String Theory 1

Lecture # 13

4. Strings in background fields

4.1 Introduction

4.2 Background field expansion and the Weyl anomaly

4.3 Including other massless fields

4.4 Space-time effective action

4.5 The dilaton revisited

4.6 Energy scales

4.2 Background field expansion and the Weyl anomaly

... continued

Recall from last lecture

NON-LINEAR σ-MODEL

NLσM

$$S_\sigma[\gamma, X] = -\frac{1}{4\pi\alpha'}\int d^2\xi \; \partial_a X^\mu \partial^a X^\nu G_{\mu\nu}(X)$$

describes an interacting 2dim QFT with couplings encoded in the target space metric $G_{\mu\nu}(X)$

complicated ! compare $G_{\mu\nu} = \eta_{\mu\nu} \Rightarrow$ free field theory

classically conformally invariant but not necessarily quantum mechanically unless

$$T_{+-} \sim \beta_{\mu\nu}\, \partial X^\mu \cdot \partial X^\nu = 0 \qquad \text{ie} \qquad \beta = 0$$

We would like to insist that the 2 dim QFT on the world sheet (ie NLσM) to be Weyl invariant at the quantum level. This implies, in particular, that the theory is conformally invariant.

(Conformal symmetry is a gauge theory we want to preserve in the quantum theory; recall this was essential for the consistency of the theory: construction of states, vertex operators and amplitudes were all based on having a CFT on the WS)

We need then to compute the β-function. The requirement $\beta = 0$ necessary to preserve Weyl invariance places restrictions on the target space fields.

However the NLσ-M is not so easy to analyze.

So how do we proceed? Return to the NLσM action & as discussed earlier, we analyze the quantum NLσM from the perturbation theory obtained by the covariant background field expansion with

$$X^\mu(\xi) = X_0^\mu + \sqrt{\alpha'}\, Y^\mu(\xi)$$

and expand around X_0^μ.

As the NLσM action is invariant under field redefinitions

$$X^\mu \longrightarrow \tilde{X}^\mu(X)$$ (target space coordinate change)

together with the corresponding transformation of $G_{\mu\nu}$ we can choose Riemann normal coordinates

$\Gamma^\mu{}_{\nu\rho}\big|_p = 0$; $R^\mu{}_{\nu\rho\sigma}\big|_p = (\partial_\rho \Gamma^\mu_{\sigma\nu} - \partial_\sigma \Gamma^\mu_{\rho\nu})\big|_p$

simplifies computations!

Then

$$G_{\mu\nu}(X_0 + \sqrt{\alpha'}\,Y) = G_{\mu\nu}(X_0) - \frac{1}{3}\sqrt{\alpha'}\,\underbrace{R_{\mu\rho\nu\sigma}(X_0)}_{\text{Riemann Tensor of } \mathcal{M} \text{ at } X_0} Y^\rho Y^\sigma + \mathcal{O}(Y^3)$$

and

$$S_\sigma[X] = -\frac{1}{4\pi}\int d^2\xi \,\Big\{ \underbrace{G_{\mu\nu}(X_0)}_{\eta_{\mu\nu}} \partial Y^\mu \cdot \partial Y^\nu \qquad \text{kinetic term}$$

$$- \underbrace{\frac{1}{3}\sqrt{\alpha'}\,R_{\mu\rho\nu\sigma}(X_0)\,Y^\rho Y^\sigma (\partial Y^\mu)\cdot(\partial Y^\nu)}_{\text{leading quartic interaction term}} + \dots \Big\}$$

$\leftrightarrow$ interacting QFT with an infinite set of coupling constants

- We are ready to read off the Feynmann rules for diagrams for the two dimensional world sheet theory.
- Moreover we can compute the (one loop) divergences that contribute to the renormalisation of the couplings.

$\hookrightarrow$ introduces a scale μ breaking conformal invariance

The Weyl anomaly:

$S_\sigma[X,\gamma]$ is classically conformally invariant but not necessarily quantum mechanically

$$T_{+-} \sim \beta_{\mu\nu}\, \partial X^\mu \cdot \partial X^\nu = 0 \qquad \text{ie} \qquad \beta = 0$$

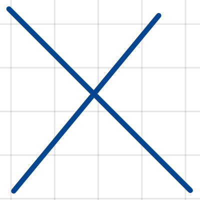
$$\beta_{\mu\nu}(G) \sim M \frac{\partial G_{\mu\nu}(X, M)}{\partial M}$$

describes how couplings (metric) depend on the energy scale M

Compute $\beta_{\mu\nu}$ to one loop

Consider 1-loop renormalisation:

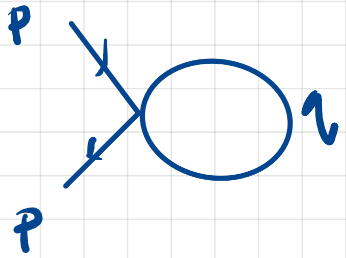
Quartic interaction (in momentum space)



$\sim R_{\mu\rho\nu\sigma}\, p^{\mu}_{a}\, p^{a\nu}$

$p^{\mu a}$ 2dim momentum of the scalar field Y^μ

One-loop divergence

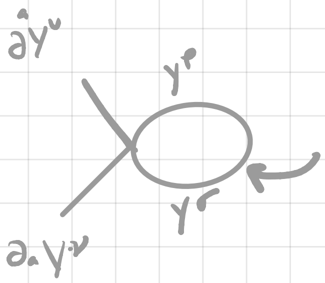


$$= \int \frac{d^2 q}{(2\pi)^2}\, R_{\mu\rho\nu\sigma}\, p^{\mu}\cdot p^{\nu}\, \eta^{\rho\sigma}\, \frac{1}{q^2}$$

$R_{\mu\nu}$ (from $R_{\mu\rho\nu\sigma}$ and $\eta^{\rho\sigma}$)

$\frac{1}{q^2}$: propagator in the loop

In position space



propagator diverges as $\xi \to \xi'$

$Y^\rho(\xi)\, Y^\sigma(\xi') \sim \frac{1}{2}\eta^{\rho\sigma} \log|\xi - \xi'|$

This divergence can be determined using "dimensional regularisation"

$$d = 2+\epsilon : \int \frac{d^{2+\epsilon} q}{(2\pi)^{2+\epsilon}} R_{\mu\rho\nu\sigma} \, p^\mu \cdot p^\nu \, \frac{\eta^{\rho\sigma}}{q^2} = \frac{1}{2\pi\epsilon} R_{\mu\nu} \, p^\mu \cdot p^\nu + \cdots$$

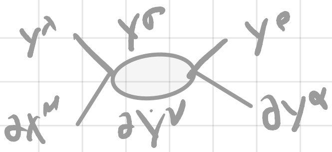
leading order divergence

$$R_{\mu\nu} = R_{\mu}{}^{\sigma}{}_{\nu\sigma} \qquad \text{target-space Ricci tensor}$$

We now cancel this divergence by adding a **counterterm** (terms that need substracted to S_σ to get a finite theory)

$$R_{\mu\rho\nu\sigma} Y^\rho Y^\sigma \partial Y^\mu \cdot \partial Y^\nu \longrightarrow R_{\mu\rho\nu\sigma} Y^\rho Y^\sigma \partial Y^\mu \cdot \partial Y^\nu - \frac{1}{\epsilon} R_{\mu\nu} \partial Y^\mu \cdot \partial Y^\nu$$

similarly for the interaction vertex



$$\rightarrow \quad CT \sim \frac{1}{\epsilon} (R^2)_{\mu\lambda\nu\kappa} \partial Y^\mu \partial Y^\nu Y^\lambda Y^\kappa$$

This divergences (and those from higher loops!) can be absorbed by a combination of

a wave function renormalisation of the fields Y^μ

$$Y^\mu \longrightarrow Y^\mu + \frac{\alpha'}{6\epsilon} R^\mu{}_\nu Y^\nu + \mathcal{O}(Y^2)$$

together with a functional renormalisation of $G_{\mu\nu}$

$$G_{\mu\nu} \longrightarrow G_{\mu\nu} + \frac{\alpha'}{2\epsilon} R_{\mu\nu}$$

not an entirely easy computation!

This gives the one-loop β-functional (obtained from the $\frac{1}{\epsilon}$ poles)

$$\beta_{\mu\nu} \propto \alpha' R_{\mu\nu}$$

The condition for conformal invariance (to leading order in α') is

$$\beta_{\mu\nu} = 0 \quad \Rightarrow \quad R_{\mu\nu} = 0 \quad !$$

tangent space must be Ricci-flat

that is, the string moves in a background spacetime which satisfies vacuum Einsteins eqs in 26 dim ($R_{\mu\nu} - \frac{1}{26} G_{\mu\nu} R = 0$).

consistency condition on the world sheet. $\longleftrightarrow$ spacetime equation of motion (spacetime dynamics !)

Higher orders in α': one gets stringy corrections to Einstein's eqs

$$\beta_{\mu\nu}(G_{\mu\nu}) = \alpha' R_{\mu\nu} + \underbrace{\frac{(\alpha')^2}{2} R_{\mu\kappa\rho\sigma} R_{\nu}{}^{\kappa\rho\sigma}}_{} = 0 \quad \text{to } \mathcal{O}(\alpha'^2)$$

string theory predicts specific small corrections to Einstein's in D=26 @ large radius.

↳ Next : including other massless modes

(eg $B_{\mu\nu}$, φ)

space time effective action

4.3 Including other massless fields

Apart from the graviton, we identified other massless fields in the closed string bosonic spectrum:

previously: identified the spacetime metric perturbations as insertions of the graviton vertex operator.

We can extend the Polyakov action further such that the effect in the path integral is to generate insertions of operators for the Ramond-Kalb $B_{\mu\nu}$ and the dilaton φ fields

- The Ramond-Kalb antisymmetric field: $B_{\mu\nu}\, dx^\mu \wedge dx^\nu$

One can add to the Polyakov action the term

$$S^{(B)}[X] = -\frac{1}{4\pi\alpha'} \int d^2\xi \; \epsilon^{\alpha\beta} B_{\mu\nu}(X)\, \partial_\alpha X^\mu \partial_\beta X^\nu$$

which is reparametrization and Weyl invariant (also power counting renormalizable).

Moreover under spacetime gauge transformations

$$B \longrightarrow B + d\Lambda \quad , \quad \Lambda \text{ a 1-form}$$

the action $S^{(B)}$ changes by a surface term (exercise).

$(d\Lambda)_{\mu\nu} \sim \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu$

• The dilaton Φ : We can add

renormalizable

$$S^{[\Phi]}[X;\gamma] = \frac{1}{4\pi\alpha'}\int d^2\xi \sqrt{\gamma}\, \Phi(X)\, R^{(2)}(\gamma)\, \alpha'$$

[For Φ = constant the integrand is a total derivative]

This term however is **not** Weyl invariant classically

$$\gamma \to e^{2\omega(\sigma)}\gamma \implies R^{(2)} \to e^{-2\omega}(R^{(2)} - 2\nabla^2\omega)$$

PS 1

$$S^{[\Phi]}[X,\gamma] \to S^{\Phi}[X,\gamma] + \frac{1}{4\pi\alpha'}\int d^2\sigma \sqrt{\gamma}\, \underbrace{\Phi(X)\,(-2\nabla^2\omega)}\alpha'$$

not a total derivative if $\Phi \neq$ const

One can show however that a classical Weyl variation of S^{Φ} can be cancelled by an $\mathcal{O}(\alpha')$ variation of $S^{(G)} + S^{(B)}$!

We have now a more general NLσM

$$S_\sigma = S^{(G)} + S^{(B)} + S^{(\Phi)}$$

$$S^{(G)}[\gamma, X] = -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\xi \; G_{\mu\nu}(X)\,\partial_a X^\mu \partial^a X^\nu$$

$$S^{(B)}[X] = -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\xi \; \epsilon^{\alpha\beta} B_{\mu\nu}(X)\,\partial_\alpha X^\mu \partial_\beta X^\nu$$

$$S^{[\Phi]}[X;\gamma] = \frac{1}{4\pi\alpha'}\int_\Sigma d^2\xi \sqrt{\gamma}\; \Phi(X)\, R^{(2)}(\gamma)\,\alpha'$$

S_σ ← field theory on Σ

with target space M which carries a geometrical structure $(G_{\mu\nu}, B_{\mu\nu}, \Phi)$.

Comment on the dilaton term:

Recall
$$S^{(\Phi)} = \frac{1}{4\pi}\int d^2\xi \sqrt{\gamma}\, R^{(2)}(\gamma)\, \Phi(X)$$

We recognize this term as a generalisation of the topological term $\lambda\chi \propto \int d^2\xi\, \lambda\, R^{(2)}$ which is related to the string coupling constant.

NLσM: string coupling not a constant (it is a field $\Phi(X)$)

$S^{(\Phi)}$ is a term in an interacting theory in a background with $\Phi = \Phi_0 =$ constant

$\Phi(X) = \Phi_0 + \sqrt{\alpha'}\, \partial_\mu \Phi\, Y^\mu + \cdots$

in the weak coupling limit where
$$\boxed{g_s = e^{\Phi_0}}$$

Generally: the string couplings are **not** parameters of the theory, they are dynamical.
(In the α'-expansion of the σ-model, we obtain an infinite set of couplings]

These considerations illustrate the fact that in string theory there are no continuous parameters.

Parameters are determined by eg expectation values of dynamical spacetime fields.

An involved computation of the β-functional extending the one-loop computation for $S^{[G]}$ gives for the full σ-model action $S^{[G]} + S^{[B]} + S^{[\Phi]}$:

$$\beta^{G}_{\mu\nu} = \alpha' \Big(\underbrace{R_{\mu\nu} - \tfrac{1}{4} H_{\mu\lambda\sigma} H_{\nu}{}^{\lambda\sigma}}_{\text{1-loop } G+B} + \underbrace{2 \nabla_\mu \nabla_\nu \Phi}_{\text{classical } \Phi} \Big)$$

$H = dB$

$H_{\mu\nu\rho} = 3\, \partial_{[\mu} B_{\nu\rho]}$

$$\beta^{B}_{\mu\nu} = \alpha' \Big(\underbrace{-\tfrac{1}{2} \nabla^\lambda H_{\lambda\mu\nu} + (\nabla^\lambda \Phi) H_{\lambda\mu\nu}}_{\text{1-loop } G+B} \Big)$$

$$\beta^{\Phi} = \underbrace{\tfrac{1}{6}(D-26)}_{\text{1-loop } G+B} + \alpha' \Big(\underbrace{(\nabla_\mu \Phi)(\nabla^\mu \Phi)}_{\text{1-loop } \Phi} \underbrace{- \tfrac{1}{2} \nabla^2 \Phi - \tfrac{1}{24} H_{\mu\nu\rho} H^{\mu\nu\rho}}_{\text{two loop } G+B} \Big)$$

references: Friedan's thesis; Callan & Thorlacius "Sigma models & string theory"; Tseytlin "Conformal anomaly in a 2dim σ-model"

4.4 Space-time effective action

We want to interpret the vanishing of the β-function as spacetime equations of motion.

Indeed, one can show that they arise as the Euler-Lagrange equations for the effective action

$$S_D^S = \frac{1}{2\kappa_0^2} \int d^D x \sqrt{-G}\, e^{-2\Phi} \left(\frac{2(26-D)}{3\alpha'} + R(G) - \frac{1}{12}|H|^2 + 4|\nabla\Phi|^2 \right)$$

↳ "string frame" action (G, B, Φ in S^S_{ec} are the fields that appear in the σ-model action)

κ_0 related to Newton's constant: see next

For space-time computations one often uses the "Einstein frame":

let $\tilde{\Phi} = \Phi - \Phi_0$, $\tilde{G} = e^{\frac{1}{6}\tilde{\Phi}} G$

$$S_D^{(E)} = \frac{1}{2\kappa^2} \int d^D x \sqrt{-\tilde{G}} \left(\frac{2(26-D)}{3\alpha'} + \tilde{R}(G) - \frac{1}{12} e^{-\frac{1}{3}\tilde{\Phi}} |H|^2 - \frac{1}{6} |\nabla \tilde{\Phi}|^2 \right)$$

$\kappa = \kappa_0 e^{\Phi_0}$

space time theory is GR coupled to additional fields

indices raised and lowered with $\tilde{G}$

Einstein-Hilbert term takes the canonical form with gravitational coupling

$\kappa = (8\pi G_N)^{1/2}$

$M_s = 1/\sqrt{\alpha'}$ (EFT with cutoff M_s), $\sqrt{\alpha'} \sim l_s \ll r_c$

The spacetime action should capture the classical limit when $E \ll M_s$.
The stringy corrections to this can be seen from the corrected β function

$$\beta = \beta^{(0)} + \alpha' \beta^{(1)} + (\alpha')^2 \beta^{(2)} + \cdots$$

↖ harder ...

(For example $\beta^G_{\mu\nu} \sim \alpha' R_{\mu\nu} + \frac{(\alpha')^2}{2} R_{\mu\nu\lambda\sigma} R_\nu{}^{\lambda\rho\sigma} + \cdots$)

The corrected β-functional is interpreted as Euler-Lagrange equation for an α' corrected action:

$$S_{26} = S_{26}^{(0)} + \alpha' S_{26}^{(1)} + (\alpha')^2 S_{26}^{(2)} + \cdots$$

- S_{26} ↑ EFT (expansion with cutoff scale M_s)
- α' ↑ $\frac{1}{M_s^2}$
- $S_{26}^{(1)}$: 4-derivative terms
- $(\alpha')^2$: $\frac{1}{M_s^4}$
- $S_{26}^{(2)}$: 6-derivative terms

↳ effective action obtained after integrating out massive modes

Remarks on the energy scales

↳ Observations about the energy scales involved in the space-time effective action obtained by requiring that its EOM are the same as the vanishing of the Beta functions.

▷ **The gravitational coupling**

The Einstein frame is constructed such that the Einstein-Hilbert term takes the canonical form with gravitational coupling

$$\kappa = \kappa_0 e^{\Phi_0} = (8\pi G_N)^{1/2} \sim (M_{\text{Planck}})^{-\frac{1}{2}(D-2)}$$

↳ related to the Planck mass

⇝ scale above which quantum gravitational effects become important

▸ We also have the string scale

$$\alpha' \sim M_s^{-2} \sim \ell_s^{+2}$$

scale at which size of the string becomes important

(Recall that we obtained an effective theory from the NLσM large radius expansion with cutoff M_s)

⇝ The string scale: controls stringy corrections (world sheet quantum corrections).

(deviations from GR in terms of higher derivative terms)

- The gravitational coupling κ and the string scale α' are related by the string coupling e^{Φ_0}

$$\kappa = \kappa_0 e^{\Phi_0} = (4\pi G_N)^{1/2}$$

We have a dimensionless ratio

$$\frac{M_s}{M_{pl}} \sim e^{+\frac{2}{D-2}\Phi_0}$$

This controls higher contributions in the genus expansion (higher loop orders)

Effective action is an action for the dynamics at energy scales $E \ll M_s$ in the limit $e^{\Phi_0} \to 0$

[suppress spacetime quantum effects]

Final remark

Thus far we have been discussing a perturbative two dimensional QFT on the worldsheet.
There is however an exact version.

$$\mathbb{R}^{1,25}_{\eta} \xrightarrow{\text{special case of}} \mathbb{R}^{1,25-d} \times M^{d}_{G} \xrightarrow{\text{special case of}} \mathbb{R}^{1,25-d}_{\eta} \times \left(\mathrm{CFT}^{2\dim}_{c}\right)$$

abstract

$\mathbb{R}^{1,25}_{\eta}$ ← free bosonic string

M^{d}_{G} ← Ricci flat to leading order

$\mathrm{CFT}^{2\dim}_{c}$ ← non-perturbative in α'

string theory goes beyond target space (semiclassical geom) → requires more formal development of CFT

free bosonic string ⟷ machinery carries over to abstract CFT (no ghost theorem, vertex operator formalism)

Next: compactifications

↳ illustrate · $\mathbb{R}^{1,24} \times S^1_R$

· T-duality