Part A.

Part A is not to be handed in. When asked to prove a statement in the notes, try to do so without looking at the proof in the notes; then compare.

1. Prove Proposition 2.5 using Zorn's lemma without assuming countability.

2. Read through section 3 of the notes. Prove 3.4 (upward Löwenheim-Skolem.) This will be similar to the exercise showing existence of a nonstandard model in HW1; but you will need many constant symbols rather than a single one.

3. Prove 3.6 (downward Löwenheim-Skolem.) You may organize your proof as follows:

Definition A theory T *admits Skolem functions* if for any n-tuple of variables $x = (x_1, \ldots, x_n)$ and any formula $\phi(x, y)$, for some term t,

$$T \models (\forall x)((\exists y)\phi(x, y) \to \phi(x, t(x)))$$

The term t is called a Skolem term for ϕ.

a) Assume T admits Skolem functions. Let $\underline{A} \leq \underline{B}$. If $\underline{B} \models T$, show that $\underline{A} \preccurlyeq \underline{B}$.

b) Let $\underline{M}$ be an L-structure. Show that there exists an expansion $\underline{M}_{sk}$ of M to a language L_{sk}, $|L_{sk}| = |L|$, such that $\underline{M}_{sk}$ admits Skolem functions. (You will need the axiom of choice, or to assume that the universe M can be well-ordered.)

Deduce that $\underline{M}$ in has an elementary submodel of cardinality $\leq |L|$.

4. Let T be a theory in a countable language, and $P_1, P_2, \cdots$ be nonprincipal partial type. Prove that T has a countable model omitting each P_i (2.20).

Part B.

1. Prove part 1 of the Lemma for the omitting types theorem, on p. 31 of the lecture notes for weeks 1-2. I.e. show that adding constants cannot by itself turn a nonprincipal partial type into a principal one.

2. Let T be a theory in a countable language, and P a nonprincipal partial type. Use P to show that T has two countable models M, N that are not isomorphic.

3. Exercise 4.10 of the class notes.

4. Let $\underline{A}$ be a finite L-structure. Assume L has finitely many non-logical symbols.

   1. Find $\sigma_1 \in Th(\underline{A})$ such that any model of σ_1 has universe of the same cardinality as $\underline{A}$.

   2. Let $\underline{B} \models Th_{\exists}(\underline{A})$. If also $\underline{B} \models \sigma_1$, show that $\underline{B} \cong \underline{A}$. In particular, any model of $Th(\underline{A})$ is isomorphic to $\underline{A}$.

   3. Show that any model of $Th_{\forall}(\underline{A}) + \sigma_1$ is isomorphic to $\underline{A}$.

   4. Find a single existential sentence σ_2 such that any model of $\{\sigma_1, \sigma_2\}$ is isomorphic to $\underline{A}$.

Part C.
1) Prove problem B4 above, clauses 1-3, without assuming the language is finite.
2) Let L be a countable language, T a theory. Let $c_1, \ldots$ be new constant symbols, $L' = L \bigcup \{c_1, \ldots\}$, and let X be the set of all complete theories t of L' with $T \subseteq t$. Define a metric on X by: $d(t, t') = 0$ if $t = t'$; otherwise, $d(t, t') = 2^{-n}$ where n is least such that $\sigma_n \in t \setminus t'$ or $\sigma_n \in t' \setminus t$.

1. Show that X is a complete metric space.
2. Let $\phi(x)$ be a formula in one variable. Show that the set of theories $t \in X$ such that $\neg(\exists x)\phi \in t$ or ϕ is witnessed in t is dense and open in X. Conclude that the set of self-witnessing theories contains a countable intersection of dense open sets.
3. Look up the Baire category theorem. Use it to show that there exists a complete, self-witnessing t containing T.
4. If $P(x)$ is a nonprincipal partial type, show similarly that there exists a complete, self-witnessing t containing T, whose canonical model omits P. (In fact 'almost all' t have this property, in the sense of Baire category.)
5. Conclude that a countable set of nonprinipal partial types can be simultaneously omitted.