String Theory 1

Lecture # 14

5 Compactifications

Target space: $\mathcal{M}_{D=26} = \mathbb{R}^{1,25-d} \times X_d$

Idea Kaluza-Klein: 1920

5 dim GR with $\mathcal{M}_5 = \mathbb{R}^{1,3} \times S^1_R$

EFT: $\hookrightarrow$ 4 dim GR + field eqs of EM

Consider S^1-compactifications of the bosonic string ie

$$\mathcal{M}_{26} = \mathbb{R}^{1,24} \times S^1_R$$

circle of radius R

fields X^μ — X^i $i = 0, \dots, 24$
— $X^{25} \sim X^{25} + 2\pi R$

parametrizes circle of radius R

We will discuss this from our two perspectives

(1) From the spacetime EFT

$\longrightarrow$ Kaluza-Klein mechanism to obtain an effective theory on $\mathbb{R}^{1,24}$

(more generally $\mathcal{M} = \mathbb{R}^{1,25-d} \times X_d$ where the geometry & topology of X_d determine the parameters of the LEET on $\mathbb{R}^{1,25-d}$)

(2) From the world sheet CFT perspective with target space $\mathbb{R}^{1,24} \times S^1_R$

[see BLT, DTong lectures]

5.1 Spacetime EFT approach

for the closed bosonic string theory

Kaluza-Klein ansatz for the fields to obtain an effective action in (1,24)-dimensions.

Fields $X^\mu \begin{cases} X^i & i=0,\dots,24 \\ X^{25} & \text{coordinate on } S^1_R \end{cases}$

metric $\quad G_{\mu\nu}dX^\mu dX^\nu = G_{ij}dX^i dX^j + e^{2\sigma}(dX^{25}+A_i dX^i)^2$

$G_{ij}(X^i, X^{25})$

$G_{25,25} = e^{2\sigma}, \quad G_{25,i} = e^{2\sigma}A_i$

KR field $\quad B_{\mu\nu}dX^\mu \wedge dX^\nu = B_{ij}\,dX^i \wedge dX^j + \tilde{A}_i\, dX^i \wedge dX^{25}$

$B_{i,25} = \tilde{A}_i$

dilaton $\quad \Phi = \Phi_{(25)} + \frac{1}{2}\sigma$

see Polchinski 8.1

▸ One then rewrites the effective action $S_{(26)}$ in terms of

$$G_{ij},\ A_i,\ e^{2\sigma}$$
$$B_{ij},\ \tilde{A}_i \qquad i,j = 0,1,\dots,24$$
$$\Phi_{(25)}$$

This is a long computation, but that is ok.

Recall
$$S_{26} = \frac{1}{2K_0^2}\int d^{26}x\,\sqrt{-G}\,e^{-2\Phi}\left(R - \frac{1}{12}|H|^2 + 4|\nabla\Phi|^2\right)$$

For example:

$$R(G_{26}) = R(G_{25}) - \frac{1}{2}e^{2\sigma}\,F(A)^{ij}F(A)_{ij} - 2e^{-\sigma}\nabla^i\nabla_i e^{\sigma}$$

etc

- All these fields depend on X^i but also on X^{25}.

Due to the identification $X^{25}(\tau,\sigma) \sim X^{25}(\tau,\sigma) + 2\pi R$

we can expand these fields in Fourier modes with respect to X^{25}:

$$F(X^i, X^{25}) = \sum_{n\in\mathbb{Z}} F_n(X^i)\, e^{in\frac{1}{R}X^{25}}$$

independent of X^{25}

- Finally we integrate $S_{(26)}$ over X^{25} to obtain a theory in 25-dimensions

We will not be able to do all this explicitly (but see below for the dilaton)

↳ long computation indeed!

$n=0$ in Fourier series for the fields

Note however that, as we will see, the zero modes (typically) give the massless sector of the theory.

The zero modes $(n=0)$ are

25 dim
- metric $G_{ij}(X^i)$
- KR field $B_{ij}(X^i)$
- 2x 1-form gauge fields

what is the gauge symmetry?

A (graviphoton) & $\hat{A}$ (KR-photon) correspond to $U(1)\times U(1)$ gauge fields

- 2 scalars: σ, $\Phi_{(25)}$

The gauge field A : under a spacetime diffeomorphism $\delta X^\mu = \epsilon^\mu(X)$

the metric changes as $\delta G_{\mu\nu} = \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu$.

Thus under $\delta X^{25} = \epsilon(X^i)$ reparametrisation of X^{25} direction

we find $\delta A_i = \partial_i \epsilon$ $(A_i = G_{25,i} \Rightarrow \delta A_i = \delta G_{25,i} = \partial_i \epsilon)$

So indeed we interpret A_i as a $U(1)$ gauge field and the gauge symmetry $\boxed{\text{descends}}$ from the 26-dimensional diffeomorphism invariance.

A_i is called the graviphoton.

The gauge field $\tilde{A}$: the KR field changes as $\delta B_{\mu\nu} = \partial_\mu \lambda_\nu - \partial_\nu \lambda_\mu$

Then under $\delta X^{25} = \lambda(X^i)$

we find $\delta \tilde{A}_i = \partial_i \lambda$ $(\tilde{A}_i = B_{i,25} \Rightarrow \delta \tilde{A}_i = \delta B_{i,25} = +\partial_i \lambda)$

$\tilde{A}_i$ is called the KR-photon

Let's look at the dilaton more carefully.

$$\Phi(X^\mu) = \Phi(X^i, X^{25})$$ recall $X^{25} \sim X^{25} + 2\pi R$

We expand this field (and any other fields) in Fourier modes with respect to X^{25}:

$$\Phi(X^\mu) = \sum_{n\in\mathbb{Z}} e^{in\frac{1}{R}X^{25}} \underbrace{\phi_n(X^i)}_{\text{independent of } X^{25}}$$

$$\underbrace{\phi_n = \phi^*_{-n}}_{\text{because } \Phi \text{ is real-valued}}$$

Dilaton terms in the action $S_{(26)}$:

$$|\nabla_{26}\Phi|^2 = \partial_i\Phi\partial^i\Phi + (\partial_{25}\Phi)^2 = \sum_{n,m} e^{i(n+m)\frac{1}{R}X^{25}} \underbrace{\left\{\partial_i\phi_n\partial^i\phi_m - \frac{nm}{R^2}\phi_n\phi_m\right\}}_{\text{Independent of } X^{25}}$$

Then

in principle there is a factor $\sqrt{G}\,e^{2\phi}$

$$\int d^{26}x\, |\nabla_{26}\Phi|^2 = \int d^{25}x\, 2\pi R \sum_{n=-\infty}^{\infty}\left\{\partial_i\phi_n\partial^i\phi_{-n} + \frac{n^2}{R^2}\underbrace{\phi_n\phi_{-n}}_{|\phi_n|^2}\right\}$$

$\to \mathbb{R}^{1,24} \times S^1_R$

$\Rightarrow$ the massless dilaton $\Phi(X^\mu)$ of the 26-dimensional EFT gives rise to a discrete infinite tower of scalar fields ϕ_n, the Kaluza-Klein modes, with mass $M_n^2 = \frac{1}{R^2} n^2$

For small R all are heavy modes except the massless mode $(n=0)$

can ignore for distance scale R with $E \ll \frac{1}{R} \sim M_{KK}$

$\rightsquigarrow$ the effective theory in $\mathbb{R}^{1,24}$ involving only the massless KK-modes.

The massive KK modes Φ_n ($n \neq 0$) are charged under the graviphoton A_i

Recall that a diffeomorphism $\delta X^{25} = \epsilon(X^i)$ results in a gauge transformation $A_i \longrightarrow A_i + \partial_i \epsilon$. This has an effect on the KK-modes:

Consider:
$$\Phi(X^\mu) \xrightarrow[X^{25} \to X^{25} + \epsilon(X^i)]{} \sum_{n \in \mathbb{Z}} e^{in\frac{1}{R}(X^{25}+\epsilon)} \phi_n(X^i)$$

reparametrisation of X^{25} direction
(rotation on the S^1_R)

$\frac{n}{R} = p^{25}$

hence $\phi_n \longrightarrow e^{in\epsilon/R} \phi_n$

electric charge under A_i

That is the graviphoton charge is the KK-momentum

One can show that there are $\boxed{\text{no}}$ excitations charged under the $U(1)$ symmetry associated to the Ramond-Kalb-photon

Massless sector of the effective 25-dimensional theory:

$$G_{\mu\nu}(x) \longrightarrow G_{\mu\nu}(x^i) : \{ \underbrace{G_{ij}(x^i)}_{\substack{\text{25 dim} \\ \text{Graviton}}}, \underbrace{G_{i,25}(x^i)}_{\substack{\text{graviphoton} \\ e^{2\sigma}A}}, \underbrace{G_{25,25}(x^i)}_{\substack{\text{radion} \\ e^{2\sigma}}} \}$$

$$B_{\mu\nu}(x) \longrightarrow B_{\mu\nu}(x^i) : \{ \underbrace{B_{ij}(x^i)}_{\substack{\text{25 dim} \\ \text{KR field}}}, \underbrace{B_{i,25}(x^i)}_{\substack{\text{KR-photon} \\ \tilde{A}}} \}$$

$$\Phi(x) \longrightarrow \Phi(x^i) \quad \text{25 dim dilaton}$$

Remark: we have introduced a new scale $M_{KK} \sim \frac{1}{R}$

At energy scales $E \ll \frac{1}{R} \sim M_{KK}$ modes with $n \neq 0$ "decouple"

We should not trust the EFT analysis for $M_{KK} \sim M_s$

However, one can perform an exact analysis of the world sheet CFT!

5.2 World-sheet perspective (closed string)

The target space for the two dimensional NLσM is $\mathbb{R}^{1,25} \times S^1_R$.

X^{25} is a field on S^1_R which is periodic ie $X^{25} \sim X^{25} + 2\pi R$

This non trivial topology has very interesting consequences.

- space-time translation by $2\pi R$: $e^{2\pi i R \hat{P}_{25}}$ should act as identity

generates translations along X^{25}

$$e^{2\pi i R \hat{P}_{25}} |\ldots, k_{25}\rangle = e^{2\pi i R k_{25}} |\ldots, k_{25}\rangle = |\ldots, k_{25}\rangle$$

iff $\boxed{k_{25} = \frac{m}{R}}$ $m \in \mathbb{Z}$

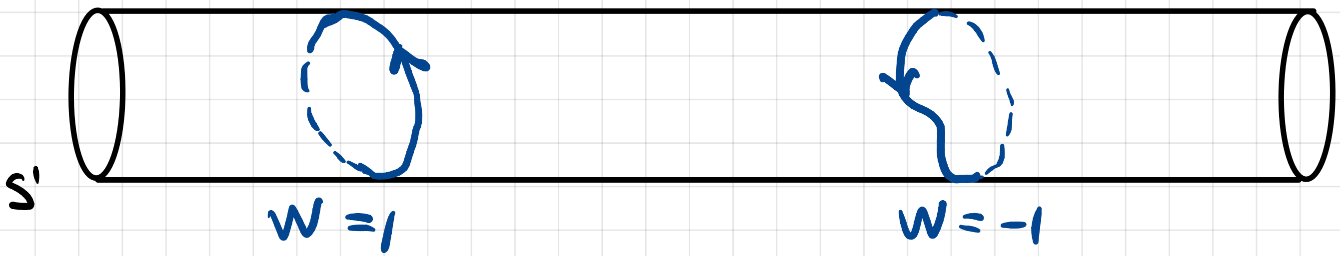
just as in the EFT analysis!

$$X^{25}(\tau, \sigma+2\pi) = X^{25}(\tau,\sigma) + 2\pi R W \qquad W \in \mathbb{Z}$$

(that is, X^{25} only needs to be periodic $\sigma \to \sigma + 2\pi$ up to $2\pi R$ shifts)

W is called the winding number

Term $2\pi R w$ ⇒ closed strings wrapped on S^1_R and counts how many times the string wraps around S^1_R



winding is a stringy effect: there is nothing like this in the EFT we discussed

In the 26 dim EFT: these are solitons!

Spectrum of the string with target space $\mathbb{R}^{1,24} \times S^1_R$

Mode expansion of X^{25} (which respects $X^{25}(\tau, \sigma+\pi) = X^{25}(\tau,\sigma) + 2\pi R w$)

$$X^{25}(\tau,\sigma) = x^{25} + \alpha' \tau p^{25} + w R \sigma + \frac{i}{2} \sum_{n\neq 0} \frac{1}{n}\left(\alpha_n^{25} e^{-in\xi^-} + \tilde{\alpha}_n^{25} e^{-in\xi^+}\right) \qquad p^{25} = \frac{n}{R}$$

$$= X_R^{25}(\xi^-) + X_L^{25}(\xi^+)$$

where

$$X_R^{25}(\xi^-) = \frac{1}{2} x_R^{25} + \frac{1}{2}\alpha' p_R \xi^- + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n} \alpha_n^{25} e^{-in\xi^-}, \qquad p_R \equiv \frac{n}{R} - \frac{Rw}{\alpha'}$$

$$X_L^{25}(\xi^+) = \frac{1}{2} x_L^{25} + \frac{1}{2}\alpha' p_L \xi^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n} \tilde{\alpha}_n^{25} e^{-in\xi^+}, \qquad p_L \equiv \frac{n}{R} + \frac{Rw}{\alpha'}$$

This is just as in $\mathbb{R}^{1,25}$ except that $\alpha_0^{25} = \sqrt{\frac{\alpha'}{2}} p_R \neq \tilde{\alpha}_0^{25} = \sqrt{\frac{\alpha'}{2}} p_L$

$$\alpha_0^{25} + \tilde{\alpha}_0^{25} = \sqrt{2\alpha'}\, p^{25} = \sqrt{2\alpha'}\, \frac{n}{r}; \qquad \alpha_0^{25} - \tilde{\alpha}_0^{25} = -\sqrt{\frac{2}{\alpha'}} R w$$

The mode expansion of X^i $i = 0, \ldots, 24$ remains unchanged.

From lecture 4

separately periodic up to a zero mode

$$X_L^\mu(\xi^+) = \frac{1}{2}x^\mu + \frac{\pi}{\ell}\alpha' p^\mu \xi^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{1}{n}\tilde{\alpha}_n^\mu \, e^{-\frac{2\pi i}{\ell} n \xi_+}$$

$$X_R^\mu(\xi^-) = \frac{1}{2}x^\mu + \frac{\pi}{\ell}\alpha' p^\mu \xi^- + i\sqrt{\frac{\alpha'}{2}} \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{1}{n}\alpha_n^\mu \, e^{-\frac{2\pi i}{\ell} n \xi^-}$$

where x^μ, p^μ, $\tilde{\alpha}_n^\mu$ and α_n^μ are the Fourier coeffts.

X^μ is real-valued : $x^\mu \in \mathbb{R}$, $p^\mu \in \mathbb{R}$, $\tilde{\alpha}_{-n}^\mu = (\tilde{\alpha}_n^\mu)^*$, $\alpha_{-n}^\mu = (\alpha_n^\mu)^*$

$\alpha_0^\mu = \tilde{\alpha}_0^\mu = \sqrt{\frac{\alpha'}{2}}\, p^\mu$ from periodicity $\sigma \to \sigma + \ell$

The string states

Because $\tilde{\alpha}_0^{25}$ & α_0^{25} only differ by a c-number, the commutators of oscillators remain unchanged.

Then the Fock space is identical to that derived for $\mathbb{R}^{1,25}$ and states are organised into levels labelled by the e-values of N & $\tilde{N}$.

The oscillator vacuum state however is now labelled by

$$|0,\tilde{0};k,\underbrace{n,w}\rangle \qquad n,w\in\mathbb{Z}$$

25 dim momentum $k\in\mathbb{R}^{1,24}$

$p^{25}=\frac{n}{R}$

from $X^{25}(\tau,\sigma+2\pi)=X^{25}(\tau,\sigma)+2\pi R w$

with
$$p^i|0,\tilde{0};k,n,w\rangle = k^i\,|0,\tilde{0};k,n,w\rangle$$
$$p^{25}_{L,R}|0,\tilde{0};k,n,w\rangle = \left(\frac{n}{R}\pm\frac{wR}{\alpha'}\right)|0,\tilde{0};k,n,w\rangle$$

General state in Fock space:

$$\prod \alpha^i_{-m}\prod\tilde{\alpha}^j_{-k}\,|0;\tilde{0};k;n;w\rangle$$

Physical spectrum: Virasoro operators & constraints

as before with some reorganisation

$$L_0 = \frac{1}{2}\left[\alpha_0 \cdot \alpha_0 + (\alpha_0^{25})^2\right] + \underbrace{\sum_{n>0} \alpha_{-n} \cdot \alpha_n + \sum_{n>0} \alpha_{-n}^{25}\alpha_n^{25}}_{N}$$

(1+24) dim inner product

$$L_m = \frac{1}{2}\sum_n \alpha_{m-n} \cdot \alpha_n + \frac{1}{2}\sum \alpha_{m-n}^{25}\alpha_n^{25}$$

Similar expressions for $\tilde{L}_m$

Physical states: $\begin{cases} L_m|\psi\rangle = 0 & \tilde{L}_m|\psi\rangle = 0 \quad \forall m>0 \\ (L_0 - 1)|\psi\rangle = 0 & (\tilde{L}_0 - 1)|\psi\rangle = 0 \end{cases}$

Because the structures are identical, the discussions about normal ordering go through as before.

The main novelty is the mass shell & level matching conditions

Mass-shell and level matching conditions

$$(L_0-1)|\phi\rangle = 0 \qquad (\tilde{L}_0-1)|\phi\rangle = 0$$

$$L_0-1 = \frac{\alpha'}{4}(\underbrace{p\cdot p}_{-M_{25}^2} + p_B^2) + N-1 \longrightarrow M_{25}^2 = p_B^2 + \frac{4}{\alpha'}(N-1)$$

$$p_B = p_{25} - \frac{Rw}{\alpha'}$$

$$\tilde{L}_0-1 = \frac{\alpha'}{4}(\underbrace{p\cdot p}_{-M_{25}^2} + p_L^2) + \tilde{N}-1 \longrightarrow M_{25}^2 = p_L^2 + \frac{4}{\alpha'}(\tilde{N}-1)$$

$$p_L = p_{25} + \frac{Rw}{\alpha'}$$

Then:

$$M_{(25)}^2 = \frac{m^2}{R^2} + \frac{R^2w^2}{\alpha'^2} + \frac{2}{\alpha'}(N+\tilde{N}-2)$$

mass shell condition

contribution to the mass from momentum along the compact direction

contribution to the mass of the string winding around the circle $|w|$ times (stringy contributions)

$$N-\tilde{N} = mw$$

level-mismatching condition

For $w=0$ this matches results from EFT

Lowest energy state: tachyon ($N=\tilde{N}=0$, $m=0, w=0$: $M_{T}^2 = -4/\alpha'$)

Massless spectrum: for $N=\tilde{N}=1$ ($\Rightarrow mw=0$) and $m=w=0$

25-dim
- graviton: $\gamma_{ij}\, \alpha_{-1}^{i}\, \tilde{\alpha}_{-1}^{j}\, |0;\tilde{0};K,0,0\rangle$
- KR B-field: $B_{ij}\, \alpha_{-1}^{i}\, \tilde{\alpha}_{-1}^{j}\, |0;\tilde{0};K,0,0\rangle$
- dilaton: scalar from $\alpha_{-1}^{i}\, \tilde{\alpha}_{-1}^{j}\, |0,\tilde{0};K,0,0\rangle$
- graviphoton and KR-photon: $(\xi\cdot\alpha_{-1}\,\tilde{\alpha}_{-1}^{25} \pm \xi\cdot\tilde{\alpha}_{-1}\,\alpha_{-1}^{25})|0,\tilde{0};K,0,0\rangle$

(graviphoton from the 26 dim metric + another photon from the 26 dim KR field)

radion $\alpha_{-1}^{25}\,\tilde{\alpha}_{-1}^{25}\,|0,\tilde{0};K\;0,0\rangle$
(identified with the scalar σ)

massless string spectrum $\longleftrightarrow$ massless spectrum from KK reduction of EFT

(For certain values of R, there are more massless states) *

As before: one constructs the state ↔ vertex operator correspondence. It turns out that the vertex operator for the graviphoton & KR-photon is given by

$$V_{\pm}(\zeta, k) = \frac{1}{\sqrt{2\alpha'}} \int d^2\xi \; \zeta \cdot \left(\partial_+ X \, \partial_- X^{25} \pm \partial_- X \, \partial_+ X^{25} \right) :e^{ik\cdot X}:$$

$U(1) \times U(1)$ symmetry

Remarks:

Problem sheet: for certain values of R there are more massless states & gauge symmetry can be enhanced.
In particular, at $R = \sqrt{\alpha'}$ gauge symmetry is $SU(2) \times SU(2)$!

States with nontrivial circle momentum (m/R) and winding (w)

In general these are not massless. (see problem sheet)

The simplest is the "winding" tachyon $|0;0;k;m=0,w\rangle$

$$M^2_{(w)} = \frac{R^2}{\alpha'^2}w^2 - \frac{4}{\alpha'} = \frac{1}{\alpha'^2}\left(R^2 w^2 - 4\alpha'\right)$$ ($N=\tilde{N}=0$ & $mw=0$)

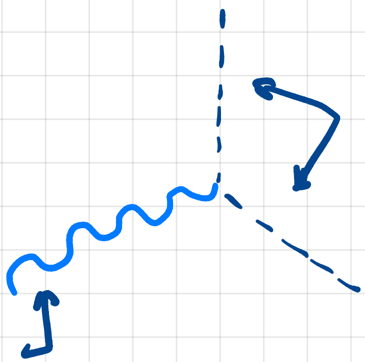
⋮

Checking that KK-modes with $m \neq 0$ & winding modes are charged under the $U(1) \times U(1)$ gauge symmetries:

A_i (graviphoton) $\qquad \tilde{A}_i$

Consider the 3-amplitude

In the 25-dim theory this is the tree level contribution to an emission/absorption of a KR-photon from a w-tachyon to a w-tachyon with winding number w



winding tachyon $(m=0)$

$|0, \tilde{0}; k, 0, w\rangle$

$M^2_{(L,R)} = \frac{1}{\alpha'^2}(R^2 w^2 - 4\alpha')$

Kalb-Ramond photon

$(\xi \cdot \alpha_{-1} \tilde{\alpha}^{25}_{-1} - \xi \cdot \tilde{\alpha}_{-1} \alpha^{25}_{-1}) |0, k; 0, 0\rangle$

Vertex operator for the KR photon

$$V_{KR}(\xi, k) = \frac{1}{\sqrt{2\alpha'}} \int d\tau d\sigma\, \xi \cdot (\partial_+ X\, \partial_- X^{25} - \partial_- X\, \partial_+ X^{25})\, e^{ik\cdot X}$$

Computing the amplitude:

$$A \sim \langle 0, -k_3; 0, w| \, V_{KR}(\xi, k_2) \, |0, k_1; 0, w\rangle$$

$$= \frac{1}{\sqrt{2\alpha'}} \underline{\langle 0, -k_3; 0, w} |(\xi \cdot \partial_+ X \partial_- X^{25} - \xi \cdot \partial_- X \partial_+ X^{25}) e^{ik_2 \cdot X} \underline{|0, k_1; 0, w\rangle}$$

$$= \frac{1}{\sqrt{2\alpha'}} \langle 0, -k_3; 0, w| \, (\xi \cdot \tilde{\alpha}_0 \, \alpha_0^{25} - \xi \cdot \alpha_0 \, \tilde{\alpha}_0^{25}) \, |0, k_1 + k_2; 0, w\rangle$$

$$= \frac{1}{\sqrt{2\alpha'}} \xi \cdot (k_1 + k_2) \, \langle 0, -k_3; 0, w| \underbrace{(\alpha_0^{25} - \tilde{\alpha}_0^{25})}_{\alpha_0^{25} - \tilde{\alpha}_0^{25} = -\sqrt{\frac{2}{\alpha'}} R w} |0, k_1 + k_2; 0, w\rangle$$

$$= \left(\frac{1}{\alpha'} R w\right) \xi \cdot k_3 \, \delta^{(25)}(k_1 + k_2 + k_3)$$

winding tachyon charged under $\tilde{A}_i$

As usual in gauge theories: winding tachyon is charged under the KR $U(1)$ gauge field $\tilde{A}$ with charge $\propto$ wR/α'

Similar computation for the graviphoton:

momentum $\frac{m}{R}$ is the charge under A_i

This agrees with the KK reduction

More generally, states with circle momentum & winding numbers (n, w) have graviphoton $U(1)$ charge $\frac{1}{2}(p_L + p_R) = \frac{n}{R}$

and KR-photon $U(1)$ charge $\frac{1}{2}(p_L - p_R) = \frac{Rw}{\alpha'}$

Remark: we have introduced a new scale R

R is called a modulus

(more general compactifications give rise to moduli spaces)

In fact, we have a one parameter family of compactifications with $R \in (0, \infty)$

Or do we? $(0,\infty)$ containts values of R which give rise to indistinguishable physical theories. !

↳ next: T duality