1 Model Theory: Introduction

- Duality:
  syntactic description $\longleftrightarrow$ structures.

- Between a universal theory and a complete theory. Quantifier elimination.

- Between a complete theory and a structure. When is a structure determined by its theory (categoricity)? Is there (in some sense) a *smallest* model? Is it unique? Is there a 'biggest' (countable) model?

2 Review of Logic: Languages

Alphabet, variables, terms, formulas.
A *language* L is specified by its **non-logical** symbols. These are relation symbols, function symbols, and constant symbols of given arities.
(0-place relation symbols: propositional constants. (We will not need them.)
0-place function symbols = constant symbols.)

The formulas of L are formed using the non-logical symbols, and the following *logical* symbols:

- $\doteq$ - the equality symbol. (We will allow writing it as $=$.)
- A (countable) set of *variables*;
- $\bot, \wedge, \neg$ - Boolean connectives;
- $\exists$ - the existential quantifier;

We now construct, successively, *terms*, *atomic formulas*, *quantifier-free formulas*, *formulas* and *sentences* of a given language L.

L**-terms** are constructed recursively from the function symbols, and variable symbols.
We write $\tau = \tau(x_1, \dots, x_n)$ to indicate that the variables occurring in τ are among $x_1, \dots, x_n$. Terms with no variables are called **closed terms**

Atomic L**-formulas** have the form
(i) $\tau_1 \doteq \tau_2$ for any L-terms τ_1 and τ_2
or
(ii) $P(\tau_1, \dots, \tau_\rho)$ for any relational L-symbol P of arity ρ and L-terms $\tau_1, \dots, \tau_\rho$.

Notice, that (i) can be seen as a special case of (ii) if we view $\doteq$ as a relation symbol of arity 2.

An L**-formula** is defined by the following recursive definition:
(i) any atomic L-formula is an L-formula;
(ii) if φ is an L-formula, so are $\neg\varphi$ and $\perp$;
(iii) if φ, ψ are L-formulas, so is $(\varphi \wedge \psi)$;
(iv) if φ is an L-formula, so is $\exists v \varphi$ for any variable v;
The set of formulas obtained using (i),(ii), (iii) along are called *quantifier-free (qf)*.

Some abbreviations

$\vee, \rightarrow, \leftrightarrow, \forall$, as defined in the Logic class, e.g.:

$(\phi \rightarrow \psi)$ is an abbreviation for $\neg(\phi \wedge \neg\psi)$;

$\forall v \psi$ for $\neg\exists v \neg\psi$.

$x \neq y$ for $\neg(x \doteq y)$

$\bigwedge_{i=1}^{4} \phi_i$ for $\phi_1 \wedge \phi_2 \wedge \phi_3 \wedge \phi_4$

$\exists^{\geq 4} x \phi(x)$ for

$(\exists x_1) \cdots (\exists x_4)(\bigwedge_{i=1}^{4} \phi(x_i) \wedge \bigwedge_{1 \leq i < j \leq 4} x_i \neq x_j)$

It is typical of logic that formulas in n-variables are discussed, and n-tuples of elements of a structure occur more frequently than single elements.
We will thus often use 'vector notation', writing a for $(a_1, \ldots, a_n)$ and x for $(x_1, \ldots, x_n)$ when possible.

Formulas that can be formed without quantifiers (Boolean combinations of atomic formulas) are called *quantifier-free*, abbreviated qf.
A formula is *universal* if it has the form $(\forall x_1) \cdots (\forall x_n)\psi$, where ψ is quantifier-free.
Similarly one of the form $(\exists x_1) \cdots (\exists x_n)\psi$ is called *existential*.

Writing $\varphi(x_1, \ldots, x_n)$ means: φ is a formula and $(x_1, \ldots, x_n)$ is a tuple of variables, including all the free variables of φ.

Free variables For an atomic formula $\varphi(v_{i_1}, \ldots, v_{i_n})$, all variables occurring in (the terms of) φ are said to be free. For more complex formulas, the set of free variables $FV(\phi)$ is defined recursively: $FV(\bot) = \emptyset$, $FV(\neg\psi) = FV(\phi)$, $FV(\phi \wedge \psi) = FV(\phi) \cup FV(\psi)$, $FV((\exists x)\phi) = FV(\phi) \setminus \{x\}$.

An L-formula with no free variables is called an **L-sentence**.

We write $|L|$ for the cardinality of the set of L-formulas.

Exercise Show that $L = \max\{\aleph_0, |Symb(L)|\}$ where $Symb(L)$ is the set of non-logican symbols of L.

Proof. We have $|L| \geq \aleph_0$ since we always have, for instance, the countably many sentences: $(\exists^{\geq n} x)(x = x)$.
Also $|L| \geq |Symb(L)|$.
To see that $|L| \leq \max(\aleph_0, |Sym(L)|)$: a formula can be viewed as a finite string of characters, taken from among the non-logical symbols of L, the finitely many logical symbols including $\doteq$, and parentheses.
So it suffices to show that the set $\cup_n X^n$ of finite sequences from a set X, itself has cardinality $\leq \max(|X|, \aleph_0)$.
If X is finite, we have $|X^n| = |X|^n < \aleph_0$ and so $|\cup_n X_n| \leq \aleph_0$.
If X is infinite, $|X^n| = |X|$ and so $|\cup_n X_n| \leq \aleph_0 |X| = |X|$. □

Proof systems

A major part of the Logic class was devoted to *proof systems.* A relation was defined between sets of sentences, and a sentence:
$\Gamma \vdash \psi$ iff there exists a formal proof of ψ, under hypotheses taken from Γ.
Formal proofs play no role in model theory, and will provide no more than silent background intuition.
But we do record the following observation:

Proposition. *If $\Gamma \vdash \psi$, then there exists a finite $\Gamma_0 \subset \Gamma$ such that $\Gamma_0 \vdash \psi$.*

This is immediate, since a formal proof has by definition a finite sequence of steps, and at each step only one hypothesis can be quoted.

3 Review of Logic: Structures

Let L be a language, consisting of relation symbols P_i $(i \in I)$, function symbols for $j \in J$, and constant symbols $c_k (k \in K)$.
An L-*structure* is an object of the form

$$\underline{A} = \left\langle A; \{P_i^{\underline{A}}\}_{i\in I}; \{f_j^{\underline{A}}\}_{j\in J}; \{c_k^{\underline{A}}\}_{k\in K} \right\rangle.$$

consisting of:
(i) a set A called the *domain* or *universe* of the L-structure;
(ii) an assignment of an r-ary relation (subset) $P^{\underline{A}} \subseteq A^r$ to each relation symbol P of L of arity r;
(iii) an assignment of an m-ary function $f^{\underline{A}} : A^m \to A$ to any function symbol f of L of arity m;
(iv) an assignment of an element $c^{\underline{A}} \in A$ to any constant symbol c of L.

$\{P_i^{\underline{A}}\}_{i\in I}$, $\{f_j^{\underline{A}}\}_{j\in J}$ and $\{c_k^{\underline{A}}\}_{k\in K}$ are called the *interpretations in* $\underline{A}$ of the predicate, function and constant symbols correspondingly.
Writing $\left\langle A; \{P_i^{\underline{A}}\}_{i\in I}; \{f_j^{\underline{A}}\}_{j\in J}; \{c_k^{\underline{A}}\}_{k\in K} \right\rangle$ implicitly specifies the language L.
For instance, $(\mathbb{R}, 0, +, -)$ is a structure for the *language of groups*, a language with a constant symbol, a unary function symbol and a binary function symbol. Similarly, $(\mathbb{R}, 0, 1, +, -, \cdot)$ is a structure for the *language of rings*; they have the same domain, but are structures for different languages.

Embeddings and isomorphisms

Fix a language L. We have defined L-structures; we will now define the notion of an *embedding* of L-structures. It is a straightforward generalization of the various cases you have seen in algebra, such as an embedding of groups, rings, or ordered fields.
Let $\underline{A}, \underline{B}$ be L-structures, with universes A, B respectively.
An *embedding* (or L-embedding) of $\underline{A}$ in $\underline{B}$ is a one-to-one function $\pi : A \to B$ which preserves corresponding relation, function and constant symbols, i.e. for any relation symbol P, function symbol F, constant symbol c of L we have:
(i) $\bar{a} \in P^{\underline{A}}$ iff $\pi(\bar{a}) \in P^{\underline{B}}$;
(ii) $\pi(F^{\underline{A}}(\bar{a})) = F^{\underline{B}}(\pi(\bar{a}))$;
As a special case of (ii) we have: (iii) $\pi(c^{\underline{A}}) = c^{\underline{B}}$.

We write in this case $\pi : \underline{A} \to \underline{B}$.
An important case occurs when $A \subseteq B$, and π is the inclusion map, i.e. $\pi(a) = a$ for $a \in A$. In this case we write $\underline{A} \leq \underline{B}$, and say $\underline{A}$ is a *substructure* of $\underline{B}$. The definition of an embedding can be rewritten as follows:
(i) $P^{\underline{A}} = P^{\underline{B}} \cap A^k$ where P is a k-place relation symbol.
(ii) $F^{\underline{A}} = F^{\underline{B}}|A^k$ where F is a k-place function symbol.
(iii) $c^{\underline{A}} = c^{\underline{B}}$ where c is a constant symbol.
Given $\underline{B}$, note that to specify $\underline{A}$ it suffices to give the universe A; the interpretation of the relation and function symbols is then completely determined by being a substructure. Moreover, a subset of B is the universe of a substructure of $\underline{B}$ if and only if it is closed under the basic functions, including the 0-place ones; more precisely:

Exercise 3.1. *A is the universe of a substructure of $\underline{B}$ if and only if $c^{\underline{B}} \in A$ for each constant symbol c, and $F^{\underline{B}}(A^k) \subset A$ for each k-place function symbol of L, $k \geq 1$.*

An *isomorphism* $\underline{A} \to \underline{B}$ is an embedding $\pi : \underline{A} \to \underline{B}$ such that $\pi : A \to B$ is bijective. In this case the inverse map $\pi^{-1} : B \to A$ is also an isomorphism from $\underline{B}$ to $\underline{A}$.
An isomorphism $\pi : \underline{A} \to \underline{A}$ of the structure onto itself is called an **automorphism** of $\underline{A}$.

4 Review of Logic: Interpretation of a formula in a structure.

Let $\underline{A}$ be an L-structure with domain A.
Then $\underline{A}$ includes an interpretation of the basic function symbols. This is extended recursively to an interpretation of *terms*, assigning to a term $\tau(v_1, \ldots, v_n)$ a function

$$\tau^{\underline{A}} : A^n \to A$$

.
$\underline{A}$ also includes interpretation of the basic relation symbols of L. In addition, the logical symbol $\doteq$ is interpreted as the *diagonal* on A, i.e. the set $\{(a, a) : a \in A\}$, a subset of A^2. We thus have an interpertation of all relation symbols, and extend this recursively to an interpretation of *formulas*; for each

assignment $x_i \mapsto c_i$ of elements of A to the free variagles of $\phi = \phi(x_1, \ldots, x_n)$, we defined the *truth value* $\phi(c_1, \ldots, c_n)^{\underline{A}}$ of the formula ϕ under the given assignment.We write $\underline{A} \models \phi(c_1, \ldots, c_n)$ in case this truth value is *true*.
The interpretation of ϕ is then, by definition, the set of all tuples $(c_1, \ldots, c_n)$ such that $\underline{A} \models \phi(c_1, \ldots, c_n)$. Thus if $\phi = \phi(x_1, \ldots, x_n)$, then $\phi^{\underline{A}} \subset A^n$. (Strictly speaking, $A^{\{x_1,\ldots,x_n\}}$.)
In case φ is a sentence, no assignment is needed. We have thus defined the truth value of φ in $\underline{A}$. If this value is **true**, we say that φ holds in $\underline{A}$, or that $\underline{A}$ is a model of φ.

Consider an L-structure $\underline{A}$ and an L-formula $\varphi(v_1, \ldots, v_n)$. Write

$$\varphi^{\underline{A}} = \{\bar{a} \in A^n : \ \underline{A} \models \varphi(\bar{a})\}.$$

The notation $\varphi(A)$ is also used. This is called a *definable set*, namely the set defined by ϕ. It is a subset of A^n, not of A! If we want to emphasize this, we refer to it as a *definable relation.*

Geometric viewpoint of the interpretation of formulas:

$$\bot^{\underline{A}} = \emptyset$$

$$(\neg\phi)^{\underline{A}} = A^n \setminus \phi^{\underline{A}}$$

$$(\phi \wedge \psi)^{\underline{A}} = \phi^{\underline{A}} \cap \psi^{\underline{A}}$$

$(\exists x_n)\phi^{\underline{A}}$ is the *projection* of $\phi^{\underline{A}}$ from n-space to $n-1$-space.

Maps between structures

Let $\underline{A}, \underline{B}$ be L-structures, and let $f : A \to B$ be a function. We say that f *preserves* a formula ϕ if for any $\bar{a} \in A^n$

$$(*) \qquad \underline{A} \vDash \varphi(\bar{a}) \text{ iff } \underline{B} \vDash \varphi(\pi(\bar{a})).$$

Equivalently, writing $f(a_1, \ldots, a_n) := (fa_1, \ldots, fa_n)$, we have:

$$f^{-1}(\phi^{\underline{B}}) = \phi^{\underline{A}}$$

f is *an embedding* iff it preserves all qf formulas.
f is *an isomorphism* if it is a bijective embedding.
f is *elementary* if it preserves all formulas.
Exercise: (1) f is an embedding iff it preserves all atomic formulas;
(2) If f is an isomorphism, it is elementary.

Example

1. Let $\mathcal{Z} = \langle \mathbb{Z}; +, -, 0 \rangle$ be the additive group of integers. Then, given an integer $m > 1$, the embedding

   $$[m] : \mathcal{Z} \to \mathcal{Z},$$

   defined as $[m](z) = m \cdot z$, is not elementary.

2. Let $\underline{\mathbb{Q}} = \langle \mathbb{Q}; +, -, 0 \rangle$ be the additive group of rational numbers. Then, given an integer $m > 1$, the embedding

   $$[m] : \mathcal{Z} \to \mathcal{Z},$$

   defined as $[m](z) = m \cdot z$, is elementary. In fact, it is an isomorphism.

3. The *inclusion* embedding of $(\mathbb{Q}, +, -, 0)$ in $(\mathbb{R}, +, -, 0)$ is also elementary; this is not obvious, but will be proved later on.

Review of Logic: Logical implication and the completeness theorem

Let Γ be a set of sentences, and σ a sentence of a language L. We say $\underline{A} \models \Gamma$ if $\underline{A} \models \phi$ for any $\phi \in \Gamma$.
A sentence σ is called **logically valid**, written $\vDash \sigma$, if $\emptyset \vDash \sigma$, i.e. $\underline{A} \vDash \sigma$ for every L-structure $\underline{A}$.

σ is a *logical consequence* of Γ (written $\Gamma \models \sigma$) if for all L-structures $\underline{A}$, if $\underline{A} \models \sigma$ then $\underline{A} \models \sigma$.
A set S of sentences is called *satisfiable* if it has a model, i.e. a structure $\underline{A}$ such that the truth value of each sentence $\sigma \in S$ is **true**. A set S is *finitely satisfiable* if every finite subset of S is satisfiable.

Theorem (Completeness). *If Γ is a consistent set of sentences of L, then it has a model of size $\leq |L|$.*

Theorem (Completeness along with Soundness). $\Gamma \models \sigma$ *iff* $\Gamma \vdash \sigma$.

The structural consequence that we will use is the Compactness theorem. We state it in two versions.

Theorem. *If* $\Gamma \models \psi$*, then there exists a finite* $\Gamma_0 \subset \Gamma$ *such that* $\Gamma_0 \models \psi$*.*

This follows immediately from the Soundness and Completeness theorem, along with the previously noted fact: if $\Gamma \vdash \psi$, then there exists a finite $\Gamma_0 \subset \Gamma$ such that $\Gamma_0 \vdash \psi$.

Theorem (Compactness Theorem)**.** *Any finitely satisfiable set of* L*-sentences* Σ *is satisfiable. Moreover,* Σ *has a model of cardinality less or equal to* $|L|$*.*

5 The compactness theorem

Here will give a direct proof of the compactness theorem. It really just involves reviewing the proof of the completeness theorem, but using the notion of *finite satisfiability* in place of *consistency*.
Fix a language L. Let Σ be a set of L-sentences.
Σ is said to be **complete** if, for any L-sentence σ, $\sigma \in \Sigma$ or $\neg\sigma \in \Sigma$.

Σ is **witnessing (by constants)** if for any formula $\phi = \phi(x)$ of L, if $(\exists x)\phi \in \Sigma$ then $\phi(c)$ belongs to Σ for some constant symbol c.

Theorem (Compactness). *Let Σ be a set of L-sentences. Assume Σ is finitely satisfiable. Then Σ has a model*

Strategy of proof: We must build a model of Σ. We will gradually enlarge Σ, keeping it finitely satisfiable, and ensuring it is also *complete* and *witnessing*. Once we obtain a complete, witnessing set of sentences, a model can be pointed to explicitly.
N.B.: To obtain witnesses, we will have to expand the language by constant symbols. We will discard them again when the proof is done.

Lemma (1)**.** *Let* Σ *be a finitely satisfiable set of sentences of* L*. Then at least one of* $\Sigma \cup \{\sigma\}$ *and* $\Sigma \cup \{\neg\sigma\}$ *is finitely satisfiable.*

Lemma (2)**.** *Assume* Σ *is a (finitely) satisfiable set of sentences of* L*. Let* σ *be a sentence. Let* c *be a new constant symbol,* $L' = L \cup \{c\}$*. Let* $\phi = \phi(x)$ *be a formula of* L*. Then either* $\Sigma \cup \neg(\exists x)\phi$ *is (finitely) satisfiable, or*

$$\Sigma \cup \phi(c)$$

is (finitely) satisfiable.

Lemma (3)**.** *A complete, witnessing set of sentences has a model.*

The proof constructs a canonical model where *every element is the interpretation of some closed term.* Such models are minimal as L-structures; they have no proper substructures.

Proof of the compactness theorem, for countable L:
Preliminaries:

1. Expand L to $L' = L \cup \{c_1, c_2, \cdots\}$. So $|L'| = \aleph_0$.
2. Enumerate all sentences of L' as $\sigma_1, \sigma_2, \cdots$.
3. Fix a variable x; enumerate all formulas $\phi = \phi(x)$ of L as $\phi_1, \phi_2, \cdots$.

Construction: Let $T_0 = \Sigma$.
We will recursively define sentences P_n of L'. and let $T_n = \Sigma \cup \{P_1, \ldots, P_n\}$.
We will make sure that T_n remains finitely satisfiable.
Claim. Assuming each T_n is finitely satisfiable, $T' := \cup_n T_n = \Sigma \cup \{P_1, P_2, \ldots\}$ is finitely satisfiable.

Definition of T_{n+1} for $n \geq 1$

At stage $2n+1$: T_{2n} has been defined, and we know inductively that it is finitely satisfiable. Using the first lemma, either $T_{2n} \cup \sigma_n$ or $T_{2n} \cup \neg\sigma_n$ is finitely satisfiable. Let $P_{2n+1} = \sigma_n$ in the first case, or if both hold; otherwise let $P_{2n+1} = \neg\sigma_n$. Note that $T_{2n+1} := T_n \cup \{P_{2n+1}\}$ is finitely satisfiable in any case; and T_{2n+1} decides σ_n.

At stage $2n+2$: If $T_{2n+1} \cup \{\neg(\exists x)\phi_n)\}$ is finitely satisfiable, let $P_{2n+2} = \neg(\exists x)\phi_n)$, so $T_{2n+2} = T_{2n+1} \cup \{\neg(\exists x)\phi_n)\}$. Otherwise, let k be least such that c_k does not occur in T_n. Let $P_{2n+2} = \phi_n(c_k)$. By Lemma 2, $T_{2n+2} := T_{2n+1} \cup \{\phi_n(c_k)\}$ remains finitely satisfiable.

Claim. T' is complete.
Claim. T' is witnessing.
By Lemma 3, T' has a countable model M'. Let $M = M'|L$. Then $M \models \sigma$ for any sentence σ of L such that $\sigma \in T'$. In particular, for any $\sigma \in T_0 = \Sigma$. So $M \models T$.
M' is a minimal L'-structure, hence countable, and so M is countable.
□

(N.B. M may not be a minimal L-structure!).

Example: $T = Th((\mathbb{Z}, +, -, 0))$. Show some model of T has an element divisible by all odd primes, but not by 2.

A set of L-sentences Σ is said to be **deductively closed** if

$$\Sigma \models \sigma \text{ implies } \sigma \in \Sigma.$$

A *theory* is a finitely satisfiable, deductively closed set of sentences of L. Though we allow the empty *structure*, we will not be interested in its theory. *We will only consider theories T such that $T \models (\exists x)(x = x)$ (i.e. the empty structure is not a model of T.)*
Remark: In practice, we often give only a subset of T. For example the axioms of the theory of groups consist of four universal sentences, namely the associate law and the axioms on the unit and inverses. The *theory of groups* is the (infinite) set of logical consequences of these; for instance

$$(\forall x, y, z, w)((xy)(zw) = x(y(zw)))$$

Since these two sets - the axioms, and the consequences of the axioms - have the same class of models, the distinction will not be important for us.

Definition Let T be a theory, $x = (x_1, \dots, x_n)$ a tuple of variables. A *partial type* $P(x)$ of a theory T in variables x is a finitely satisfiable set P of formulas in the variables x, containing T and closed under logical deduction.
Here *finitely satisfiable* means: for any $\phi_1, \dots, \phi_k \in P$, there exists a model $\underline{A} \models T$ and $c \in A^n$ such that $\underline{A} \models \phi_i(a)$ for each $i \leq k$. (Equivalently, there exists a model of $T \cup (\exists x)(\bigwedge_{i=1}^{k} \phi_i(x))$.)
An n-tuple c from a model $\underline{A}$ of T is said to *realise* P if $\underline{A} \models \phi(a)$ for each $\phi \in P$.
$\underline{A}$ is said to *realise* P if some n-tuple from A does.
$\underline{A}$ is said to *omit* P otherwise.
We saw that any partial type is *realised* in some model.
When does there exist a model *omitting* P?

Example: Let $L = \{\cdot, ^{-1}, 1\}$ be the language of abelian groups.
Let $P(x)$ be the partial type: $x \neq 1, x^2 \neq 1, x^3 \neq 1, \cdots$.
Let $\underline{A}$ be the Abelian $\mathbb{C}^*$ (nonzero complex numbers with the usual multiplication.)
Does $Th(\underline{A})$ have a model omitting P?
I.e. is there a model of $Th(\underline{A})$ where every element has finite order?
(We will later have tools to give a positive answer; indeed to show that the subgroup of $\mathbb{C}^*$ whose universe consists of roots of unity, is an elementary substructure. For now we are interested in the *question*; it is an omitting types question.

Definition A set of formulas $P(x)$ is *principal* if there exists a formula θ such that $T \cup \exists x \theta(x)$ is satisfiable, and for any $\phi \in P \quad T \models \forall x(\theta(x) \rightarrow \phi(x))$.
If T is a complete theory, a principal partial type is realised in *every* model.

Example. *For $Th(\mathbb{Z}, +, -, 0)$, the partial type: $2|x, 3|x, 4|x, \cdots$ is principal. The formula $x = 0$ implies all of these!*

We now show that the property of being nonprincipal cannot be destroyed by adding finitely many sentences consistent with T, or by adding new constants.

Lemma. *L be a language, T a theory in L, $P = P(x)$ a set of formulas L in the variables x. Assume P is nonprincipal for T.*

1. *Let $L' = L \cup \{c\}$, where c is a new constant symbol. Let T' be the set of logical consequences of T in L'. Then P' is nonprincipal for T'*

2. *Let L' be obtained from L by adding some new constant symbols, and let c be any constant of L'. Assume $T \cup \{\sigma\}$ is satisfiable. Then for some $\phi \in P$, $T \cup \{\sigma\} \cup \{\neg\phi(c)\}$ is satisfiable.*

Proof. (1) Left as an exercise. Hint: any L'-formula $\theta'(x)$ can be written as $\theta(c, x)$, where $\theta(y, x)$ is a formula of L. Show that if $T \cup \theta'(x) \models P$ then $T \cup (\exists y)\theta(y, x) \models P$.
(2) We may assume L' is L augmented with the finite number of constant symbols mentioned in σ, along with c. By applying (1) finitely many times, we see that P remains nonprincipal for T in L'.
Let θ be the formula $\sigma \wedge (x = c)$. Certainly $T \cup \{\exists x\theta\}$ is satisfiable. Since P is not principal, there exists $\phi \in P$ such that $T \cup \{\theta(x)\}$ does not imply $\phi(x)$. So $T \cup \{\theta(x)\} \cup \{\neg\phi(x)\}$ is satisfiable. Equivalently, $T \cup \{\sigma, \neg\phi(c)\}$ is satisfiable.

□

Theorem 5.1 (Omitting a partial type)**.** *Assume L is a countable language, T a theory, P a partial type for T. If $P(x)$ is non-principal, there exists a countable model M omitting P.*

Proof. Preliminaries:

1. Expand L to $L' = L \cup \{c_1, c_2, \cdots\}$; these are distinct constant symbols, not in L. So $|L'| = \aleph_0$.

2. Enumerate all sentences of L' as $\sigma_1, \sigma_2, \cdots$.

3. Fix a variable x; enumerate all formulas $\phi = \phi(x)$ of L as $\phi_1, \phi_2, \cdots$.

We will recursively define sentences P_n of L'. and let $T_n = \Sigma \cup \{P_1, \ldots, P_n\}$. We will make sure that T_n remains finitely satisfiable.
Claim. Assuming each T_n is finitely satisfiable, $T' := \cup_n T_n = \Sigma \cup \{P_1, P_2, \ldots\}$ is finitely satisfiable.
Construction: Let $T_0 = \Sigma$. At stage n we will define T_n.
At stages $3n+1$ we assure $\sigma_n \in T_n$ or $\neg\sigma_n \in T_n$.
At stages $3n+2$ we assure $\neg(\exists x)\phi_n(x) \in T_n$ or some $\phi_n(c_k) \in T_n$.
So far, all is as in the proof of completeness/compactness.
At stage $3n$ (with $n \geq 1$):
Note that T_{3n-1} was obtained by adding constants to L, and then adding finitely many sentences. By the Lemma, T_{3n-1} is consistent with $\neg\phi(c_n)$, for some $\phi \in P$. Let $P_n = \neg\phi(c_n)$ and let $T_{3n} = T_{3n-1} \cup \{P_n\}$.

□

Now T' is complete and witnessing. By Lemma 3, T' has a countable model M'. Let $M = M'|L$. Then $M \models \sigma$ for any sentence σ of L such that $\sigma \in T'$. In particular, for any $\sigma \in T_0 = \Sigma$. So $M \models T$. By construction, each element a of M has the form $a = c_n^{M'}$ for some $n \geq 1$; and (for some $\phi \in P$),

$$\neg\phi(c_n) \in T_{3n}.$$

So $M \models \neg\phi(a)$. Hence no a from M can realise P. □