String Theory 1

Lecture # 11

3 Interactions

3.1 Generalities

3.2 Vertex operators: introduction

3.3 Vertex operators: open string

3.4 The state vertex correspondence open strings

3.5 Vertex operator: closed string

3.6 3-point interactions

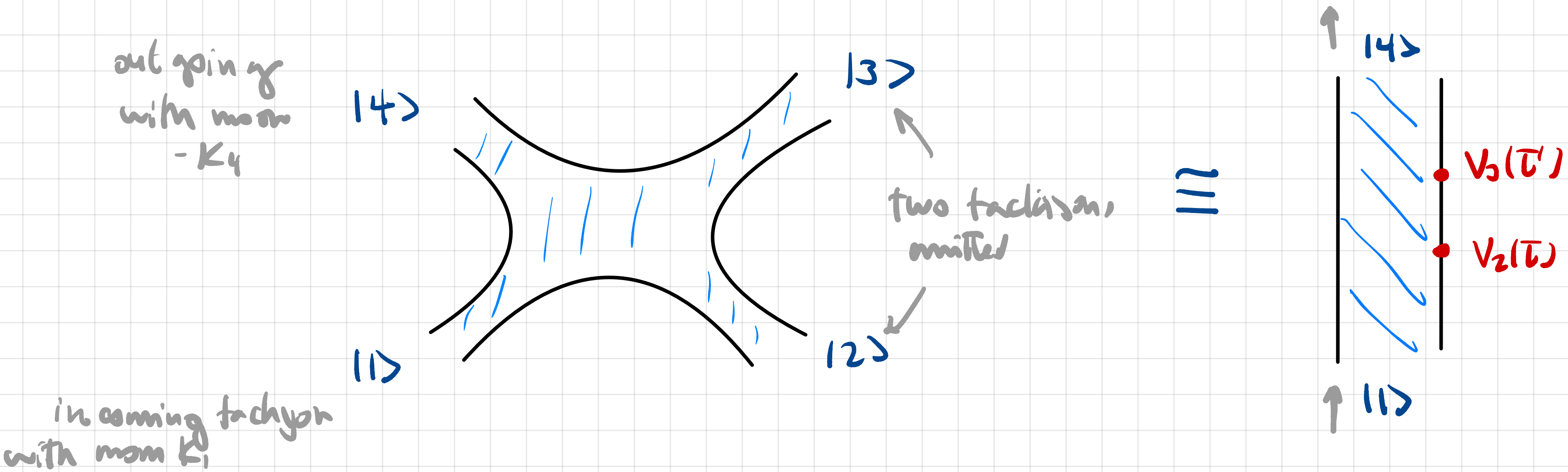
3.7 4-point tachyon amplitude

3.8 Comments on the general picture

3.7 The Veneziano amplitude continued

or 4-tachyon amplitude

[This section: features of amplitudes that can be generalised to n-point amplitudes in open & closed string amplitudes]



$$\mathcal{A}_4(k_1,k_2,k_3,k_4) = g_o^2 \int_{\tau'>\tau} d\tau\, d\tau' \langle 0;-k_4|\, V_3(\tau')\, V_2(\tau)\, |0;k_1\rangle \Big/ \mathrm{Vol(conf)}$$

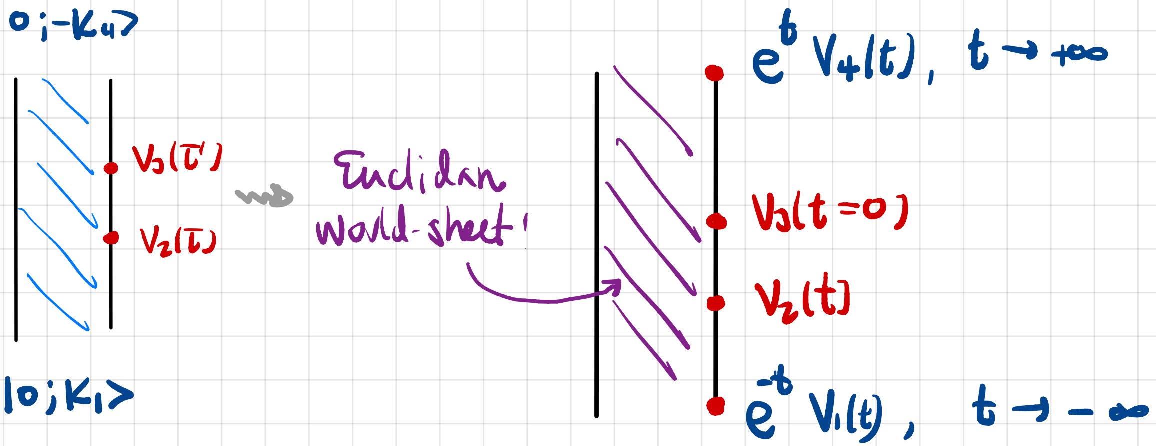
Lorentzian WS

$$= g_o^2 \int_{-\infty}^{0} d\tau\, \langle 0;-k_4|\, V_3(0)\, V_2(\tau)\, |0;k_1\rangle$$

$\tau = -it$

$$= g_o^2 \int_{-\infty}^{0} dt\, \langle 0;-k_4|\, V_3(0)\, V_2(-it)\, |0;k_1\rangle$$

Euclidean WS



use the state-vertex correspondence

- Similar expressions for n-point amplitudes!
- Amplitudes now have an interpretation in Euclidean worldsheet

↳ Next: rework amplitudes in Euclidean WS and

- exploit conformal symmetries (to learn about general properties of amplitudes)

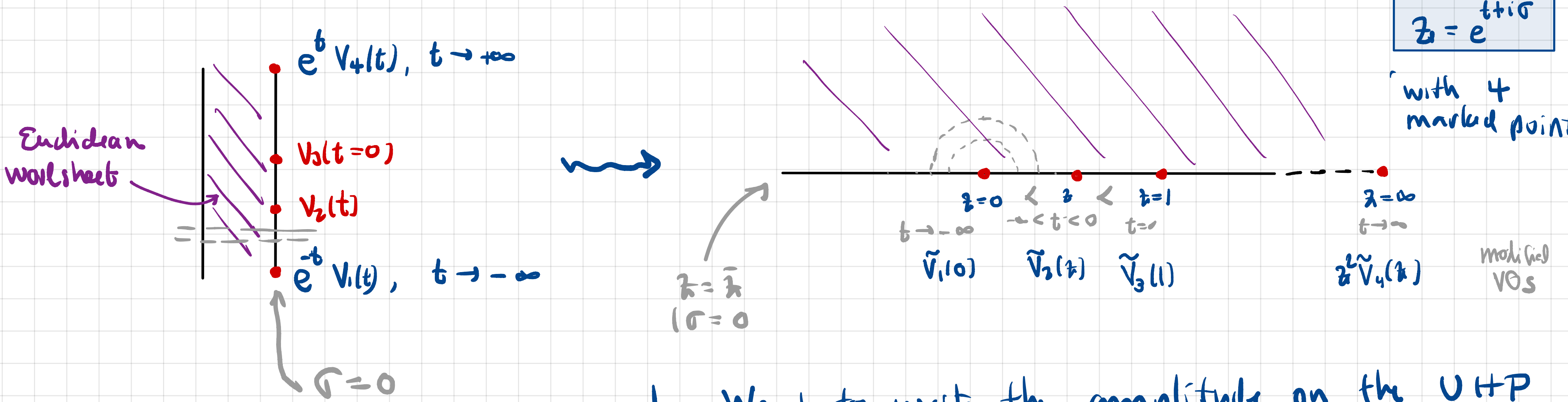
Consider now a Euclidean conformal map

$$z = e^{t+i\sigma}, \quad \bar{z} = e^{t-i\sigma}$$

new coordinates on EWS

UHP $z = e^{t+i\sigma}$

with 4 marked points



↳ Want to write the amplitude on the UHP but now write it covariantly and fix the gauge using the Faddeev-Popov "trick"

↳ need to learn more about conformal transformations

Recall

The group of global conformal transformations of the UHP is $PSL(2,\mathbb{R})$

Mobius group

$$z \longmapsto \frac{az+b}{cz+d}, \qquad a,b,c,d \in \mathbb{R}, \quad \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1$$

This is a three dimensional group of residual gauge symmetries.

(1-1 mappings of UHP $\to$ UHP generated by $\{L_{-1}, L_0, L_1\}$)

$z \mapsto z+b$ $\quad$ $z \mapsto az$ $\quad$ $z \mapsto \dfrac{z}{1+cz}$

$\left(w = -\frac{1}{z} \mapsto w - c\right)$

Important properties of $PSL(2;\mathbb{R})$

- One can find a transformation which maps any distinct three points $\{z_1, z_3, z_4\}$ to the points $\{0, 1, \infty\}$

Indeed
$$z \longmapsto \frac{z_{43}}{z_{31}} \frac{z - z_1}{z_4 - z} \qquad \text{where } z_{ij} = z_i - z_j \ (i \neq j)$$

One can easily prove this by asking which $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in PSL(2,\mathbb{R})$ maps $z_1 \to 0$, $z_2 \to 1$, $z_4 \to \infty$

1) $0 = \dfrac{a z_1 + b}{c z_1 + d} \Rightarrow b = -a z_1$

2) $1 = \dfrac{a z_3 - (a z_1)}{c z_3 + d} \Rightarrow c z_3 + d = a z_{31} \Rightarrow d = a z_{31} - c z_3$

3) $0 = \dfrac{c z_4 + d}{a z_4 + b} \Rightarrow c z_4 + a z_{31} - c z_3 = 0 \Rightarrow a z_{31} = -c z_{43}$

$$z \mapsto \frac{a z - a z_1}{-\frac{a z_{31}}{z_{43}} z + a z_{31} + \frac{a z_{31}}{z_{43}} z_3} = \frac{z - z_1}{\frac{z_{31}}{z_{43}}\left(-z + z_{43} + z_3\right)} = \frac{z_{43}}{z_{31}} \frac{z - z_1}{z_4 - z}$$

One can use this to gauge fix the three point amplitude

- Of particular interest for us is the fact that $PSL(2,\mathbb{R})$ preserves the cyclic ordering of any four points on the boundary $(\sigma=0)$

Consider the four points $\{z_1, z_2, z_3, z_4\}$ on the boundary st

$$z_1 < z_2 < z_3 < z_4$$

Then the map above maps

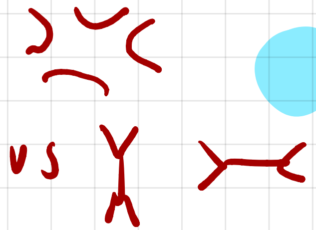
$$z_2 \longmapsto \frac{z_{21}\, z_{43}}{z_{31}\, z_{42}} \in (0,1)$$

conformal cross ratio

so it maps $(z_1, z_2, z_3, z_4) \longrightarrow \left(0, \frac{z_{12}\, z_{34}}{z_{13}\, z_{24}}, 1, \infty\right)$

ie fixing three points $\{z_1, z_3, z_4\}$ at $\{0, 1, \infty\}$ the fourth point $0 < z_2 < 1$.

The preservation of the cyclic ordering of four points on the boundary implies a cyclic symmetry of the four point amplitude (see eg GSW)

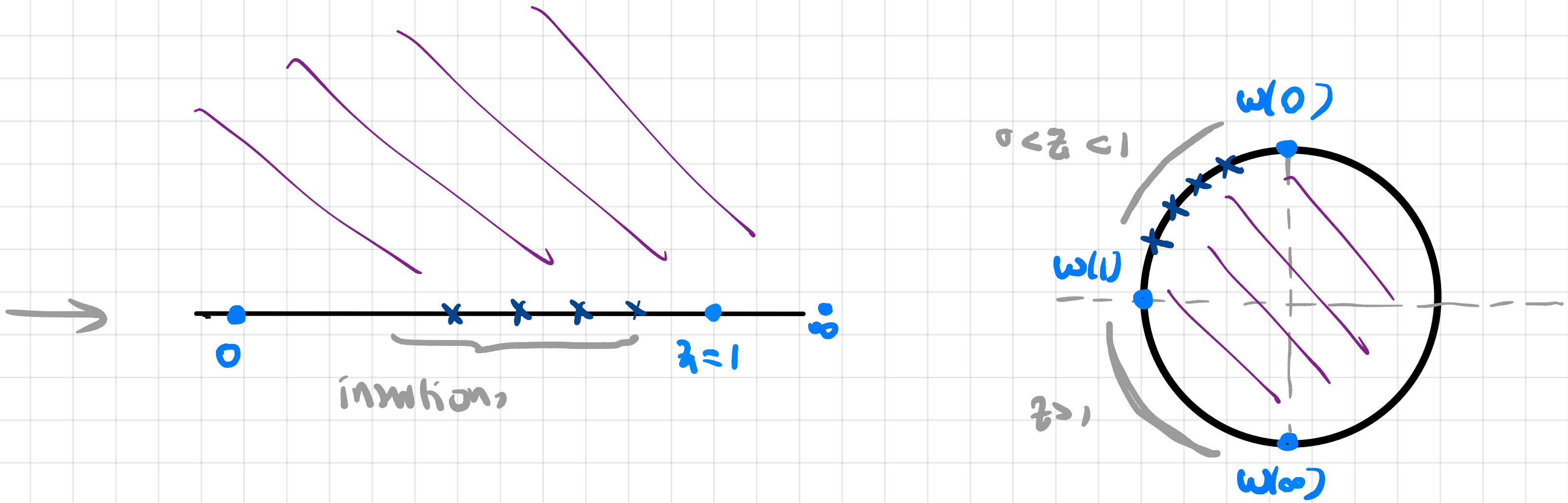


Remark

(see GSW chapter 7)

cyclic symmetry : an elegant way to make the cyclic symmetry of amplitudes manifest is by introducing the (new) map $z \longmapsto w = \frac{1-z}{1-iz}$

which maps UHP $\longrightarrow$ unit disk



- boundary $\longrightarrow$ boundary of the unit disk

(z real $\quad |w|^2 = \frac{z\bar{z}+1}{z\bar{z}+1} = 1$)

Return to 4-point amplitude: $z = e^{t+i\sigma}$

$$A_4(k_1, k_2, k_3, k_4) = g_0^2 \int_{-\infty}^{0} dt \, \underbrace{\langle 0; -k_4 \,|\, V_3(0)\, V_2(it) \,|\, 0; k_1 \rangle}_{\lim_{z' \to \infty} z'^2 \langle \tilde{V}_4(\frac{1}{z'}) \tilde{V}_3(1) \tilde{V}_2(z) \tilde{V}_1(0) \rangle}$$

we have the interval on the boundary $\int_{-\infty}^{0} dt \longrightarrow \int_0^1 dz$

We say that the interval on the boundary $(0,1)$ the is the moduli space of conformal structures on the UHP with four marked points.

We will return to this notion of moduli space later

Covariance and the Fadeev-Popov gauge fixing:

We could have written the four-point amplitude covariantly,
(if we had done the Wick rotation & conformal transformation before...)

$$g_0^2 \int \underbrace{d\tau_1\, d\tau_2\, d\tau_3\, d\tau_4}_{\text{overcounts equivalent configurations}} \langle \tilde{V}_4(\tau_4)\, \tilde{V}_3(\tau_3)\, \tilde{V}_2(\tau_2)\, \tilde{V}_1(\tau_1) \rangle \Big/ \underbrace{\mathrm{Vol}(SL(2;\mathbb{R}))}_{\text{need this here due to the overcounting of configurations}}$$

and then use the Faddeev-Popov trick to fix the gauge,
(that is to deal with any residual gauge symmetries).

see Peskin-Schroeder Chapter 9 for details
(also in AQFT)

Faddeev-Popov trick : fix the gauge by setting

$z_1 = z_1^0 \qquad z_3 = z_3^0 \qquad z_4 = z_4^0$

recall we only have the freedom to fix 3 points

Thm :

$$g_0^2 \int dz_1\, dz_2\, dz_3\, dz_4\, \delta(z_1 - z_1^0)\, \delta(z_3 - z_3^0)\, \delta(z_4 - z_4^0)$$

to account for the gauge choice

$$\times \left| \mathrm{Det}\, \frac{\partial(z_1, z_3, z_4)}{\partial(\lambda_{-1}, \lambda_0, \lambda_1)} \right|$$

Faddeev-Popov determinant

$\{\lambda_{-1}, \lambda_0, \lambda_1\}$ parameters of $PSL(2,\mathbb{Z})$

new measure for the amplitude

$$\times \langle \tilde{V}_4(z_4)\, \tilde{V}_3(z_3)\, \tilde{V}_2(z_2)\, \tilde{V}_1(z_1) \rangle$$

and we drop $\mathrm{Vol}(SL(2,\mathbb{R}))$

$$\left| \mathrm{Det} \, \frac{\partial(z_1, z_3, z_4)}{\partial(\lambda_{-1}, \lambda_0, \lambda_1)} \right| = \text{Jacobian of the transformation}$$

from z_1, z_2, z_3 to $\underbrace{\lambda_1, \lambda_0, \lambda_{-1}}_{\text{parameter of the gauge group}}$

$$= z_{43} \, z_{31} \, z_{41}$$

where $\delta z = \lambda_{-1} + \lambda_0 z + \lambda_1 z^2$ infinitesimal Möbius Transf

expand $\tilde{z} = \dfrac{az+b}{cz+d}$ around identity matrix $\begin{matrix} a = d = 1 \\ b = c = 0 \end{matrix}$

$\lambda_{-1} = \delta b$, $\lambda_0 = 2\delta a$, $\lambda_1 = -\delta c$

L_{-1} translations, L_0, L_1

$$\mathrm{Det} \, \frac{\partial(z_1, z_3, z_4)}{\partial(\lambda_{-1}, \lambda_0, \lambda_1)} = \begin{vmatrix} 1 & 1 & 1 \\ z_1 & z_3 & z_4 \\ z_1^2 & z_3^2 & z_4^2 \end{vmatrix} \begin{matrix} \leftarrow \partial z_i / \partial \lambda_{-1} \\ \leftarrow \partial z_i / \partial \lambda_0 \\ \leftarrow \partial z_i / \partial \lambda_1 \end{matrix} \qquad i = 1, 3, 4$$

$$= z_3 z_4^2 + z_4 z_1^2 + z_1 z_3^2 - (z_1^2 z_3 + z_3^2 z_4 + z_1 z_4^2)$$

$$= (z_4 - z_3)(z_3 - z_1)(z_4 - z_1)$$

$d\mu$ = measure on the UHP

$$g_o^2 \int \boxed{dz_1\, dz_2\, dz_3\, dz_4\, |(z_4-z_3)(z_3-z_1)(z_4-z_1)|}\, \delta(z_1 - z_1^0)\,\delta(z_3 - z_3^0)\,\delta(z_4 - z_4^0)$$

$$\times \langle \tilde{V}_4(z_4)\, \tilde{V}_3(z_3)\, \tilde{V}_2(z_2)\, \tilde{V}_1(z_1) \rangle_{UHP}$$

choose $z_1^0 = 0$, $z_3^0 = 1$, $z_4^0 = \Lambda \to \infty$

$$= \lim_{\Lambda\to\infty} g_o^2 \int dz_2\, \underbrace{\overset{z_{43}}{(\Lambda-1)}\overset{z_{41}}{\Lambda} \langle \tilde{V}_4(\Lambda)}_{\Lambda^2 \langle \tilde{V}_4(\Lambda)}\, \tilde{V}_3(1)\, \tilde{V}_2(z_2)\, \underbrace{\tilde{V}_1(0) \rangle_{UHP}}_{|1\rangle}$$

$$= g_o^2 \int_0^1 dz\, \langle 4 |\, \tilde{V}_3(1)\, \tilde{V}_2(z)\, | 1 \rangle$$

PS 3 This is the Veneziano amplitude.

$$A(k_1,k_2,k_3,k_4) = g_0^2\, \delta(k_1+k_2+k_3+k_4)$$

$$\times \Big(B(-\alpha(s), -\alpha(u)) + B(-\alpha(u), -\alpha(t)) + \underbrace{B(-\alpha(s), -\alpha(t))}_{A(s,t)} \Big)$$

where $\quad \alpha(x) = 1 + \alpha' x$

$$B(a,b) = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)} \qquad \text{Beta function}$$

$$\Gamma(a) = \int_0^\infty dt\; t^{x-1} e^{-t} \qquad \text{Euler's Gamma function}$$

Mandelstam variables
$$\begin{cases} s = -(k_1+k_2)^2 \\ u = -(k_1+k_4)^2 \\ t = -(k_1+k_3)^2 \end{cases}$$

$$s+u+t = \Sigma m_i^2$$

Consider $A(s,t) = B(-\alpha(s), -\alpha(t))$

- poles at $-\alpha(s) = 0, -1, -2, \dots$; $-\alpha(t) = 0, -1, -2, \dots$

(from numerator of the Gamma-function)

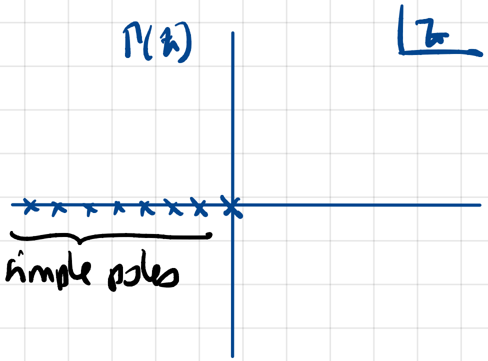
that is at $s = \frac{1}{\alpha'}(n-1)$, $n = 0, 1, 2, \dots$

precisely the masses of level n open string states (infinitely many)

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt, \quad \mathrm{Re}(z) > 0$$

$$\Gamma(z+1) = z\,\Gamma(z)$$

(use to analytically continue Γ to the whole z-plane)



$\Gamma(z)$ has no zeros

Behaviour near singularities: near $z = -n$, n non-negative integer

$$\Gamma(z) = \frac{\Gamma(z+n+1)}{z(z+1)\cdots(z+n-1)(z+n)} \sim \frac{(-1)^n}{n!}\,\frac{1}{z+n}$$

↳ using $\Gamma(z+1) = z\,\Gamma(z)$ repeatedly

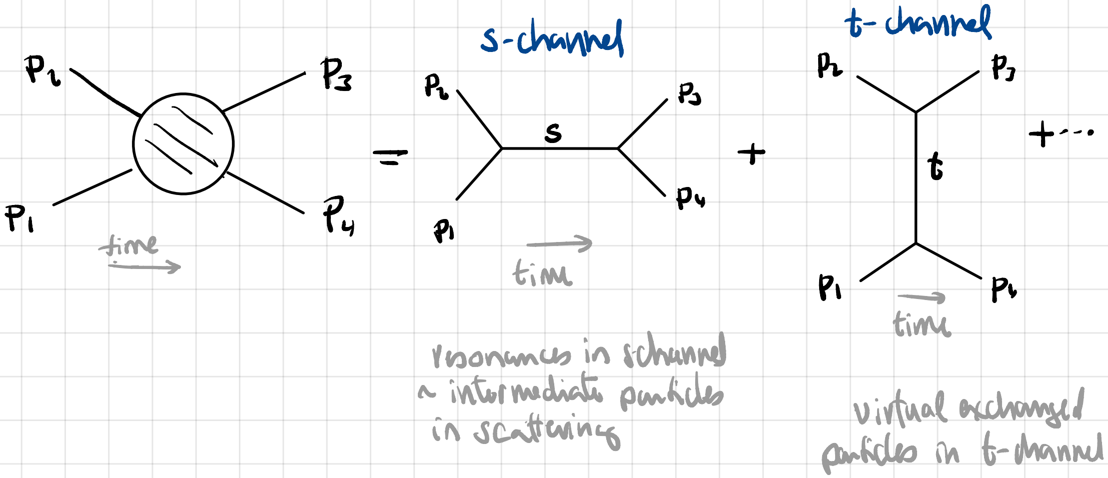
▶ Dolen-Horn-Schmid duality (see GSW chapter 1)

$$A(s,t) = A(t,s)$$

obvious from expression above in terms of B-function

Compare with QFT

leading (non trivial) contributions: come from the tree diagr



$A(s,t)$ invariant under $s \leftrightarrow t$?

$$A(s,t) = A(t,s) = -\sum_{n=0}^{\infty} \frac{1}{n!} (\alpha(s)+1)(\alpha(s)+2)\cdots(\alpha(s)+n) \frac{1}{\alpha(t)-n}$$

obvious

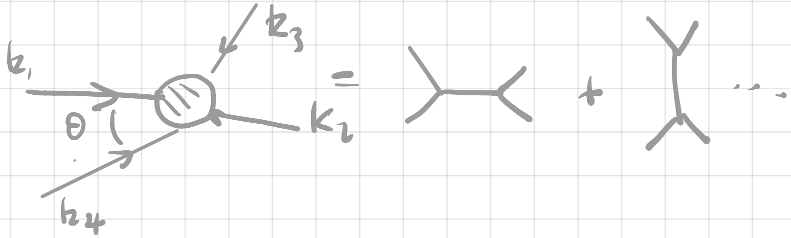
OTOH, can expand as a series to exhibit the poles in the t-channel
" " " " " s-channel !

UV behaviour (see GSW chapter 1)

$A(s,t) \sim F(\theta_s)^{-\alpha(s)}$ exponentially soft behaviour

↳ the function of an angle θ_s

$\frac{t}{s} \approx -\sin^2\frac{\theta}{2}$ $\quad s \to \infty$



Compare QFT: fixed angle scattering of point particles

$$A(s,t) \sim -\sum_{J=0}^{J_{max}} \frac{(-s)^J}{t-M_J^2} \underset{\substack{s\to\infty \\ t \text{ fixed}}}{\sim} s^{J_{max}}$$

t-channel

- exchange of spin J & M_J particles
- bad UV behaviour (also at higher loops)
- no s-channel poles!

string interpretation: infinite tower of states & UV divergences "cancelled"

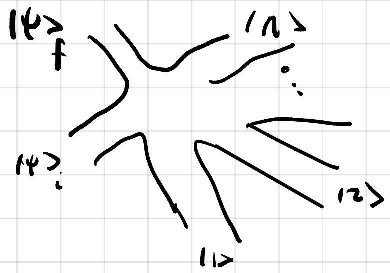
3.8 Comments on the general picture

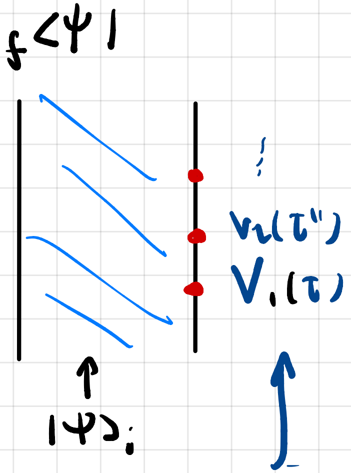
In string perturbation theory we are interested in the amplitude for the scattering of asymptotic in and out states (the S-matrix)

We have discussed a number of ideas and tools for computing amplitudes. (& so far tree level OS)

Wrap up this chapter on interactions with a number of comments on the lessons learned and on the general picture for scattering amplitudes

open strings (tree level)

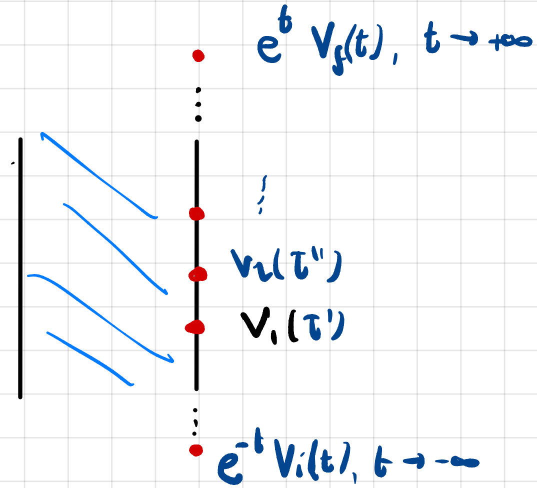




vertex operators inserted on the boundary of Σ

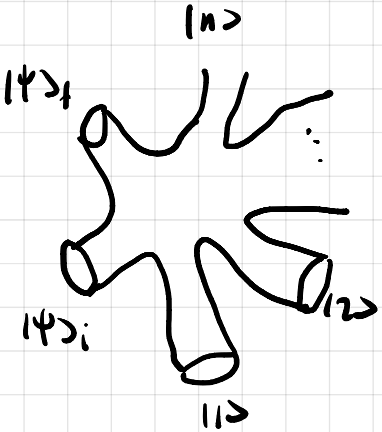
$t = i\tau$

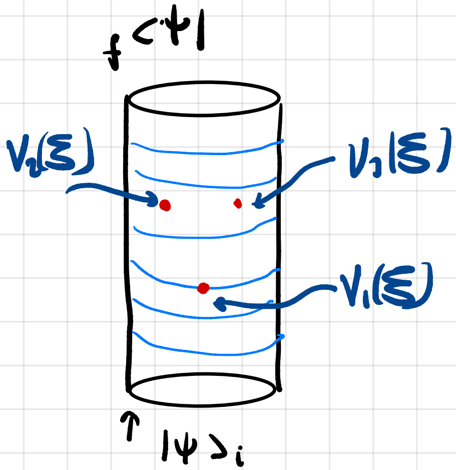
$=$



Euclidean World sheet

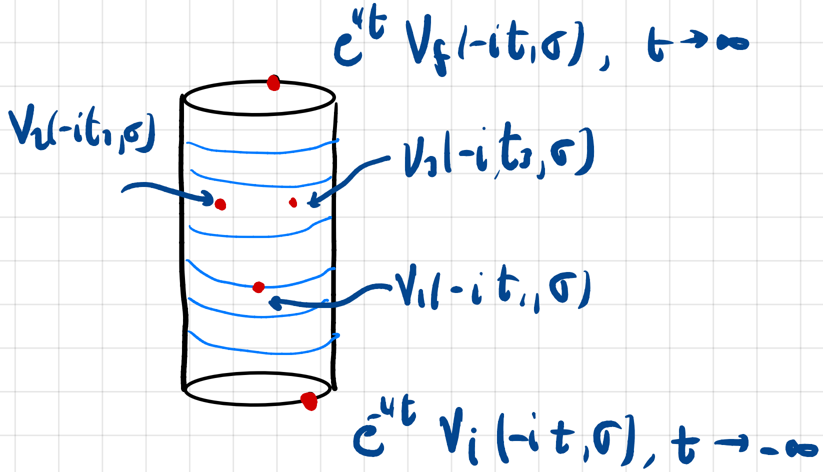
closed strings (tree level)



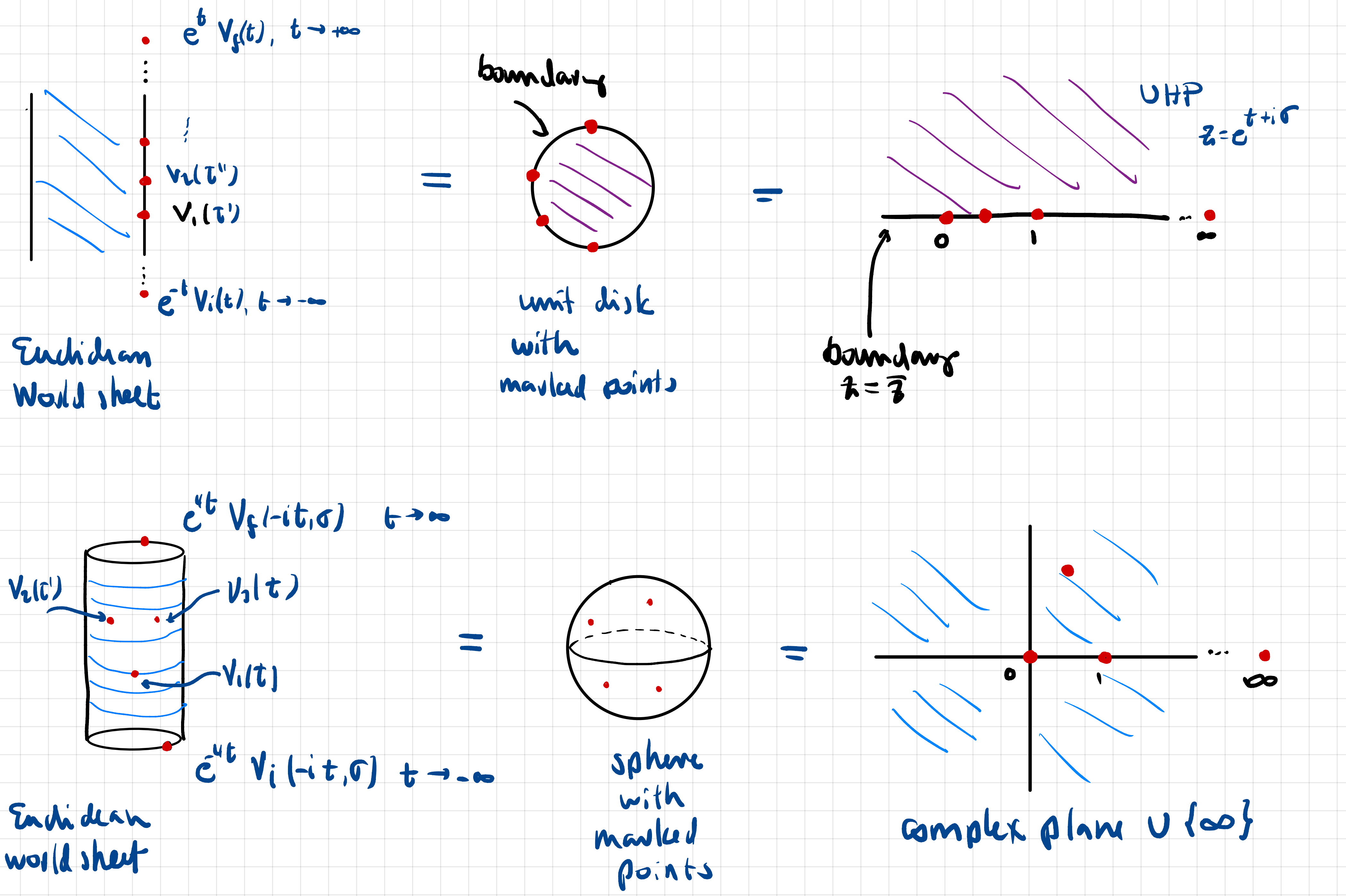


vertex operators inserted on the interior of Σ

$=$



Euclidean world sheet

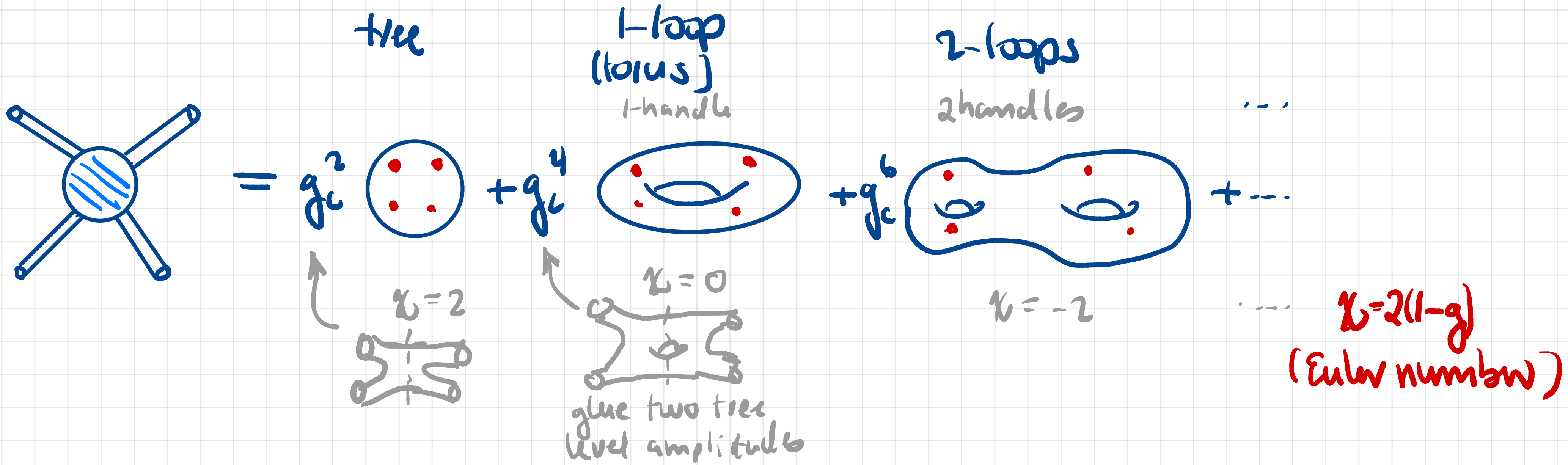


$e^{t} V_f(t)$, $t \to +\infty$
$V_2(\tau'')$
$V_1(\tau')$
$e^{-t} V_i(t)$, $t \to -\infty$
Euclidean World sheet
=
boundary
unit disk with marked points
=
UHP
$z = e^{t+i\sigma}$
0
1
∞
boundary $z = \bar{z}$
$e^{4t} V_f(-it, \sigma)$ $t \to \infty$
$V_2(t')$
$V_3(t)$
$V_1(t)$
$e^{4t} V_i(-it, \sigma)$ $t \to -\infty$
Euclidean world sheet
=
sphere with marked points
=
0
1
∞
complex plane $\cup \{\infty\}$

Beyond tree level $\rightsquigarrow$ string perturbation theory:

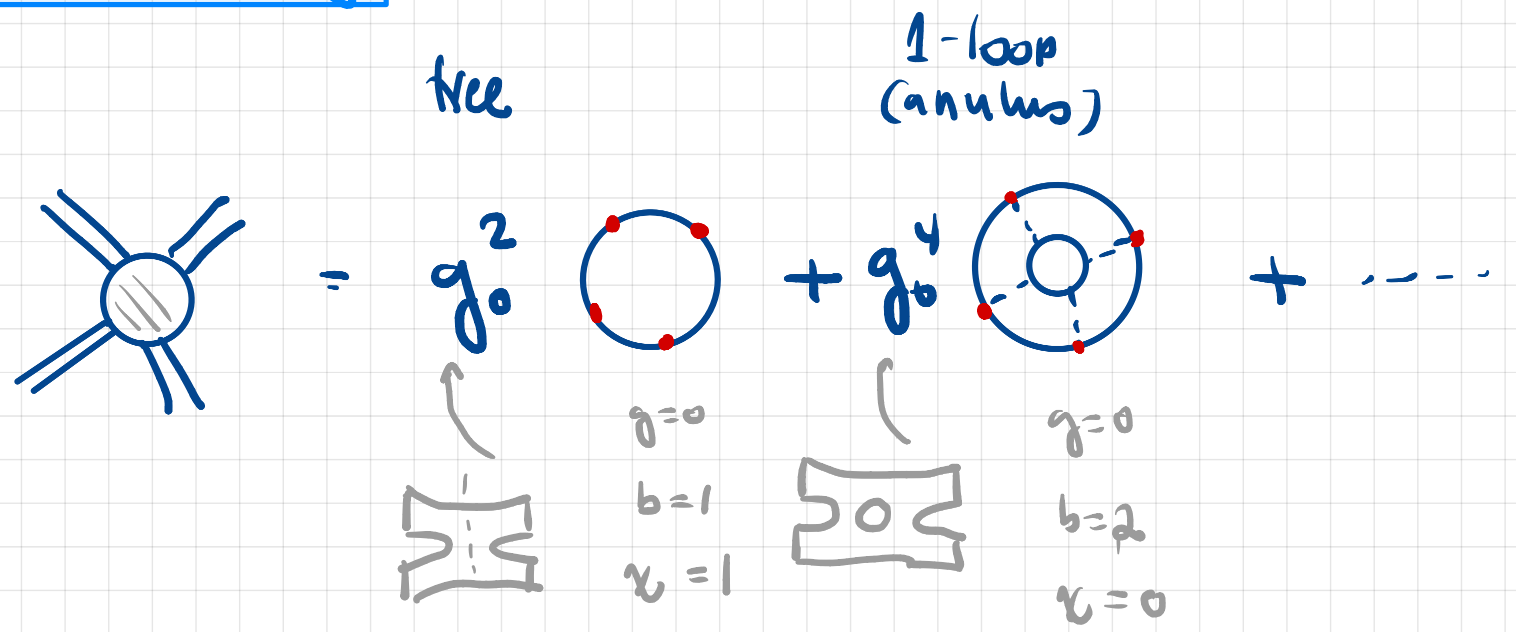
The string perturbation series is a genus expansion that is, a sum of Euclidean world sheets with different topology.

Closed string



- sum over all topologies (Riemann surfaces) $\boxed{\text{without}}$ boundaries
- these surfaces are classified by the number of handles g
- <u>one</u> diagram at each loop

Open string



$\chi = 2(1-g) - b$

Euler number

$b = \#$ of boundary components

- sum over all topologies (Riemann surfaces) $\boxed{\text{with}}$ boundaries
- these surfaces are classified by the number of handles g and the number of boundaries b
- one diagram at each order in perturbation theory

The relation between couplings

Recall: we cannot add any interaction terms to S_P without breaking conformal and Weyl invariance except for

$$\frac{1}{4\pi}\int_\Sigma d^2\xi \sqrt{-\det\gamma}\, R[\gamma] + \frac{1}{2\pi}\int_{\partial\Sigma} ds\, K[\gamma] = \chi = 2-2g-b$$

PS1

topological invariant

Consider then the action $S = S_P + \lambda\chi$, $\lambda \in \mathbb{R}$

S has the same dynamics as S_P

However, in the path integral formalism

integrated vertex insertion for $|i\rangle$

$$\mathcal{A}(|1\rangle, \dots, |n\rangle) = \sum_{\text{topologies}} \int \frac{\mathcal{D}[X,\gamma]}{\mathrm{Vol}(\mathrm{conf}_{\mathrm{lo}})} e^{-S[X,\gamma]} \prod_{i=1}^{n} V_{|i\rangle}$$

$$= \sum_{\text{topologies}} (e^{\lambda})^{-\chi} \int \frac{\mathcal{D}[X,\gamma]}{\mathrm{Vol}(\mathrm{conf}_{\mathrm{lo}})} e^{-S_P[X,\gamma]} \prod_{i=1}^{n} V_{|i\rangle}$$

Same series expansion as above with expansion parameter $\boxed{g_s = e^{\lambda}}$.

TBC ...

↳ next strings in background fields

strings propagating in non trivial backgrounds end of lecture 11

To study string amplitudes we use

physical state $\longleftrightarrow$ vertex correspondence

$|\psi\rangle \in \mathcal{H}_{phys}$ $\longleftrightarrow$ V_ψ operator of conformal

so that V_ψ maps $\mathcal{H}_{phys} \partial$ and $\mathcal{H}_{null} \partial$ $\Rightarrow$ weight $\begin{cases} h = 1 & \text{open strings} \\ h = \tilde{h} = 1 & \text{closed strings} \end{cases}$

V_ψ well def (eg normal ordering) so phys state cond.

V_ψ represents emission/absorption of a physical string state $|\psi\rangle$ from a point on the world sheet

and incoming/outgoing states are represented by

$$|\psi\rangle = \lim_{z \to 0} z^{-1} V_\psi(z) |0;0\rangle$$

action of V_ψ on zero momentum vacuum state in the infinite Euclidean past

Euclidean inf future

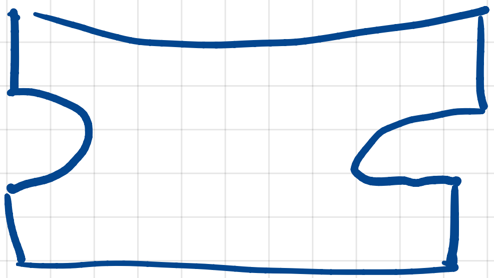
$$\langle\phi| = \lim_{z \to \infty} z \langle 0;0| V_\psi(z)$$

Remarks:

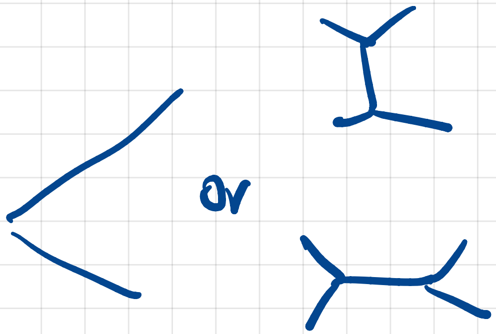
$\boxed{1}$ One diagram per order in perturbation theory

$\boxed{2}$ Degeneration limits look like many Feynman diagrams

eg



Veneziano



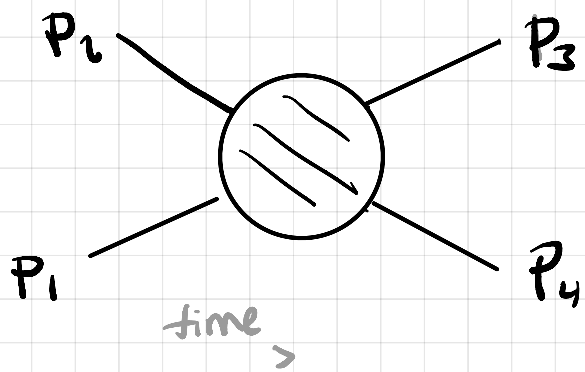
etc

s channel or t channel

•• Generalization of DHS duality

String theory appeared first in the 60s as a theory of strong interactions (the dual resonance models)

Consider elastic scattering 4 scalars



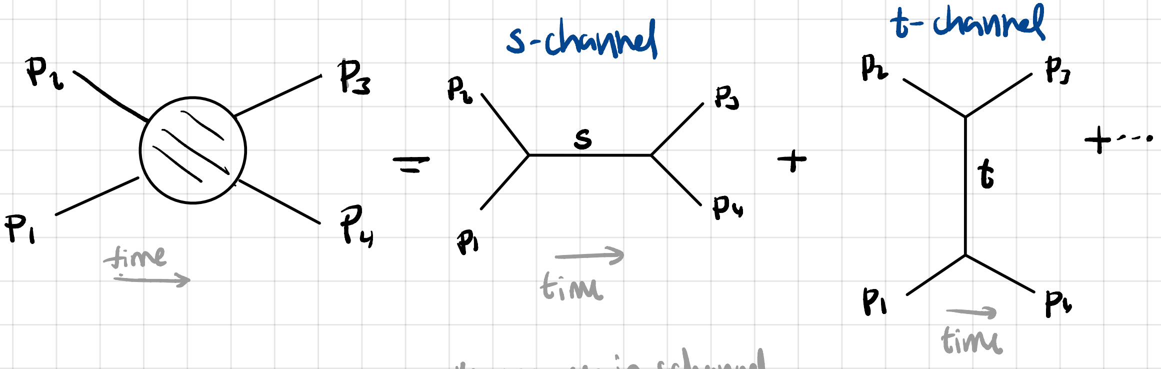
Mandelstam variables

$$s = -(p_1+p_2)^2 \quad (>0 \text{ for physical elastic scattering})$$
$$t = -(p_1+p_4)^2 \quad (<0 \text{ for physical elastic scattering})$$
$$u = -(p_1+p_3)^2 \quad (>0 \text{ for physical elastic scattering})$$

with $s+t+u = \sum m_i^2$

Amplitude $A(s,t)$ depends only on two of Mandelstam variables

leading (non trivial) contributions: come from the tree diagr



$A(s,t)$ invariant under $s \leftrightarrow t$

resonances in s-channel ~ intermediate particles in scattering

virtual exchanged particles in t-channel

$\boxed{\text{Dolen - Horn - Schmid duality}}$ (1968) proposed

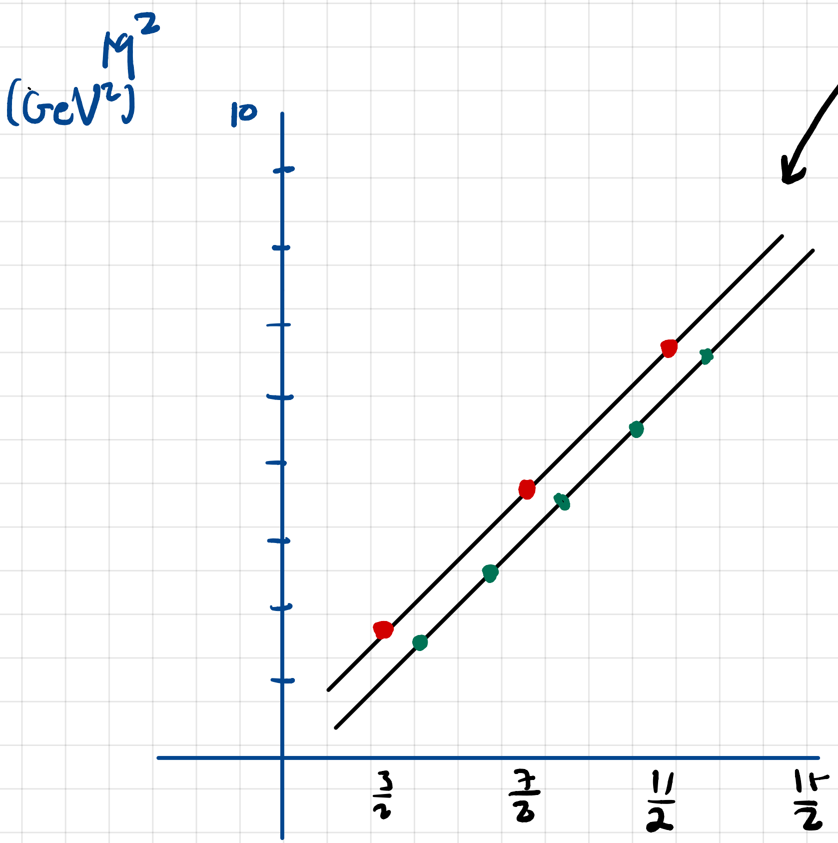
Dual (resonance) model (channel duality of S-matrix)

$$A(s,t) = A_t(s,t) = A_s(s,t)$$

and more over!

t & s channels give dual description of the same physics

One of the most important observations was that hadronic resonances appeared in families along curves in the (J, M^2) plane
(the Regge trajectories)



Chew-Frautschi plot

Regge trajectories

$$J = \alpha(0) + \alpha' M^2$$

$\alpha(0)$ Regge intercept

$\alpha' \simeq 1\ \mathrm{GeV}^{-2}$ slope (universal)

lightest scalar: $M^2 = -\frac{\alpha(0)}{\alpha'}$
$(J=0)$

$$e^{\lambda_1 L_1} : \quad z \longmapsto \frac{z}{1-\lambda_1 z} = z(1+\lambda_1 z)$$

$$e^{\lambda_0 L_0} : \quad z \longmapsto e^{\lambda_0} z = z + \lambda_0 z$$

$$e^{\lambda_{-1} L_{-1}} : \quad z \longmapsto z + \lambda_{-1}$$

5. Argue from the above result that the Einstein–Hilbert term on a closed string worldsheet would indeed be conformally invariant.

Note: By contrast, on an open string worldsheet Σ with boundary $\partial\Sigma$, only the combination

$$\chi := \frac{1}{4\pi}\int_{\Sigma} d^2\xi\sqrt{-\det h}\mathcal{R} + \frac{1}{2\pi}\int_{\partial\Sigma} ds\mathcal{K} \tag{5}$$

is conformally invariant. Here, the *extrinsic curvature* $\mathcal{K}$ is

$$\mathcal{K} = \pm t^a\, n_b\, \nabla_a t^b\,, \tag{6}$$

with t^a a unit vector tangent to the boundary, and n^a an outward pointing unit vector orthogonal to t^a. The sign choice corresponds to timelike/spacelike boundaries. This integral is the *Euler characteristic* for a 2d manifold Σ with boundary.

$$n\cdot\left(t^a \nabla_a t^b\right)$$