String Theory 1

Lecture #15

5 Compactifications

$\mathbb{R}^{1,24} \times S^1_R$ continued...

Two perspectives

① Spacetime EFT approach

↳ Kaluza-Klein mechanism on the 26 dim EFT

- We obtained a massless sector for a $\mathbb{R}^{1,24}$ EFT

$$G_{\mu\nu}(X) \longrightarrow G_{\mu\nu}(X^i) : \{ \underbrace{G_{ij}}_{\text{25 dim graviton}} , \underbrace{e^{2\sigma} A}_{\text{graviphoton}} , \underbrace{e^{2\sigma}}_{\text{radion}} \}$$

$$B_{\mu\nu}(X) \longrightarrow B_{\mu\nu}(X^i) : \{ \underbrace{B_{ij}}_{\text{25 dim KR field}} , \underbrace{\tilde{A}}_{\text{KR-photon}} \}$$

2 $U(1)$ gauge symmetries

$$\Phi(X) \longrightarrow \Phi(X^i) \quad \text{25 dim dilaton}$$

- $\boxed{\text{Plus}}$ a discrete infinite tower of massive states (KK-modes)

For example: the 26 dimensional dilaton gives rise to a discrete infinite tower of scalar fields (KK modes)

$$\phi_n \quad \text{with mass} \quad M_n^2 = \frac{n^2}{R^2} \qquad \forall n \in \mathbb{Z}$$

These are charged ($n \neq 0$) under the graviphoton: charge $\frac{n}{R}$

(KK-momentum)

There are no modes charged under the $U(1)$ corresponding to the KR photon.

Of course this introduces a new scale $M_{KK} \sim \frac{1}{R}$

In fact one can show that $\langle \sigma \rangle = R$

(see Blumenhagen + Lüst + Theisen)

We should not trust the EFT analysis for $M_{KK} \sim M_s$.

- However one can perform an exact analysis of the worldsheet CFT.

② The world sheet perspective (closed strings)

2dim World sheet NLσM with target space with a nontrivial topology

$$X^i \longrightarrow X^i \qquad i = 0, \dots, 24$$

$$X^{25} \sim X^{25} + 2\pi R \qquad (X^{25} \text{ parametrises a circle } S^1_R)$$

States in the string Hilbert space are similar to those of $\mathbb{R}^{1,25}$ however:

we have states on S^1_R with quantized momentum $\left(p^{25} = \frac{m}{R}\right)$ ($\rightsquigarrow$ KK-modes)

and quantized winding modes.

The winding modes come from the periodicity condition

$$X^{25}(\tau, \sigma + 2\pi) = X^{25}(\tau, \sigma) + 2\pi R w \qquad w \in \mathbb{Z} \qquad (X^{25}(\tau,\sigma) \text{ periodic up to } 2\pi R w)$$

The mode expansion of $X^i(\tau,\sigma)$, $i=0,\dots,24$, is as for $\mathbb{R}^{1,25}$, but the expansion for X^{25} changes

$$X^{25}(\tau,\sigma) = x^{25} + 2\alpha' p^{25}\tau + \underline{2R\omega\sigma} + \text{oscillator modes}, \qquad p^{25} = \frac{m}{R} \qquad m\in\mathbb{Z}$$

The states are of the form: $\prod \alpha_{-n}^{\mu} \prod \tilde{\alpha}_{-m}^{\nu} \, |K; m; \omega\rangle$

K: 25-dim momentum evalue

m: $p^{25} = \frac{m}{R}$

ω: $X^{25}(\tau,\sigma+2\pi) = X^{25}(\tau,\sigma) + 2\pi R\omega$

and physical states must satisfy the conditions:

$$(L_0-1)|\phi\rangle=0 \qquad (\tilde{L}_0-1)|\phi\rangle=0$$

$$\begin{cases} M^2_{(25)} = \dfrac{m^2}{R^2} + \underbrace{\dfrac{R^2\omega^2}{\alpha'^2}} + \dfrac{2}{\alpha'}(N+\tilde{N}-2) \\ \\ N-\tilde{N} = m\omega \end{cases}$$

contribution to the mass from momentum along the compact direction

contribution to the mass of the string winding around the circle $|\omega|$ times $\left(\frac{R\omega}{\alpha'}\right)^2 = (2\pi R T\omega)^2$

mass shell condition

$\boxed{M_{(25)} \text{ depends on } R\,!}$

level-(mis)matching condition

Also $L_m|\phi\rangle = 0 \quad \forall m>0$ (and similarly for $\tilde{L}_m$)

Massless spectrum: for any R (states with $m=w=0$, $N=\tilde{N}=1$)

- 25 dim graviton $\quad \gamma_{ij}\, \alpha_{-1}^{i}\, \tilde{\alpha}_{-1}^{j} |0,k;0,0\rangle$
- 25 dim B-field $\quad B_{ij}\, \alpha_{-1}^{i}\, \tilde{\alpha}_{-1}^{j} |0,k;0,0\rangle$
- dilaton: scalar from the trace part of γ: $\Phi_{(25)}$
- 2x 25 dim $U(1)\times U(1)$ gauge fields $\quad \xi_i (\alpha_{-1}^{i} \tilde{\alpha}_{-1}^{25} \pm \tilde{\alpha}_{-1}^{i} \alpha_{-1}^{25}) |0,k;0,0\rangle$

(graviphoton from the 26 dim metric + another photon from the 26 dim KR field)

- scalar ("radion") $\quad \alpha_{-1}^{25}\, \tilde{\alpha}_{-1}^{25} |0,k\rangle \otimes |0,0\rangle$

identified with a scalar σ

massless string spectrum $\longleftrightarrow$ massless spectrum from KK reduction of EFT

For certain values of R (eg $\alpha' = \sqrt{R}$!) there are more!

Remark: we have introduced a new scale R

In fact, we have a one parameter family of compactifications with $R \in (0, \infty)$. R is called a modulus

(More general compactifications give rise to a moduli space)

However $(0, \infty]$ contains values of R which give rise to indistinguishable physical theories.

$\boxed{5.3}$ **T-duality** (closed strings)

Returning to the mass formulas

$$M^2_{(25)} = \frac{m^2}{R^2} + \frac{1}{(\alpha')^2} w^2 R^2 + \frac{2}{\alpha'}(N+\tilde{N}-2) \,, \qquad N-\tilde{N} = mw$$

Limiting cases:

- $R \to \infty$, $w \to 0$ (keeping wR finite)

  "decompactification", strings cannot wind anymore

  continuum of KK modes $\to$ sign of 26th dimension

  expect theory on $\mathbb{R}^{1,25}$

- $R \to 0$, $m \to 0$ (keeping m/R finite)

  "genuine" 25 dim theory

  no momentum along 25th dim

  continuum of winding modes

$\boxed{\text{Symmetry}}$ of the spectrum: observe that the formulas

$$M^2_{(2\pi)} = \frac{m^2}{R^2} + \frac{1}{(\alpha')^2} W^2 R^2 + \frac{2}{\alpha'}(N+\tilde{N}-2) \quad , \qquad N-\tilde{N} = mW$$

are invariant under: $m \leftrightarrow W$ & $R \leftrightarrow \frac{\alpha'}{R} = \hat{R}$

$\Rightarrow$ compactifications on S_R & $S_{\alpha'/R}$ have the $\boxed{\text{same}}$ spectrum.

[Note that $R = \sqrt{\alpha'}$ is a fixed point of this transformation: something special happens at this point.]

This is in fact an exact symmetry of the CFT

$$\boxed{\text{T-duality}}$$

so compactifications on S_R & $S_{\hat{R}}$ with $\hat{R} = \frac{\alpha'}{R}$ are indistinguishable as physical theories.

The interchange $m \leftrightarrow w$ means that

momentum excitations $\longleftrightarrow$ winding mode excitations

so continuum of KK modes for $R \to \infty$ $\longleftrightarrow$ continuum of winding modes for $\hat{R} \to 0$

T-duality as an exact symmetry of the CFT

BUT, we have $\boxed{\text{only}}$ shown that the spectrum is the same for two theories where

$$R \longleftrightarrow \hat{R} = \frac{\alpha'}{R}$$

and simultaneously

$$(m, w) \longleftrightarrow (w, m)$$

We need to consider the $\boxed{\text{full CFT}}$ to prove this is an $\boxed{\text{exact}}$ symmetry of the CFT

any transformation which reorganises the CFT data must not modify the physics (null state conditions & correlation functions)

Recall $\alpha_0^{25} = \sqrt{\frac{\alpha'}{2}}\left(m\frac{1}{R} - \frac{R}{\alpha'}w\right)$

$$\tilde{\alpha}_0^{25} = \sqrt{\frac{\alpha'}{2}}\left(m\frac{1}{R} + \frac{Rw}{\alpha'}\right)$$

Then under $R \leftrightarrow \frac{\alpha'}{R}$ and $m \leftrightarrow w$

$$\alpha_0^{25} \longrightarrow \sqrt{\frac{\alpha'}{2}}\left(w\frac{R}{\alpha'} - \frac{1}{R}m\right) = -\alpha_0^{25}$$

$$\tilde{\alpha}_0^{25} \longrightarrow \sqrt{\frac{\alpha'}{2}}\left(w\frac{R}{\alpha'} + \frac{1}{R}m\right) = \tilde{\alpha}_0^{25}$$

interchanges the graviphoton with the KK-photon!

2× 25 dim

$U(1)\times U(1)$ gauge fields

$$\zeta\cdot(\alpha_{-1}\tilde{\alpha}_{-1}^{25} \pm \tilde{\alpha}_{-1}\alpha_{-1}^{25})|0,K;0,0\rangle$$

We extend action of the transformation to the oscillator modes

$$X^{25}_R(\xi^-) \longleftrightarrow \hat{X}^{25}_R(\xi^-) = -X^{25}_R(\xi^-)$$

$$X^{25}_L(\xi^+) \longleftrightarrow \hat{X}^{25}_L(\xi^+) = X^{25}_L(\xi^+)$$

$$X^i(\xi^\pm) \longleftrightarrow \hat{X}^i(\xi^\pm) = X^i(\xi^\pm) \qquad i=0,\dots,24$$

new coordinate fields

Under $\sigma \to \sigma + 2\pi$: $\hat{X}^{25}(\tau, \sigma+2\pi) = X^{25}(\tau,\sigma) + 2\pi \frac{\alpha'}{R} n$

$2\pi\hat{R}$ periodic! $\qquad \hat{R}$

Mode expansions :

$$X^{25}(\tau,\sigma) = X^{25}_L(\xi^+) + X^{25}_R(\xi^-) \longleftrightarrow \hat{X}^{25}(\tau,\sigma) = X^{25}_L(\xi^+) - X^{25}_R(\xi^-)$$

$$= x^{25} + 2\alpha' \frac{m}{R}\tau + 2Rw\sigma + \dots \qquad = \hat{x}^{25} + 2\alpha' \frac{m}{R}\sigma + 2Rw\tau + \dots$$

circle radius R $\qquad$ circle radius $\frac{\alpha'}{R}$

conj momentum $p^{25} = \frac{m}{R}$ $\qquad$ conjugate momentum $\hat{p}^{25} = \frac{wR}{\alpha'}$

X & $\hat{X}$ have the same energy momentum tensor

$$T_{\pm\pm} = \partial_\pm X \cdot \partial_\pm X = \partial_\pm \hat{X} \, \partial_\pm \hat{X}$$

$$\partial_+ \hat{X}^{\tau\sigma} = \partial_+ X_L^{\tau\sigma} = \partial_+ X^{\tau\sigma}$$
$$\partial_- \hat{X}^{\tau\sigma} = -\partial_- X_R^{\tau\sigma} = -\partial_- X^{\tau\sigma}$$

So one can recover L_m & $\tilde{L}_m$ as Fourier modes

$\Rightarrow$ CFTs of X & $\hat{X}$ are the $\boxed{\text{same}}$ with $\hat{R} = \dfrac{\alpha'}{R}$

As a consequence of this duality the moduli space of circle compactifications of the bosonic string is not $(0, \infty)$ but instead

$$\boxed{R \in (0, \sqrt{\alpha'}]} \quad \text{or equivalently} \quad R \in [\sqrt{\alpha'}, \infty)$$

Fixed point of the duality transformation:

$$R \longleftrightarrow \hat{R} = \frac{\alpha'}{R} \quad \text{when} \quad \boxed{R = \sqrt{\alpha'}}$$

$R = \sqrt{\alpha'}$ is special $\rightsquigarrow$ more massless states and enhanced gauge symmetry

$R = \sqrt{\alpha'}$

$$M^2_{(25)} = \frac{m^2}{R^2} + \frac{1}{(\alpha')^2} w^2 R^2 + \frac{2}{\alpha'}(N + \tilde{N} - 2) = \frac{1}{R^2}\left(m^2 + w^2 + 2(N + \tilde{N} - 2)\right)$$

so $M^2_{25} = 0$ when $m^2 + w^2 + 2(N + \tilde{N}) = 4$

& $mw = N - \tilde{N}$

$$m^2 + w^2 + 2(N+\tilde{N}) = 4 \qquad\qquad R = \sqrt{-\alpha'}$$

$$\& \quad mw = N - \tilde{N}$$

$$(m-w)^2 = 4 - 2(N+\tilde{N}) - 2mw = 2(2 - (N+\cancel{\tilde{N}}) - (N-\cancel{\tilde{N}}))$$

$$= 4(1-N) \qquad \text{so} \qquad N = 0,1$$

$$(m+w)^2 = 4 - 2(N+\tilde{N}) + 2mw = 2(2 - (\cancel{N}+\tilde{N}) + (\cancel{N}-\tilde{N}))$$

$$= 4(1-\tilde{N}) \qquad \text{so} \qquad \tilde{N} = 0,1$$

possible cases: $(N,\tilde{N}) = (0,0), (0,1), (1,0), (1,1)$

$$\underset{\substack{mw=0 \\ m^2+w^2=4}}{(0,0)}, \quad \underset{\substack{\underbrace{mw=-1} \\ m=-w \\ w=\pm 1}}{(0,1)}, \quad \underset{\substack{\underbrace{mw=1} \\ m=w \\ w=\pm 1}}{(1,0)}, \quad \underset{mw=0}{(1,1)}$$

$\underline{N = \tilde{N} = 0} \implies mw = 0 \quad m^2 + w^2 = 4$

$m = 0 \quad w = \pm 2$
$w = 0 \quad m = \pm 2$ $\Big\}$ $|0; 0, \pm 2; K\rangle$, $|0; \pm 2, 0; K\rangle$ 4 new scalars

$\underline{N = 1 \quad \tilde{N} = 0} \implies mw = 1 \; \Big\langle \begin{matrix} m = w = 1 \\ m = w = -1 \end{matrix}$

$\lambda \cdot \alpha_{-1} |0; \pm 1, \pm 1, K\rangle$ 2x (1 gauge field + 1 scalar)

$\underline{N = 0 \quad \tilde{N} = 1} \implies mw = -1 \; \Big\langle \begin{matrix} m = -w = 1 \\ m = -w = -1 \end{matrix}$

$\tilde{\lambda} \cdot \tilde{\alpha}_{-1} |0; \mp 1, \pm 1, K\rangle$ 2x (1 gauge field + 1 scalar)

$\underline{N = \tilde{N} = 1} \implies \begin{matrix} mw = 0 \\ m^2 + w^2 = 0 \end{matrix} \implies m = w = 0$ ✓

$\gamma_{\mu\nu} \alpha^{\mu}_{-1} \tilde{\alpha}^{\nu}_{-1} |0; 0, 0; K\rangle$ graviton, KR, 2 gauge fields, 2 scalars

There are in fact

4 extra massless vectors which enhance the $U(1) \times U(1)$ symmetry to

$SU(2) \times SU(2)$

$\xi \cdot \alpha_{-1}^{25} \tilde{\alpha}_{-1} |0;0,0;K\rangle$, $\tilde{\xi} \cdot \alpha_{-1} \tilde{\alpha}_{-1}^{25} |0;0,0;K\rangle$

$\lambda \cdot \alpha_{-1} |0;\pm 1, \pm 1, K\rangle$ $\quad$ $\tilde{\lambda} \cdot \tilde{\alpha}_{-1} |0;\mp 1, \pm 1, K\rangle$

and

8 additional scalar fields $\longrightarrow$ $(\underline{3}, \underline{3})$ rep of $SU(2) \times SU(2)$

φ, σ

$|0;0,\pm 2;K\rangle$ $\quad$ $|0;\pm 2, 0;K\rangle$

$\alpha_{-1}^{25} |0;\pm 1, \pm 1, K\rangle$ $\quad$ $\tilde{\alpha}_{-1}^{25} |0;\mp 1, \pm 1, K\rangle$

(BLT for details)

5.4 Open strings and T-duality

What happens to T-duality?

closed strings it was crucial that strings can wind around S^1

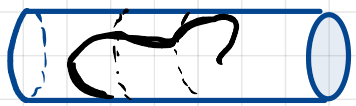
Recall: open string boundary conditions compatible with Poincaré invariance in 26 dimensions

$$\frac{\partial}{\partial \sigma} X^\mu(\tau,\sigma) = 0 \quad \text{at} \quad \sigma = 0, \pi$$

Neumann boundary conditions

(ends of the string are free to move in spacetime)

Consider now compactifying on a circle



→ no winding modes !

while KK-momentum momentum modes still make sense

Recall, for open strings:

Soln of $\partial_+ \partial_- X^\mu = 0$

$$X_L^\mu(\xi^+) = \frac{1}{2} x^\mu + \alpha' p^\mu \xi^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{m\neq 0} \frac{1}{m} \tilde{\alpha}_m^\mu e^{-im\xi^+}$$

$$X_R^\mu(\xi^-) = \frac{1}{2} x^\mu + \alpha' p^\mu \xi^- + i\sqrt{\frac{\alpha'}{2}} \sum_{m\neq 0} \frac{1}{m} \alpha_m^\mu e^{-im\xi^-}$$

Neumann boundary condition $\quad \partial_\sigma X^\mu \big|_{\sigma=0,\pi} = 0 \implies \alpha_m^\mu = \tilde{\alpha}_m^\mu$

Dirichlet boundary condition $\quad X^\nu \big|_{\sigma=0,\pi} = c^\nu$

$$\implies x^\nu = c^\nu , \quad p^\nu = 0, \quad \alpha_m^\nu = -\tilde{\alpha}_m^\nu$$

compactify on a circle with X^{25} parametrising the circle of radius R.
& consider an open string with NN boundary conditions in the coordinate X^{25} ($2\pi R$ periodic)
so both ends of the string move freely on the circle S^1_R
Follow the same procedure as for the bosonic string.

What happens when interchanging

$$X_L^{25} \longleftrightarrow X_L^{25} \qquad X_R^{25} \longleftrightarrow -X_R^{25} \qquad ?$$

What should we expect?

Should we expect a dual string for which there is a winding quantum number but no KK-momentum?

The proposed dual coordinate is

$$\hat{X}^{25}(\tau,\sigma) = X_L^{25}(\xi^+) - X_R^{25}(\xi^-)$$

$2\pi\hat{R} = \frac{2\pi\alpha'}{R}$

periodic

$\hat{R} = \frac{\alpha'}{R}$ dual circle $S^1_{\hat{R}}$

$$= 2\alpha' p^{25}\sigma + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^{25} e^{-in\tau}\sin(n\sigma)$$

$$= \frac{2\alpha'}{R} m\,\sigma + osc = 2\hat{R}m\,\sigma + osc$$

- no term linear in τ ie the dual string has no momentum in the circle direction: translation invariance along S^1 is broken

- Moreover dual string wraps around the dual circle m times

Boundary conditions of the dual string: at $\sigma = 0, \pi$

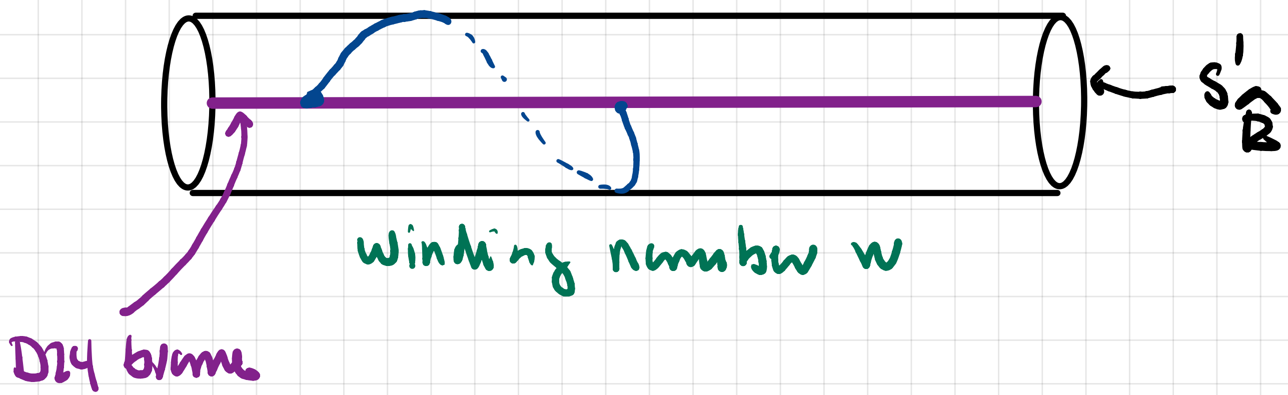
$$\hat{X}^{25}(\tau, \sigma)\Big|_{\sigma=0} = 0$$

$$\hat{X}^{25}(\tau, \sigma)\Big|_{\sigma=\pi} = 2\alpha' \frac{m}{R} \pi = 2\pi m \hat{R}$$

position of the end points of the dual string are fixed.

→ This is a Dirichlet boundary condition!

The dual open string is attached to a (1+24) dimensional hyperplane, a $\boxed{\text{D24-brane}}$



Under a T-duality transformation:

open string with Newman boundary condition on S'_R $\longleftrightarrow$ open string with Dirichlet boundary condition on $S'_{\hat{R}}$

[momentum $\frac{m}{R}$ along S'_R $\longleftrightarrow$ no momentum along $S'_{R'}$

no winding around S'_R $\longleftrightarrow$ winding around $S'_{R'}$]

The subspace where the string ends are attached to is called a D-brane

convention: a Dp-brane is a D-brane with p spatial dimensions

(so it is p+1 dimensional)

T-duality

$\longleftrightarrow$

open string with Neumann boundary conditions compactified on S^1_R

dual open string with Dirichlet boundary conditions compactified on $S^1_{\hat{R}}$, $\hat{R} = \alpha'/R$

D25 space-filling brane
↳ open string ends are free to move on spacetime

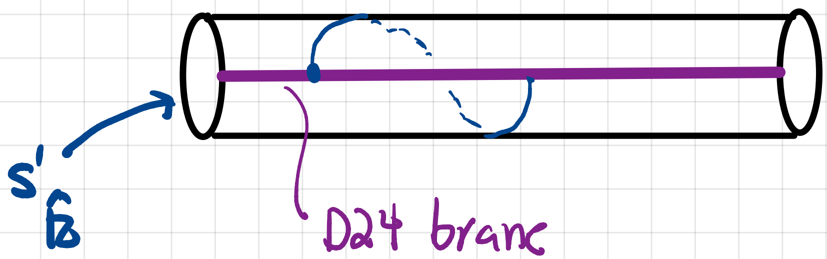
endpoints of the string live on a D24 brane

$p^{25} = \frac{m}{R}$ quantized

no translational symmetry along $S^1_{\hat{R}}$

no winding

string can wind around $S^1_{\hat{R}}$



massless vector: (both sides)
25 dimensional U(1) gauge fields

Final remark:

$\mathbb{R}^{1,d-1} \times M_d$ is not the most general possibility

eg: for the superstring

- BH sln.
- $AdS_3 \times M_7$
- ⋮

↳ next: Epilogue on D-branes.