String Theory 1

Lecture # 12

3 Interactions

3.8 Comments on the general picture continued

... Wrapping up this chapter on interactions with a number of comments on the lessons learned and on the general picture for scattering amplitudes

A comment from last lecture

Faddeev-Popov trick : fix the gauge by setting

Then: $z_1 = z_1^0$ $\quad z_3 = z_3^0$ $\quad z_4 = z_4^0$

recall we only have the freedom to fix 3 points

$$g_0^2 \int dz_1\, dz_2\, dz_3\, dz_4\, \delta(z_1 - z_1^0)\, \delta(z_3 - z_3^0)\, \delta(z_4 - z_4^0) \times \left| \mathrm{Det}\, \frac{\partial(z_1, z_3, z_4)}{\partial(\lambda_{-1}, \lambda_0, \lambda_1)} \right|$$
$$\times \langle \tilde{V}_4(z_4)\, \tilde{V}_3(z_3)\, \tilde{V}_2(z_2)\, \tilde{V}_1(z_1) \rangle$$

to account for the gauge choice

$\{\lambda_{-1}, \lambda_0, \lambda_1\}$ parameters of $PSL(2,\mathbb{Z})$

Faddeev-Popov determinant

new measure for the amplitude

and we drop $\mathrm{Vol}(SL(2,\mathbb{R}))$

- FP det: When choosing to fix 3 point we can c transf $z \to \frac{az+b}{cz+d}$ which sends $z_1 \to z_1^0$, $z_3 \to z_3^0$, $z_4 \to z_4^0$

- see textbooks eg Polchinski Vol I p 86-88

$d\mu$ = measure on the UHP

$$g_o^2 \int \boxed{dz_1\, dz_2\, dz_3\, dz_4\, |(z_4-z_3)(z_3-z_1)(z_4-z_1)|}\, \delta(z_1 - z_1^0)\, \delta(z_3 - z_3^0)\, \delta(z_4 - z_4^0)$$

$$\times \langle \tilde{V}_4(z_4)\, \tilde{V}_3(z_3)\, \tilde{V}_2(z_2)\, \tilde{V}_1(z_1) \rangle_{UHP}$$

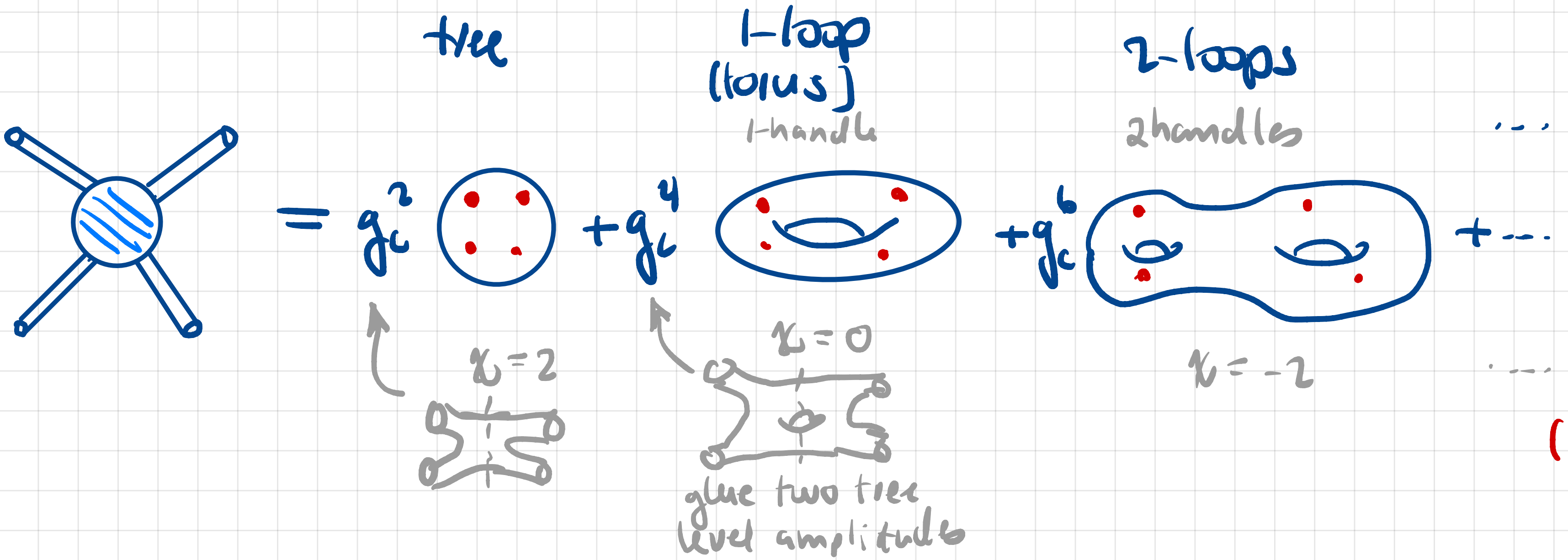
choose $z_1^0 = 0$, $z_3^0 = 1$, $z_4^0 = \Lambda \to \infty$

$$= \lim_{\Lambda \to \infty} g_o^2 \int dz_2\, \underbrace{\overset{z_{43}}{(\Lambda - 1)}\, \overset{z_{41}}{\Lambda}\, \langle \tilde{V}_4(\Lambda)}_{\substack{\Lambda^2 \langle \tilde{V}_4(\Lambda) \\ \downarrow \\ \langle 4 |}}\, \tilde{V}_3(1)\, \tilde{V}_2(z_2)\, \underbrace{\tilde{V}_1(0) \rangle_{UHP}}_{|1\rangle}$$

Last lecture ⇝ string perturbation theory:

The string perturbation series is a genus expansion that is, a sum of Euclidean world sheets with different topology.

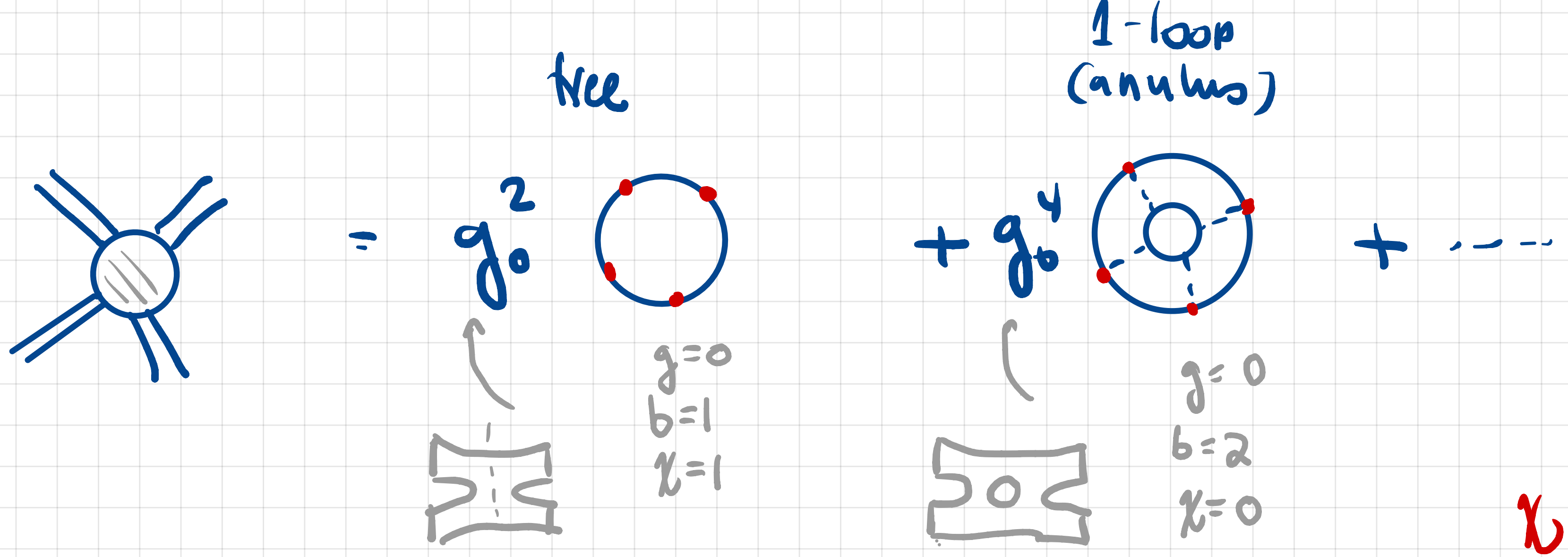
Closed string



$\chi = 2(1-g)$
(Euler number)

- sum over all topologies (Riemann surfaces) $\boxed{\text{without}}$ boundaries
- these surfaces are classified by the number of handles g
- one diagram at each loop

Open string



$\chi = 2(1-g) - b$

Euler number

$b = \#$ of boundary components

- sum over all topologies (Riemann surfaces) $\boxed{\text{with}}$ boundaries
- these surfaces are classified by the number of handles g and the number of boundaries b
- <u>one</u> diagram at each order in perturbation theory

The relation between couplings

Recall: we cannot add any interaction terms to S_p without breaking conformal and Weyl invariance except for

PSI

$$\frac{1}{4\pi}\int_\Sigma d^2\xi \sqrt{-\det\gamma}\, R(\gamma) + \frac{1}{2\pi}\int_{\partial\Sigma} ds\, K(\gamma) = \chi = 2-2g-b$$

topological invariant

Consider then the action $S = S_p + \lambda\chi$, $\lambda \in \mathbb{R}$

S has the same dynamics.

However, in the path integral formalism

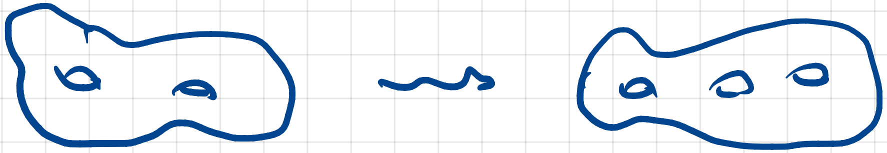
integrated vertex insertion for $|i\rangle$

$$\mathcal{A}(|1\rangle,\dots,|n\rangle) = \sum_{\text{topologies}} \int \frac{\mathcal{D}[X,\gamma]}{\mathrm{Vol}(\mathrm{conf}_{\text{re}})} e^{-S[X,\gamma]} \prod_{i=1}^n V_{|i\rangle}$$

$$= \sum_{\text{topologies}} (e^\lambda)^{-\chi} \int \frac{\mathcal{D}[X,\gamma]}{\mathrm{Vol}(\mathrm{conf}_{\text{re}})} e^{-S_p[X,\gamma]} \prod_{i=1}^n V_{|i\rangle}$$

Same series expansion as above with expansion parameter $\boxed{g_s = e^\lambda}$.

Add a handle so new diagram has an extra closed string loop



$$\chi \longrightarrow \chi - 2$$

$$(g_s)^{-\chi} \longrightarrow (g_s)^{-\chi} \cdot g_s^2$$

$\chi = 2(1-g)$

$\chi(g+1) - \chi(g)$

$= -2$

Identify $\quad g_s = g_c = e^{\lambda}$

Add an interior boundary so new diagram has an extra open string loop

$$\chi \longrightarrow \chi - 1$$

$$g_s^{-\chi} \longrightarrow (g_s)^{-\chi} g_s$$

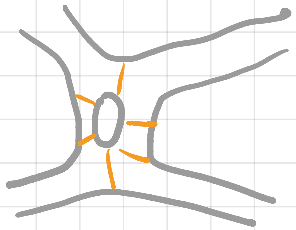
$\chi = 2(1-g) - b$

$\chi(b+1) - \chi(b) = -1$

Identify $\quad g_o^2 = g_s$

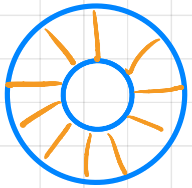
Then $\quad \boxed{g_s = g_o^2 = g_c}$

Open-closed duality: a single geometry can have two interpretations



open string

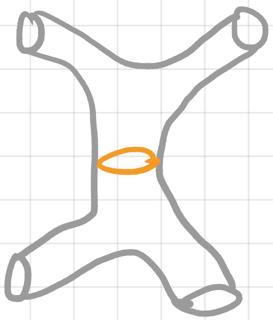
$\rightsquigarrow$



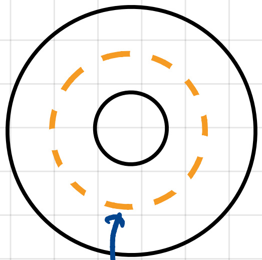
g_o^4 one loop open string amplitude

↳ topology of a cylinder!

Reinterpret: tree level amplitude of a closed string!



$\rightsquigarrow$



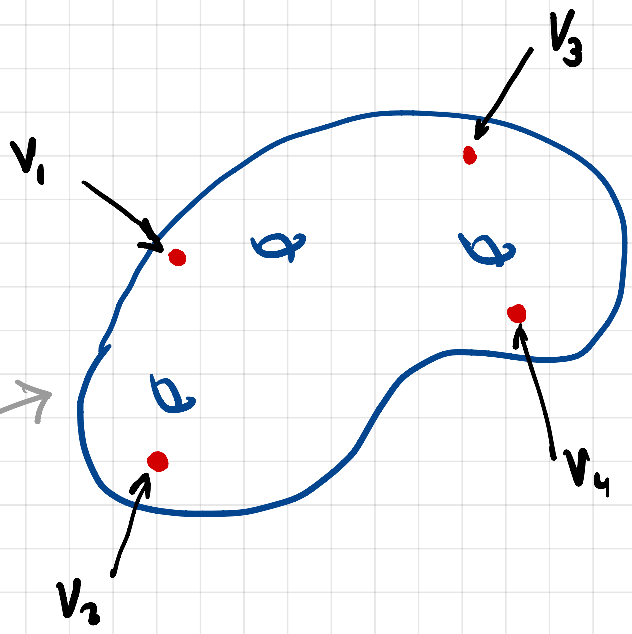
closed string

g_c^2 tree level closed string amplitude

Consistency: $g_c = g_o^2$ ✓

(See GSW for the computation)

General scattering process: say for the closed string



genus g Riemann surface (complex surfaces)

$$\rightsquigarrow \mathcal{A}_g(\psi_1, \dots, \psi_n)$$

$$= \int_{\mathcal{M}_{g,n}} \underbrace{[d\mu]}_{\text{measure on moduli space}} \langle V_1 V_2 V_3 \cdots V_n \rangle_{\Sigma_{g,n}(\mu)}$$

$\mathcal{M}_{g,n}$: Moduli space of Riemann surfaces with marked points, (parameters which characterise the world sheet as a complex 2d manifold without boundary)

$\Sigma_{g,n}(\mu)$: parametrizes Weyl equiv class of metrics $\cong$ $\mathbb{C}$-structures

parametrises Weyl equivalent class of metrics (complex structure)

- M_Σ & $[d\mu]$ $\to$ complicated

however low genus isn't so bad (we did tree level examples)
(1-loop amplitude calculations are rather interesting)

↳

Next: strings in background fields

strings propagating in non trivial backgrounds

4. Strings in background fields

4.1 Introduction

For strings propagating in $M^{1,25}$, we have identified various massless fields in the bosonic string spectrum, including a graviton.

We expect then that a theory of space-time gravity should emerge, so spacetime should be allowed a nontrivial metric (or indeed a a nontrivial topology)

We expect a <u>D=26 dim theory of gravity emerging with a Hilbert-Einstein action</u>.

Moreover, we should be able to describe the dynamics of string excitations propagating in non-trivial backgrounds.

The action for a string propagating in a spacetime with metric $G_{\mu\nu}(X)$ is

$$S_\sigma[\gamma, X] = -\frac{1}{4\pi\alpha'} \int d^2\xi \sqrt{\gamma}\, \gamma^{ab} \partial_a X^\mu \partial_b X^\nu \underbrace{G_{\mu\nu}(X)}_{\text{target space metric}}$$

So far we have only considered a flat target spacetime $G_{\mu\nu} = \eta_{\mu\nu}$

Classically this is Weyl invariant so taking $\gamma_{ab} = e^{2\phi(\sigma)} \eta_{ab}$

$\boxed{\text{NON-LINEAR } \sigma\text{-MODEL}}$

NLσM

$$S_\sigma[\gamma, X] = -\frac{1}{4\pi\alpha'} \int d^2\xi \; \partial_a X^\mu \partial^a X^\nu G_{\mu\nu}(X)$$

describes an <u>interacting</u> 2dim QFT with couplings encoded in the target space metric $G_{\mu\nu}(X)$

complicated ! compare $G_{\mu\nu} = \eta_{\mu\nu} \Rightarrow$ free field theory

In this chapter we discuss how a $D=26$ dimensional gravitational theory emerges: we will do this from the effective field theory point of view.

KEY: we require that the quantum theory is Weyl invariant.

First however, we use this action to try to make sense of the graviton states in the spectrum of the free string

(We will generalise S_p later to include the other massless states)

To get some intuition consider $G_{\mu\nu}(X) = \eta_{\mu\nu} + h_{\mu\nu}(X)$

small perturbation of flat space

$$\Rightarrow \quad e^{-S_\sigma} = e^{-S_0}\Big(1 + \frac{1}{4\pi\alpha'}\int_\Sigma d^2\xi \, h_{\mu\nu}(X)\, \partial_a X^\mu \partial^a X^\nu + \dots\Big)$$

$G=\eta$

In the path integral $\rightsquigarrow \int \mathcal{D}X \mathcal{D}\gamma \, e^{-S_0} \, \mathcal{V}$

ie insertion of an operator $\mathcal{V} \sim \int_\Sigma d^2\xi \, h_{\mu\nu}(X)\, \partial_a X^\mu \partial^a X^\nu$ in the path integral

This must be a vertex operator corresponding to a physical state, the graviton, if $h_{\mu\nu}$ satisfies the appropriate conditions, ie if

$$h_{\mu\nu} = \gamma_{\mu\nu} : e^{ik\cdot X} \dots, \qquad \gamma_{\mu\nu} \text{ traceless symmetric}$$

$\mathcal{V}$ generates a gravitational plane wave with polarization $\gamma_{\mu\nu}$

4.2 Background field expansion and the Weyl anomaly

To analyze the quantum NLσM we use the covariant background field expansion, which is a perturbation theory in which one separates the 2 dim fields as

$$X^\mu(\xi) = X_0^\mu(\xi) + \sqrt{\alpha'}\, Y^\mu(\xi) \qquad [\sqrt{\alpha'}] = L, \quad Y \text{ dimensionless}$$

X_0^μ: background part or "expectation value" satisfying EOM. For our purposes we take this to be a constant.

Y^μ: dynamical quantum fluctuation

One then expands the NLσM action around X_0^μ and get an expansion in powers of the quantum field Y about X_0

$$G_{\mu\nu}(X)\,\partial_a X^\nu \partial^a X^\mu = \alpha' \Big(G_{\mu\nu}(X_0^\mu) + \sqrt{\alpha'}\,\partial_\rho G(X_0) Y^\rho(\xi) + \frac{\alpha'}{2}\partial_\rho \partial_\sigma G_{\mu\nu}(X_0) Y^\rho(\xi) Y^\sigma(\xi) + \cdots \Big) \partial_a Y^\mu \partial^a Y^\nu$$

Each term represents an interaction for the fluctuations Y.

What is the expansion parameter?

The quantum perturbation theory is an expansion in in powers of $\sqrt{\alpha'}$ ($\sqrt{\alpha'}$ is an $\hbar$-like parameter)

We need to expand in terms of an effective dimensionless parameter: noting that $a_p G \sim 1/r_c$

r_c = characteristic radius of the curvature of target space, our effective dimensionless coupling constant is of order $\sqrt{\alpha'}/r_c$.

Then we obtain a perturbative expansion if

$$\sqrt{\alpha'} \sim l_s \ll r_c$$

string length << typical space-time length scales

Remark: this means that perturbative string theory has a double expansion in g_s & α'

For $l_s \ll r_c$ we then work with a weakly coupled σ-model perturbation theory (in the usual sense of a perturbative QFT framework; from this one can read-off Feynmann rules for diagrams ...).

In other words, we have a large radius expansion corresponding in spacetime to an EFT-like expansion with cutoff $M_s \sim (\alpha')^{-1/2}$.

(When $l_s \approx r_c$ this interpretation breaks down and instead we have a strongly coupled theory.)

Returning to $S_\sigma[G]$: this is classically conformally invariant however this is not necessarily true after quantisation because the NLσM is an interacting theory.

The interactions typically lead to (unphysical) divergences of the WS correlation functions. To deal with this we resort to the regularisation & renormalisation techniques. Fortunately the theory with action S_σ is renormalisable.

However these techniques inevitably introduce an explicit scale dependence of the correlation functions (see AQFT) hence the theory is no longer conformally invariant.

(YM theory is classically conformally invariant but on quantisation the theory develops a scale dependence)

The lack of scale invariance in a QFT is described in terms of the β-function (which arises when computing the UV divergences in Feynman diagrams)

Recall $\quad T_{ab} = -\frac{2}{T}\frac{1}{\sqrt{\gamma}}\frac{\delta S}{\delta \gamma^{ab}} = 0 \quad$ & in particular $T_{+-} = 0$

classically $\quad T_{+-} = 0 \longleftrightarrow$ Weyl invariance

At the quantum level however

$$T_{+-} = -\frac{1}{2\alpha'}\beta_{\mu\nu}\,\partial X^\mu \cdot \partial X^\nu$$

gets corrected at 1-loop

$\mu \frac{\partial G}{\partial \mu}$

Beta function

[In fact, even for $G_{\mu\nu} = \eta_{\mu\nu}$: $\quad T_{+-} = -\frac{1}{12}(D-26)\, R^{(2)}$]

The theory is conformal invariant if $\boxed{\beta = 0}$

We would like to insist that the 2 dim QFT on the world sheet (ie NLσM) to be Weyl invariant at the quantum level. This implies, in particular, that the theory is conformally invariant.

(Conformal symmetry is a gauge theory we want to preserve in the quantum theory; recall this was essential for the counting of the theory: construction of states, vertex operators and amplitudes were all based on having a CFT on the WS)

We need then to compute the β-function. The requirement $\beta = 0$ necessary to preserve Weyl invariance places restrictions on the target space fields.

However the NLσ-M is $\boxed{\text{not}}$ so easy to analyze.

↳ next

So how do we proceed? Return to the NLσM action & as discussed earlier, we analyze the quantum NLσM from the perturbation theory obtained by the <u>covariant background field expansion</u> with

$$X^\mu(\xi) = X_0^\mu + \sqrt{\alpha'}\, Y^\mu(\xi)$$

and expand around X_0^μ.