \langle 0;-k_4|\, V_3(\tau')V_2(\tau)\, |0;k_1\rangle \Big/ \text{Vol(conf)}$$

joining two 3-point diagrams

Use the residual gauge freedom to fix $\tau' = 0$

$$A_4(k_1,k_2,k_3,k_4) = g_0^2 \int_{-\infty}^{0} d\tau \, \langle 0;-k_4 | V_3(0) V_2(\tau) | 0;k_1 \rangle$$

(so no need to divide by Vol(conf))

$$= g_0^2 \int_{-\infty}^{0} d\tau \, \langle 0,-k_4 | V_3(0) \, e^{i\tau L_0} V_2(0) \underbrace{e^{-i\tau L_0} |0;k_1\rangle}_{e^{-i\tau} |0;k_1\rangle}$$

$$= g_0^2 \int_{-\infty}^{0} d\tau \, \langle 0;-k_4 | V_3(0) \, e^{i\tau(L_0-1)} V_2(0) | 0;k_1 \rangle$$

$$= g_0^2 \langle 0;-k_4 | V_3(0) \underbrace{\left(\int_{-\infty}^{0} d\tau \, e^{i\tau(L_0-1)} \right)}_{\text{how do we deal with this?}} V_2(0) | 0;k_1 \rangle$$

$\rightarrow$ propagator!

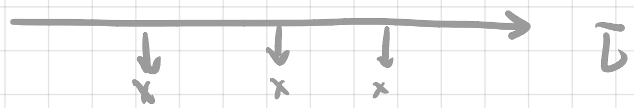
$$\Phi_4(K_1, K_2, K_3, K_4) = g_0^2 \langle 0; -K_4 | V_3(0) \left(\int_{-\infty}^{0} d\tau \, e^{i\tau(L_0-1)} \right) V_2(0) | 0; K_1 \rangle$$

$$\left[\overset{?}{=} g_0^2 \langle 0; -K_4 | V_3(0) \left(\frac{-i + \text{osc. modes}}{L_0 - 1} \right) V_2(0) | 0; K_1 \rangle \right]$$

poles

To obtain a convergent diagram, we consider instead

$$\int_{-\infty}^{0} d\tau \, e^{i\tau(L_0 - 1 - i\epsilon)} |\cdots\rangle$$



and we "rotate" the contour: $t = i\tau$

$$\int_{-\infty}^{0} d\tau \, e^{i\tau(L_0 - 1 - i\epsilon)} |\cdots\rangle \longrightarrow \int_{-\infty}^{0} dt \, e^{t(L_0-1)} |\cdots\rangle$$

$$A_4(k_1,k_2,k_3,k_4) = g_0^2 \langle 0;-k_4| V_3(0) \left(\int_{-\infty}^{0} d\tau\, e^{i\tau(L_0-1)} \right) V_2(0) |0;k\rangle$$

$$= g_0^2 \langle 0;-k_4| V_3(0) \left(\int_{-\infty}^{0} d\tau\, e^{i\tau(L_0-1-i\epsilon)} \right) V_2(0) |0;k_1\rangle$$

"rotate" contour $\tau = -it$

$$= g_0^2 \langle 0;-k_4| V_3(0) \left(\int_{-\infty}^{0} dt\, e^{t(L_0-1)} \right) V_2(0) |0;k_1\rangle$$

pull out integral

$$= g_0^2 \int_{-\infty}^{0} dt\, \langle 0;-k_4| V_3(0) \underbrace{e^{t(L_0-1)} V_2(0] |0;k_1\rangle}_{\text{back track (but now with } t \text{ instead of } i\tau)}$$

$$= g_0^2 \int_{-\infty}^{0} dt\, \langle 0;-k_4| V_3(0) \underbrace{e^{tL_0} V_2(0) e^{-tL_0}}_{\text{Vertex operator at } -it} |0;k_1\rangle$$

ie $V_2(-it]$

$$A_4(k_1,k_2,k_3,k_4) = g_0^2 \int_{-\infty}^{0} dt \, \langle 0; -k_4 | V_3(0) \, V_2(-it) | 0; k_1 \rangle$$

- Compare with original expression

$$A_4(k_1,k_2,k_3,k_4) = g_0^2 \int_{-\infty}^{0} d\tau \, \langle 0; -k_4 | V_3(0) \, V_2(\tau) | 0; k_1 \rangle$$

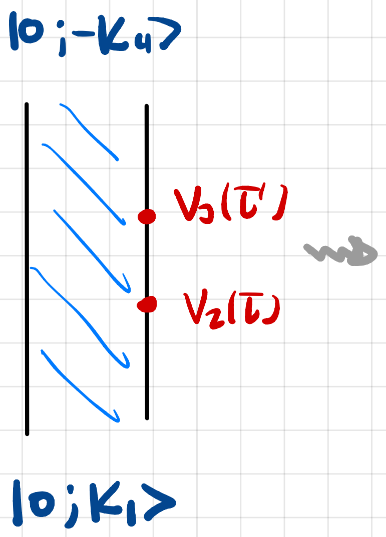
Now all operators are in Euclidean worldsheet time

(We have manipulated amplitude st we work instead in Euclidean WS time)

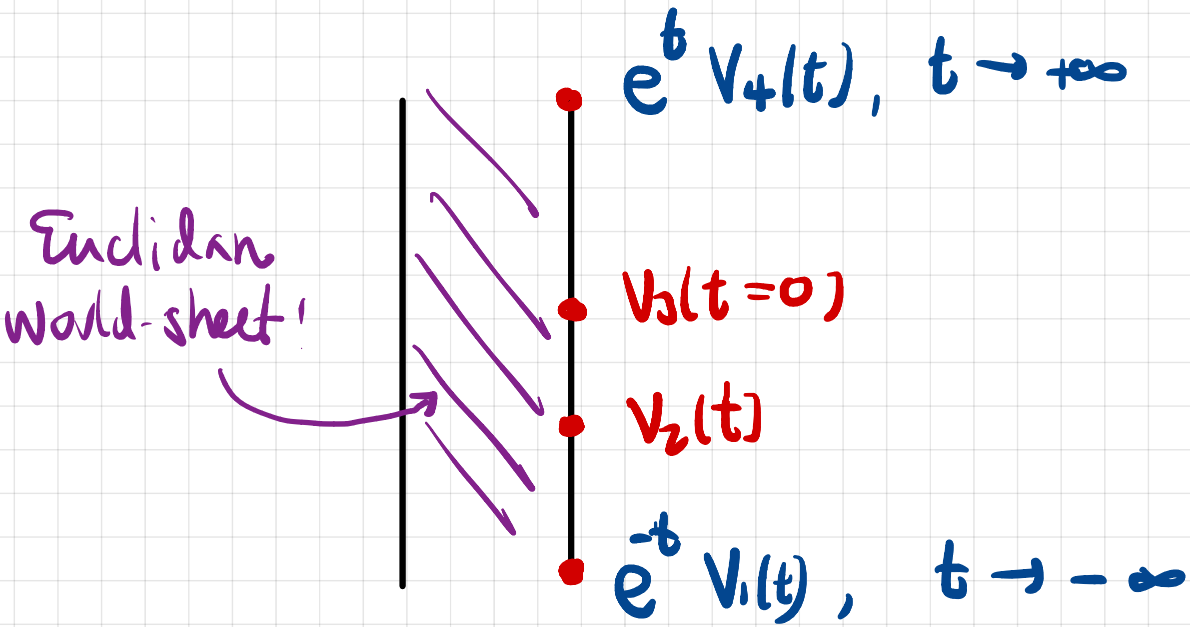
↳ Next: We will rework this in Euclidean WS

The amplitude now has an interpretation in Euclidean worldsheet time, so we discover a Euclidean interpretation of the amplitude

$$\mathcal{A}_4(k_1, k_2, k_3, k_4) = g_o^2 \int_{-\infty}^{0} dt \, \langle 0; -k_4 | V_3(0) \, V_2(-it) \, | 0; k_1 \rangle$$



amplitude in a Lorentzian world-sheet



↳ $|0;k\rangle = \lim_{t \to -\infty} e^{-t} V_T(t) \, |0;0\rangle$

physical state obtained by inserting the appropriate VO in past infinity

Consider now a Euclidean conformal map

$$z = e^{t+i\sigma} \quad , \quad \bar{z} = e^{t-i\sigma}$$

new coordinates on EWS

What happens to the strip?

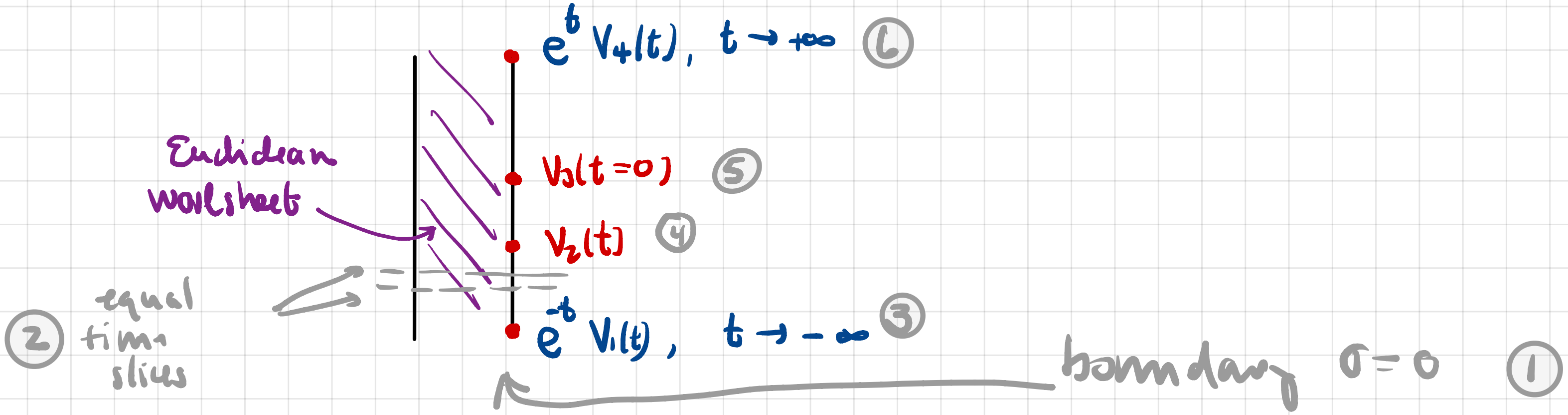
metric: $dz\,d\bar{z} = e^{2t}(dt^2 + d\sigma^2)$ (Weyl equivalent to $dt^2+d\sigma^2$)

so $\dfrac{dz\,d\bar{z}}{|z|^2} = dt^2 + d\sigma^2$ metric on the upper-half plane (UHP)

Indeed: $\mathrm{Im}\, z = \frac{1}{2i}(z - \bar{z}) = \frac{1}{2i} e^t (e^{i\sigma} - e^{-i\sigma}) = e^t \sin\sigma$

but $0 \leq \sigma < \pi \implies \sin\sigma \geq 0 \implies \mathrm{Im}\, z \geq 0$

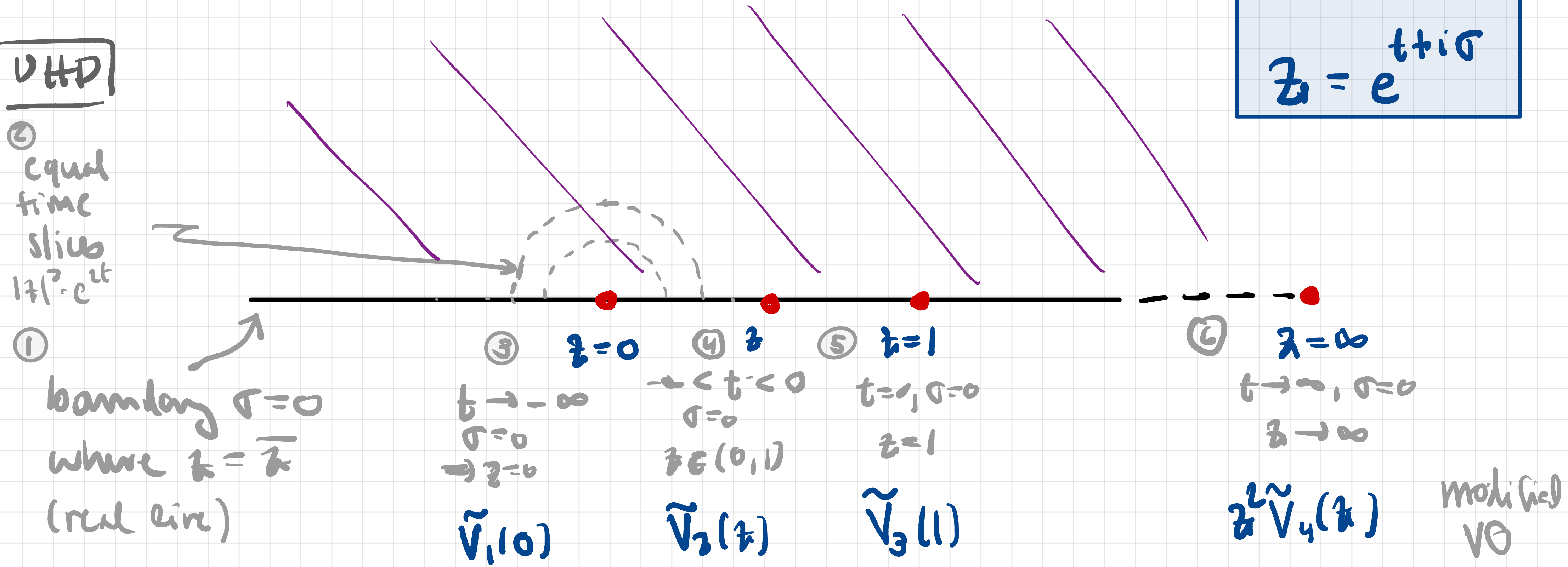
The strip



is mapped to the UHP with four marked point

UHP

$$z = e^{t+i\sigma}$$



Modification of the vertex operators due to the conformal transformation:

Recall that a vertex operator $V(\tau)$ transforms as

$$V(\tau) \longrightarrow V(\tilde{\tau}) = \left(\frac{d\tau}{d\tilde{\tau}}\right)^{1} V(\tau)$$ primary with $h=1$

Then

$$\tilde{V}(\underbrace{z=\bar{z}}_{\sigma=0}) = \frac{dt}{dz} V(t) = z^{-1} V(t)$$

$z = e^{t+i\sigma} \Longrightarrow \frac{dz}{dt} = z$

So, the conformal transformations modify the VO according to $\tilde{V}(z=\bar{z}) = \frac{1}{z} V(t)$

$\boxed{z = e^{t}}$

- incoming state $\quad |0; k_1\rangle = \lim_{t\to-\infty} e^{-t} V_1(t) |0;0\rangle$

$$\Rightarrow \quad \lim_{\substack{t\to-\infty \\ z\to 0}} \underset{z^{-1}}{e^{-t}} \underset{z\tilde{V}_1(z=\bar{z})}{V_1(t)} = \lim_{z\to 0} \tilde{V}_1(z)$$

- $V_2(t)$: $\quad \underbrace{V_2(t)\,dt}_{\text{we have this in the amplitude}} = \underbrace{z\,\tilde{V}_2(z)}_{V_2(t)} \frac{1}{z} dz = \tilde{V}_2(z)\,dz$

$dt = \frac{dt}{dz} dz = \frac{1}{z} dz$

- $V_3(t=0)$: $\quad V_3(t=0) = 1 \cdot \tilde{V}_3(1)$

as $t=0 \rightarrow z=1$

- outgoing state $\quad \langle 0; -k_4| = \lim_{t\to\infty} e^{t} V_4(t)$

$$\Rightarrow \quad \lim_{t\to\infty} \underset{z}{e^{t}} \underset{z\tilde{V}_4}{V_4(t)} = \lim_{z\to\infty} z^2 \tilde{V}_4(z)$$

We can now write the amplitude on the strip as an amplitude on the UHP...

Let's discuss first conformal transformations and the gauge fixing procedure in more generality (which is usefull when computing more general amplitudes)

The group of global conformal transformations of the UHP is $PSL(2,\mathbb{R})$

Mobious group

$$z \longmapsto \frac{az+b}{cz+d}, \qquad a,b,c,d \in \mathbb{R}, \quad \det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = 1$$

This is a three dimensional group of residual gauge symmetries.

One can check that this group maps 1) the UHP into itself

2) 1-1 mappings of the UHP into itself are of the form $z \to \tilde{z}(z)$ ie analytic functions of z.

Hence, $PSL(2,\mathbb{R})$ is a subgroup of the full conformal group

3) Generators of $PSL(2,\mathbb{R})$

L_{-1} : $z \to z+b$

L_0 : $z \to az$

L_1 : $z \to \dfrac{z}{1+cz}$ special conf transformation $\left(w = -\frac{1}{z} \longmapsto w - c\right)$

Important properties of $PSL(2;\mathbb{R})$

▷ One can find a transformation which maps any distinct three points $\{z_1, z_3, z_4\}$ the points $\{0, 1, \infty\}$

Indeed $\quad z \longmapsto \dfrac{z_{43}}{z_{31}} \dfrac{z-z_1}{z_4-z} \quad$ where $z_{ij} = z_i - z_j$

(one can easily prove this by asking which $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{R})$ maps $z_1 \to 0$, $z_2 \to 1$, $z_4 \to \infty$)

One can use this to gauge fix the three point amplitude

↳ Next properties of $PSL(2,\mathbb{Z})$

covariant amplitudes & the Faddeev-Popov trick