ST 2

#8

Chapter 7 — Compactifications to 4 dim

10 dim superstring on $\mathbb{R}^{1,3} \times M_6$

$\hookleftarrow$ compact 6 dim

↳ KK reduction to obtain an effective QFT in 4 dim

Aim: physics of string theory $\longleftrightarrow$ geometry of M_6

(for example: enhancement of gauge symmetries for specific choice of Λ (ie params of T^N) ⇝ Bosonic [T^N])

Require: preservation of some SUSY

- controlled calculations

  (non renormalisation statements)

M_6 has to be a <u>spin</u> manifold

KK ansatz : $(G_{MN}) = \left(\begin{array}{c|c} \eta_{\mu\nu} & * \\ \hline * & G_{mn} \end{array}\right)$ G_{mn} metric on M_6

10 dim Lorentz group $SO(1,9) \longrightarrow SO(1,3) \times SO(6)$

$\simeq SU(2) \times SU(2)$
4 dim Lorentz group

rotation group in 6 dim
$SO(6) \simeq SU(4) \to Spin(6)$

spinors decompose $\epsilon \longrightarrow \sum \epsilon^{4d} \otimes \epsilon^{6d}$ (tensors also decompose)

$$\underline{16}_{MW} \rightsquigarrow \left((\underline{2},\underline{1})_W ; \underline{\bar{4}} \right) \oplus \left((\underline{1},\underline{2})_W , \underline{4} \right)$$

ϵ : 10 dim MW spinor
susy transf parameter

spinors of $Spin(6) \sim SU(4)$

T^6 : $4 \times \quad (2,1)_W \oplus (1,2)_W$ $\quad N=4$ SUSY

preserves all of them (T^6 is <u>flat</u>)

Supersymmetric backgrounds

ϵ supersymmetry transf parameter in 10 dim

Q corresponding supercharges

Supersymmetric vacuum: $\underline{\epsilon \cdot Q \,|\, vacuum \rangle = 0}$

So if there are background fields $\Phi = \{ G_{MN}, B_{MN}, C_p\text{-forms}, \phi, \ldots \}$

then $\langle \delta_\epsilon \Phi \rangle_{vacuum} = 0$

▸ $\delta_\epsilon \Phi_{bosonic} \sim \boxed{\Phi_{fermionic} \equiv 0}$ ⇝ no fermion condensates in the background

so $\langle \Phi_{fermions} \rangle_{vacuum} = 0$

▸ Then: $\boxed{\delta_\epsilon \Phi_{fermionic} \equiv 0}$

$\Phi_{fermionic} = \{ \text{Gravitino, dilatino, gaugino (Het)} \}$

Supersymmetric variation of the gravitino ψ_M

$$\delta_\epsilon \psi_M = \nabla_M \epsilon + \cancel{H}_M \epsilon + e^{\Phi} \sum_p \cancel{F}_p \Gamma_M \epsilon = 0$$

$= [\text{bosonic fields}] \cdot \epsilon$

$$\nabla_M = \partial_M + \frac{1}{2} \omega_M{}^{AB} \Gamma_{AB}$$

Γ_A 10 dim Γ-matrices, $\Gamma_{AB} = \Gamma_{[A}\Gamma_{B]}$

$$\cancel{H}_M \equiv H_{MNP} \Gamma^{NP}, \quad \cancel{F}_p \equiv F_{p\, M_1 \dots} \Gamma^{M_1 \dots}$$

spin connection

$$\omega_M{}^{ab} = \frac{1}{2} (\Omega_{MNR} - \Omega_{NRM} + \Omega_{RMN}) e^{Na} \Omega^{Rb}$$

$e \to$ 10-bein

$$\Omega_{MNR} = (\partial_M e_N^a - \partial_N e_M^a) e_{aR}$$

Need to solve $\boxed{\delta_\epsilon \psi_M = 0}$ for susy vacuum configurations of the fields: e_M^a (metric), H, F, ϕ, ϵ

Many interesting solutions!

(Het: also $\delta_\epsilon(\text{gaugino}) = 0$)

- Set $\Phi = 0$, $H=0$, $F=0$, $\omega = 0$ (simplest)

  $\uparrow$ flat space

  $\partial_M \epsilon = 0$ constant spinors

  M_6 is flat ie $M_6 = T^6$

  preserves all susy

- $H = F = \phi = 0$ $\boxed{\omega \neq 0}$ $\Longrightarrow$ $\boxed{\nabla_M \epsilon = 0}$

  (no background fluxes) non zero curvature

  covariantly constant spinors (Killing spinor eq)

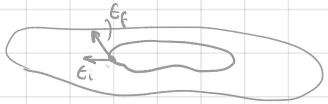
$[\nabla_M, \nabla_N]\epsilon = \frac{i}{2} R_{MN}{}^{AB}\Gamma_{AB}\epsilon$

$\nabla_m \epsilon = 0 \Longrightarrow R_{mn}{}^{ab}\Gamma_{ab}\epsilon^{6d} = 0$ $\Longleftarrow$ condition on the curvature

(We still have a spinor of $SO(1,3)$)

- Γ_{ab} generate a subgroup of Cliff(6) leaving one component of ϵ invariant. $SO(6) \simeq SU(4) \supset SU(3) = \text{Hol}$

  Holonomy

[ϵ_i, ϵ_f after parallel transport are related by an $SO(6)$ transf. ]

- $\nabla_m \epsilon^{6d} = 0 \Rightarrow R_{mn} = 0$ M_6 is a Ricci-flat manifold.

Spinors

$SO(6) \longrightarrow SU(3)$

$\underline{4} \rightarrow \underline{1} \oplus \underline{3}$

component of 6^{6d} which is inv. under parallel transport

$\Rightarrow \quad N=1$ SUSY in 4 dims

$\epsilon \rightarrow \underline{16}_{MW} \longrightarrow ([2,1],\bar{1}) \oplus ((1,2), 1)$ cov. const

Type I, Het on $M_6 \longrightarrow$ 4 dim $N=1$

IIA, IIB $\longrightarrow$ 4 dim $N=2$ $\leftarrow$ from two ten dimensional spinors

M_6 Ricci-flat $\rightsquigarrow$ Calabi-Yau

Full classification : compactify to other dimensions

M_{2N} $\quad Hol = SU(N) \subset SO(2N)$ $\quad$ CY $\quad$ T^2 ..., K3 ..., 3-fold ..., ⋮, n-fold

Other dimensions

M_7 $\quad Hol = G_2 \subset SO(7)$
M_8 $\quad Hol = Spin(7) \subset SO(8)$
$\Big\}$ manifolds with exceptional holonomy

Can have $\quad$ CY × torus.

Non-zero fluxes :

IIB $\quad AdS_5 \times S^5$ $(\lambda = -1)$ $\quad F_5 \neq 0$ along S_5 & AdS_5

Het $\quad AdS_3 \times G_2$ $\quad H_3 \neq 0$

⋮

Next $\quad$ IIA / IIB on a CY & mirror symmetry

CALABI-YAU MANIFOLDS

Mathematical objects of interest: 6 dim

$\underbrace{\text{algebraic varieties}}_{\text{set of solutions of}}$ with certain special properties

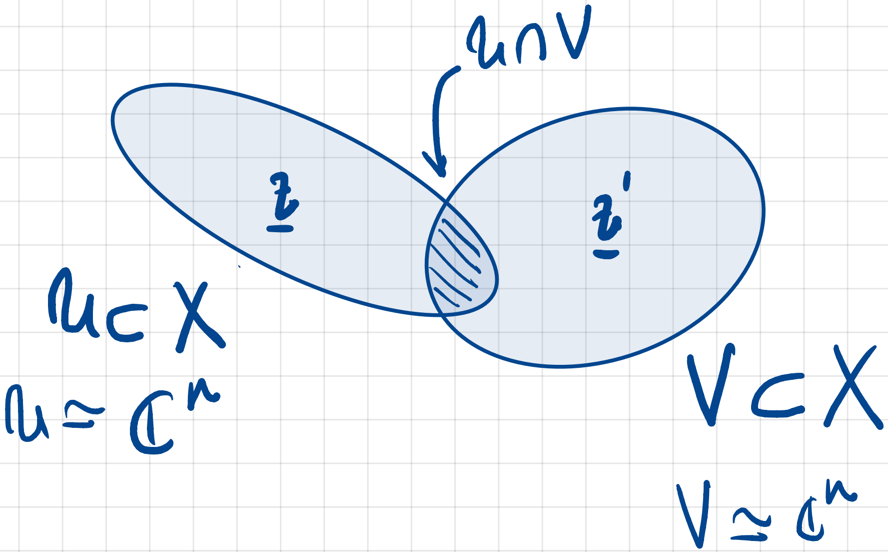
Calabi-Yau manifolds

$$\boxed{P(\varphi, \underline{x}) = 0, \quad \underline{x} \in \mathbb{A}}$$

polynomials with complex coefficients φ

Definition: a Calabi-Yau manifold is a complex manifold which is Kähler and admits a Ricci-flat metric (that is, $c_1 = 0$)

==complex manifold==: an n-dim complex manifold X is a $2n$-dimensional real differentiable manifold on which one can choose $\mathbb{C}$-local coordinates $\{z^1, \dots, z^n\}$ st the transition functions between two coordinate patches are holomorphic maps $\mathbb{C}^n \longrightarrow \mathbb{C}^n$



On $\mathcal{U} \cap V$

$$\boxed{\underline{z}' = \underline{f}(\underline{z})}$$

holomorphic functions

The complex coordinate system with holomorphic transition functions is called a ==complex structure on X==

An important example for us today is

$\mathbb{P}^n$ n-dim complex projective spaces

$$(z^1, \dots, z^{n+1}) \in \mathbb{C}^{n+1} \setminus \{\underline{0}\}$$

subject to the identifications

$$(z^1, \dots, z^{n+1}) \simeq \lambda (z^1, \dots, z^{n+1}) \qquad \forall \ \lambda \in \mathbb{C}, \ \lambda \neq 0$$

<u>Exercise</u>: $\mathbb{P}^1 = S^2$

Forms decompose in (p,q)-type $\Omega^{(p,q)}$

$$\underset{r\text{-form}}{C} = \sum_{\substack{p,q \\ p+q=r}} \frac{1}{p!q!} C_{i_1 \cdots i_p \bar{j}_1 \cdots \bar{j}_q} dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \cdots \wedge d\bar{z}^{j_q}$$

$z = x + iy$

For example: ω a two form $\to$ $\underset{\omega_{ij}}{(2,0)} + \underset{\omega_{i\bar{j}}}{(1,1)} + \underset{\omega_{\bar{i}\bar{j}}}{(0,2)}$

exterior derivation $d = \partial + \bar{\partial}$

$\partial : \Omega^{p,q} \longrightarrow \Omega^{p+1,q}$ $\qquad$ $\bar{\partial} : \Omega^{p,q} \longrightarrow \Omega^{p,q+1}$

$\partial = \sum dz^i \dfrac{\partial}{\partial z^i}$ $\qquad$ $\bar{\partial} = \sum d\bar{z}^j \dfrac{\partial}{\partial \bar{z}^j}$

Cohomology

$$H_d^{(r)} = \frac{\mathrm{Ker}(d)}{\mathrm{Im}(d)}$$

$\mathrm{Ker}(d)$ ← d-closed forms

$\mathrm{Im}(d)$ ← d-exact form

$[\alpha] \in H^r$

$(d\alpha = 0 \quad \alpha \sim \alpha + d\beta)$

$\dim H^{(r)} = b_r$ Betti numbers ← topological invariants

For complex manifolds

$$H_\partial^{(p,q)} = \frac{\mathrm{Ker}(\partial)}{\mathrm{Im}(\partial)} \qquad H_{\bar\partial}^{p,q} = \frac{\mathrm{Ker}(\bar\partial)}{\mathrm{Im}(\bar\partial)}$$

$h^{p,q} = \dim H_{\bar\partial}^{(p,q)}$ Hodge numbers of M

arrange in a Hodge diamond

$$\begin{array}{ccccccc}
 & & & h^{00} & & & \\
 & & h^{10} & & h^{01} & & \\
 & h^{20} & & h^{11} & & h^{02} & \\
h^{30} & & h^{21} & & h^{12} & & h^{03} \\
 & h^{31} & & h^{22} & & h^{13} & \\
 & & h^{32} & & h^{23} & & \\
 & & & h^{33} & & &
\end{array}$$

symmetry *

$h^{00} = 1 = h^{33}$

similar for $\bar{h}^{p,q}$

<u>Definition</u>: a Calabi-Yau manifold is a complex manifold which is <u>Kähler</u> and admits a Ricci-flat metric ($c_1 = 0$)

A <u>Kähler</u> manifold is a complex Riemannian manifold with a (hermitian) metric g which can be written as

($g_{ij} = 0$, $g_{\bar{i}\bar{j}} = 0$ hermitian metric)

$$g_{i\bar{j}} = \frac{\partial}{\partial z^i} \frac{\partial}{\partial \bar{z}^j} K(z, \bar{z})$$

real valued "function" (Kähler potential)

That is, the (1,1)-form

$$\omega \equiv i\, g_{i\bar{j}}\, dz^i \wedge d\bar{z}^j \qquad \text{(hermitian form)}$$

is closed:

$$\boxed{d\omega = 0}$$

In fact, ω determines a <u>cohomology class</u>

$$[\omega] \in H^2(X) \qquad (\omega \sim \omega + d\alpha)$$

called the <u>Kähler class</u>.

The equation $\quad d\omega = 0$

is a linear differential equation which can have many solutions $\{e_1, \dots, e_N\}$ $\quad (N = b_2)$

so the most general solution is

$$\boxed{\omega = \sum_{i=1}^{N} t^i e_i \qquad \{t^1, \dots, t^N\} \text{ Kähler class parameters}}$$

($\mathbb{P}^n$ is Kähler.)

Theorem: If M is Kähler then $h^{pq} = \overline{h}^{pq}$

using complex conjugation $\overline{h}^{pq} = h^{qp} \Longrightarrow \underline{h^{pq} = h^{qp}}$

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & h^{10} & & h^{01} & & \\
 & h^{20} & & h^{11} & & h^{02} & \\
h^{30} & & h^{21} & & h^{12} & & h^{03} \\
 & h^{31} & & h^{22} & & h^{13} & \\
 & & h^{32} & & h^{23} & & \\
 & & & 1 & & &
\end{array}
$$

symmetry *

symmetry

$h^{00} = b_0 = 1$ constant function

$h^{33} = 1$ volume form

Definition: a Calabi-Yau manifold is a complex manifold which is Kähler and admits a Ricci-flat metric

(that is, $c_1 = 0$)

$c_1 \equiv \left[\frac{1}{2\pi} \mathcal{R} \right] \in H^2(X)$ (first Chern-class)

analytic invariant

one can easily prove that $R_{mn} = 0 \Rightarrow c_1 = 0$

1957, E Calabi: conjectured that c_1 is the only topological obstruction for there to exist a Ricci flat metric (ie $c_1 = 0 \Rightarrow \exists\, g_{mn}$ with $R_{mn} = 0$)

1977, S-T Yau proved this conjecture (& won the Fields medal)

$\boxed{\text{CY condition } (c_1=0) : \quad h^{30}=h^{03}=1}$ $\quad (a=0)$

$\exists^1$ well defined nowhere vanishing holomorphic $(3,0)$-form

$\bar\partial\Omega=0$

$h^{1,0}=h^{0,1}=0 \quad (\Rightarrow h^{20}=h^{02}=0)$ if X_6 has $\mathrm{Hol}=SU(3)$

h^{11} = # Kähler class deformations

[recall $[\omega]\in H^2$ in fact $H^{1,1}$

$\Rightarrow \omega=\sum_{i=1}^{h_{11}} t^i e_i$ Kähler class params]

$b_2=h^{11}$

$b_3=2(1+h^{12})$

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{11} & & 0 & \\
1 & & h^{21} & & h^{12} & & 1 \\
 & 0 & & h^{11} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

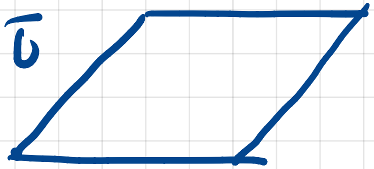
h^{12} = # of $\mathbb{C}$-structure deformations

$\chi=\sum(-1)^r b_r = 2(h^{11}-h^{21})$

Lower dims

- T^2 1 dim CY T^2/Λ $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$

τ



$Hol = 1$

$$\begin{array}{ccc} & 1 & \\ 1 & & 1 \\ & 1 & \end{array}$$

$\chi = 0$

Kähler structure: changing the volume but keeping the shape τ fixed

complex structure τ "shape parameter"

- K3 2 dim CY $Hol = SU(2)$

Kümmer + Kähler + Kodaira

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

$\chi = 24$

//

Mathematical objects of interest: CY 3fold

$\underbrace{\text{algebraic varieties}}_{\text{set of solutions of}}$ with certain special properties

$$\boxed{P(\varphi, \underline{x}) = 0, \quad \underline{x} \in \mathbb{A}}$$

Calabi-Yau manifolds

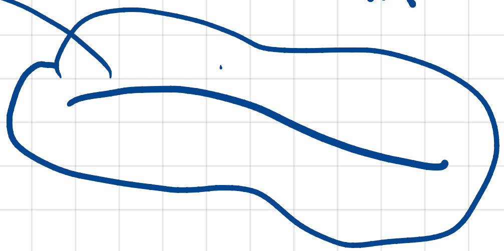
polynomials with complex coefficients φ

CY condition encoded in the properties of P

Examples : (the simplest!)

$P(x)$
$\deg P = n+1$

$A = \mathbb{P}^n$



$\mathbb{P}^n[n+1]$ has $c_1 = 0$ and inherits its Kähler class from $\mathbb{P}^n$

↳ degree $n+1$ polynomial $\Leftrightarrow c_1 = 0$

3-fold $\quad \mathbb{P}^4[5] \quad$ quintic 3-fold

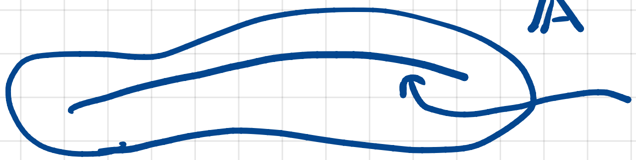
eg $\quad P(x,\psi) = \sum_{i=1}^{5} x_i^5 - 5\psi\, x_1 x_2 x_3 x_4 x_5 + \cdots$

$(x^1, \dots, x^5) \in \mathbb{P}^4 = \{ (x_1, \dots, x_5] \in \mathbb{C}^5, (x_1, \dots, x^5] \simeq (\lambda x^1, \dots, \lambda x^5), \lambda \in \mathbb{C}^* \}$

More sofisticated examples :

toric varieties
products of $\mathbb{P}^n$, etc

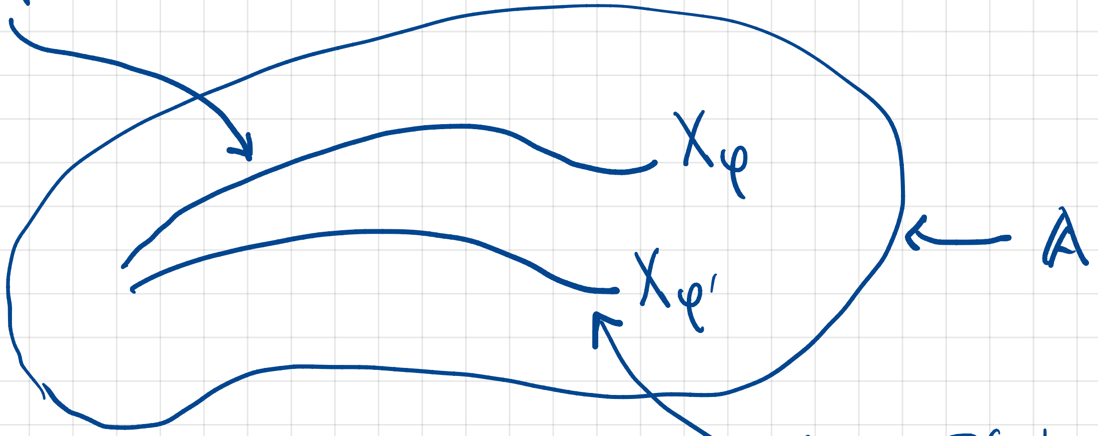
A



$P(\underline{x}) = 0$

- complex structure parameters $\rightarrow$ coefficients of P

$X_\varphi : \; P(\varphi, \underline{x}) = 0$



$X_{\varphi'} : \; P(\varphi', \underline{x}) = 0$

Quintic : $A = \mathbb{P}^4$

$h^{12} = 101 \qquad h^{11} = 1$

# of polynomial deformations (modulo identifications in $\mathbb{P}^4$)

Kähler class induced from ambient $\mathbb{P}^4$

The number of Kähler class parameters and $\mathbb{C}$-structure parameters are given by $\boxed{\text{topological invariants}}$

- Kähler class : $\omega = \sum_{i=1}^{N} t^i e_i$, $N = h^{(1,1)} = b_2$

  $d\omega = 0$, $\{e_1, \dots, e_N\}$ basis of $H^{1,1}$

- $\mathbb{C}$-structure :

  # of complex structure parameters $= h^{(2,1)}$

  $= \frac{1}{2} b_3 - 1$

  for a CY

Mirror symmetry

Mirror symmetry was uncovered by String Theory. We say that a pair of CY manifolds (X, Y) is a mirror pair if

$$\boxed{\Gamma(\mathrm{IIA}[X]) = \Gamma(\mathrm{IIB}[Y])}$$

where $\Gamma(\cdot\cdot)$ = effective QFT of a Type II string compactified on a CY

The mirror symmetry conjecture means that the physical theory of IIA strings propagating in the spacetime $\mathbb{M}_4 \times X$ is indistinguishable from that of strings propagating in $\mathbb{M}_4 \times Y$.

Indistinguishable means that the spectrum of particles as well as the quantum physical quantities (correlation functions) are the same in both theories.

MS was conjectured in the late 80's by
L. Dixon (87) & W. Lerche, C. Vafa, N. Warner (89)
based on the fact that in the CFT associated to CY compactifications, it is a matter of convention which parameters are associated to the $\mathbb{C}$-structure or the Kähler class

The most basic consequence of mirror symmetry is the equivalence of the massless spectrum.
This implies that for a mirror pair (X, Y)

massless scalar fields $\leftrightarrow$ cohomology classes

$$\boxed{\begin{aligned} h^{(1,1)}(X) &= h^{(2,1)}(Y) \\ h^{(2,1)}(X) &= h^{(1,1)}(Y) \end{aligned}}$$

topological numbers!

(and therefore $\chi(X) = -\chi(Y) \rightsquigarrow |\chi(X)| = |\chi(Y)|$)

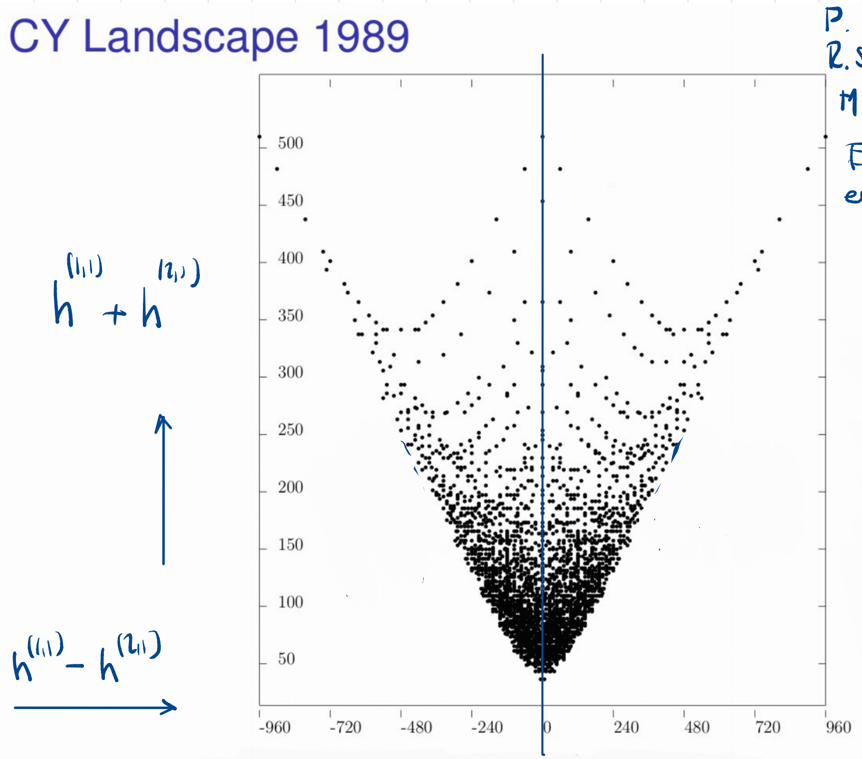
This simple fact is the first glimpse at how non-trivial MS is: X & Y are topologically different!

From the point of view of classical geometry MS seems very mysterious.

The mathematical community was very skeptical originally and there weren't many examples of CY manifolds in the 80's.

In the late 1980's evidence for mirror symmetry started to accumulate.

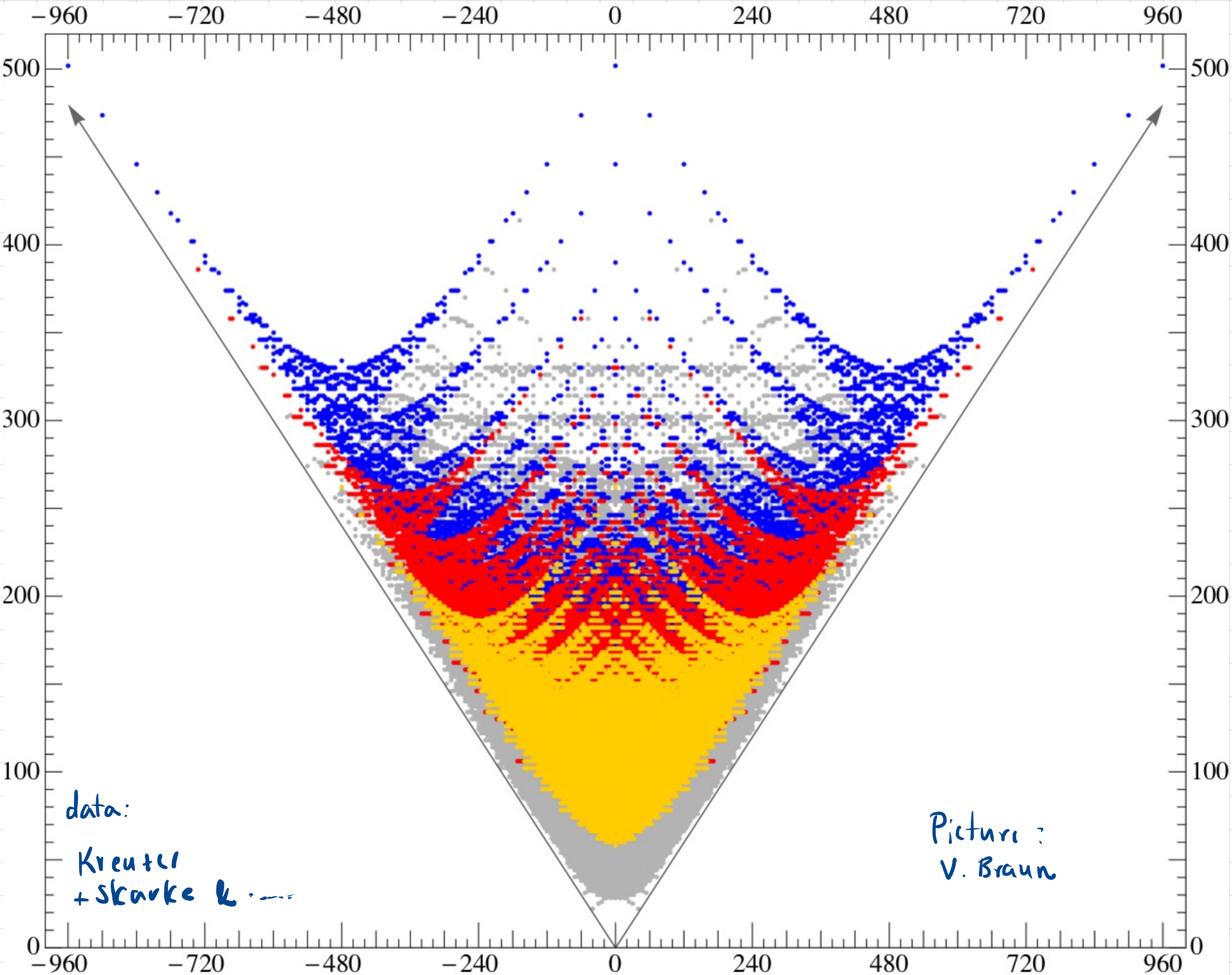
- Experimental evidence was obtained by constructing thousands of examples as hypersurfaces in projective spaces & products of projective spaces.



P. Candelas,
R. Schimmrigk
M. Lynker
Experimental evidence

- B. Greene & R. Plesser: constructive evidence where the mirror of a CY was obtained explicitly for some examples (eg the mirror of the quintic 3-fold)

landscape today



data:

Kreuzer
+ Skarke k ...

Picture:
V. Braun

Parameter spaces and mirror symmetry

Evidence of a deeper structure in relation to MS came from the study of the geometry of the moduli space of IIA and IIB string theories compactified on CY manifolds.

[XD & P. Candelas , A Skroninger 90]

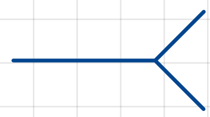
The understanding of this geometry is crucial to calculate the necessary physical quantities to obtain the 4 dim effective theory

For example,

dimension of the moduli space $\longleftrightarrow$ number of massless particles in the effective theory

metric on the moduli space $\longleftrightarrow$ kinetic terms in the 4 dim effective action

certain natural cubic forms $\longleftrightarrow$ Yukawa couplings



(3-point correlation function)

⋮

As we have said, CY manifolds have two types of parameters. One can show that

$$\mathcal{M} = \mathcal{M}_{cs} \times \mathcal{M}_{KC}$$

$\mathbb{C}\cdot\dim \mathcal{M}_{cs} = h^{(2,1)}$

parameters of the complexified Kähler class

$B + i\omega$

$\mathbb{C}\cdot\dim \mathcal{M}_{KC} = h^{(1,1)}$

The complexification of the Kähler class comes from string theory. (the massless spectrum includes a closed 2-form B)

This is the first step away from the classical geometry of CY manifolds.

Both $\mathcal{M}_{cs}$ & $\mathcal{M}_{KC}$ are Kähler with a holomorphic prepotential ("special geometry")

The mirror symmetry conjecture implies that for a mirror pair (X, Y)

$$\boxed{\mathcal{M}_{cs}(Y) = \mathcal{M}_{KC}(X)}$$

$$\mathbb{C}\dim \mathcal{M}_{cs}(X) = h^{(2,1)}(Y) = \mathbb{C}\dim \mathcal{M}_{KC}(X) = h^{11}(X)$$

isomorphism map = <u>mirror map</u>

$$\psi \longmapsto t(\psi)$$

cs of Y   KC of X

complex structure parameters

[IIB[Y]]

Let Y be a CY manifold.

$\mathcal{M}_{cs}(Y)$ is Kähler with "holomorphic prepotential" $\mathcal{G}$

Let $\{z^0, \ldots, z^{h^{(2,1)}}\}$ be complex projective coordinates on $\mathcal{M}_{cs}(Y)$. The Kähler metric on $\mathcal{M}_{cs}(Y)$

$$G_{a\bar{b}} = \frac{\partial}{\partial z^a} \frac{\partial}{\partial \bar{z}^{\bar{b}}} \mathcal{K} \qquad a, b = 1, \ldots, h^{(2,1)}$$

where the Kähler potential $\mathcal{K}$ is given by

$$e^{-\mathcal{K}} = i\left(z^a \frac{\partial \bar{\mathcal{G}}}{\partial \bar{z}^a} - \bar{z}^a \frac{\partial \mathcal{G}}{\partial z^a} \right)$$

$\mathcal{G}$ also determines the Yukawa couplings of the 4 dim effective theory of IIB strings on Y

Mathematically: these are natural cubic forms on Y

$$y : H^{1,2} \times H^{1,2} \times H^{1,2} \longrightarrow \mathbb{C}$$

One finds that

$$\boxed{y_{abc} = \partial_a \partial_b \partial_c \mathcal{G}}$$

complicated functions of the c-structure parameters. But calculable

In general correlation functions receive quantum corrections.

However in IIB[Y] these couplings are exact (Distler & Greene 88).

In fact: the classical geometry of $\mathcal{M}_{cs}(Y)$ is exact.

example of a "non renormalisation" theorem for supersymmetric theories

Kähler class parameters

IIA [X]

Let X be a CY manifold. The surprising fact is that similar considerations apply to $\mathcal{M}_{KC}(X)$

$\mathcal{M}_{KC}(X)$ is also Kähler with a holomorphic prepotential

But there are differences.

The main one is that the classical geometry of $\mathcal{M}_{KC}(X)$ is not enough: one has to compute the quantum corrections.

Let $\{e_i\}$ $i=1,\dots,h''$ be a basis of $H^2(X)$. Then

$$B+i\omega = \sum_{i=1}^{h''} t^i e_i$$

where $\{t^1,\dots,t^{h''}\}$ are complex coordinates of $\mathcal{M}_{KC}(X)$

Let $\{w^0, w^1, \dots, w^{h''}\}$ be complex projective coordinates on $\mathcal{M}_{KC}(X)$ with $t^i = \dfrac{w^i}{w^0}$ $i=1,\dots,h''$

Then there is a classical prepotential

$$\mathcal{F}^{cl} = \frac{1}{3!}\,\hat{y}^{cl}_{ijk}\,\frac{w^i w^j w^k}{w^0}$$

such that

- $\hat{y}^{cl} : \underbrace{H^{1,1}\times H^{1,1}\times H^{1,1}}_{\text{intersection number}} \longrightarrow \mathbb{Z}$ ← y^{cl} is $\mathbb{Z}$ valued

and (trivially): $\hat{y}^{cl}_{ijk} = \partial_i \partial_j \partial_k \mathcal{F}^{cl}$

- $$G^{cl}_{i\bar{j}} = \partial_i \partial_{\bar{j}} K^{cl}$$
$$e^{-K^{cl}} = i\left(W^I \bar{\mathcal{F}}^{cl}_I - \bar{W}^I \mathcal{F}^{cl}_I\right)$$

the surprise is that this $\mathcal{F}^{cl}$ also determines the classical metric on $\mathcal{M}_{KC}(X)$

RECAP

- $$\Gamma(\Pi A[X]) = \Gamma(\Pi B[Y])$$

- $$\mathcal{M}_{KC}(X) = \mathcal{M}_{CS}(Y)$$

  $$t \longmapsto \psi(t) \qquad \text{mirror map}$$

- classical computations are not exact vs classical computations are exact

$\hat{y}(t) = \hat{y}^{cl}(t) + \cdots$ $\;\hat{y}^{cl}$: simple

$y(\psi)$ is exact but a complicated function of the C-structure (but calculable)

$$\hat{y}(t) = \hat{y}^{cl}(t) + \cdots \overset{MS}{\longleftrightarrow} y(\psi)$$

$\cdots$: typically hard to compute

(P Candelas & XD)

Suppose $h^{(2,1)} = 1$ (one parameter example)

Let ψ be a (affine) c-coordinate $\mathcal{M}_{cs}(Y)$

t " " " " " $\mathcal{M}_{kc}(X)$

$\psi(t)$ is the mirror map

Then $$\hat{y}_{ttt} = \hat{y}^{cl}_{ttt} + \Delta y_{ttt} = \left(\frac{\partial\psi}{\partial t}\right)^3 \underbrace{y_{\psi\psi\psi}}_{\partial^3_\psi \mathcal{G}}$$

So the classical computation on $\mathcal{M}_{cs}(Y)$ (IIB [Y]) together with the mirror map gives the quantum corrections of $\hat{y}^{cl}_{ttt}$ (IIA [X])

More on $\mathcal{M}_{KC}(X)$

To appreciate the power of MS, let's try to understand better $\mathcal{M}_{KC}(X)$ without using MS.
In particular we want to understand where the "quantum corrections" come from.

For simplicity let $X \in \mathbb{P}^4[5]$ $\qquad h^{(1,1)} = 1$
$(h^{(2,1)} = 101)$

so $\quad B + i\omega = t\, e$

t ← C-Kähler class parameter

Yukawa couplings in physics are calculated using a path integral:

Let Σ = 2-dim surface swept out by the string moving in X

and $x : \Sigma \longrightarrow X$ embedding of Σ into X

$$y_{ijk} = \frac{\int [\mathcal{D}x] \; e_i e_j e_k \; e^{-S[x]}}{\int [\mathcal{D}x] \; e^{-S[x]}}$$

action for a string moving in X

where

$$S[x] = t \int_\Sigma e$$

(pull back of Kähler class to Σ)

Today: $\Sigma = \mathbb{P}^1 = S^2$

To compute the PI:

1) expand PI around a classical solution (minimum of the action)

2) compute quantum corrections

supersymmetry $\Rightarrow$ quantum corrections to g^{ij} can only come from saddle points of the action: these are called instantons (Distler & Greene)

(saddle points: stationary points of S)

Distler & Greene proved that the result is then exact

What is this mathematically?

Stationary points of the action are

$$\frac{\partial x^\mu}{\partial \bar{z}} = 0 \qquad x : \Sigma = \mathbb{P}^1 \longrightarrow X$$

$(z, \bar{z})$ coordinates on $\mathbb{P}^1$

that is, it is a holomorphic embedding

$$\mathbb{P}^1 \longrightarrow X$$

Then, $x(\Sigma)$ can be

- a point in X (x = constant, classical contribution)
- an algebraic curve in X
- multiple cover of an algebraic curve in X

} rational curves

A rational curve of degree k is a holomorphic embedding of degree k

- For example a rational curve of degree 2 would be

$$\mathbb{P}^1 \ni (u, v) \longmapsto (u^2, v^2, uv, 0, 0)$$

$$\xrightarrow{\text{or}} (u^2, v^2, 0, 0, 0)$$

2 double cover of rational curve of deg 1

$$(u, v) \longmapsto (u, v, 0, 0, 0)$$

- Example of a rational curve of degree 1 in X = quintic

$$P = \sum x_i^5 - 5\psi x_1 x_2 x_3 x_4 x_5 \qquad (u, v) \longrightarrow (u, -\alpha^k u, v, -\alpha^k v, 0), \quad \alpha^5 = 1$$

Evaluating the PI

$$\hat{y}_{ttt} = \hat{y}^{cl}_{ttt} + \sum_k \cdots$$

classical contribution

sum over all inequivalent holomorphic embeddings of $\mathbb{P}^1$ into X

$$\boxed{y_{ttt} = 5 + \sum_k n_k\, k^3 \frac{q^k}{1-q^k}\,, \quad q = e^{2\pi i t} \quad \text{for the quintic}}$$

n_k: # of (irreducible) rational curves of degree k in X (hard)

The problem in the late 80's was that the numbers n_k where extremely hard to compute using traditional mathematics.

By 1991 only n_1 & n_2 where known correctly.

→ 1994 n_1 (Schubert) 2875

→ 1986 n_2 (Katz) 609250

→ Ellingsrud + Strømme calculated n_3 incorrectly due to an error in their computing code; in 1992 the gave the right number.

$$\boxed{\text{MS} \rightsquigarrow \text{give a generating function for the } n_k!}$$

Mirror symmetry and the numbers n_k

For X = quintic 3-fold & Y its mirror

[Candelas, de la Ossa, Green, Parkes 91]

$$y = 5\frac{\psi^2}{\varpi_0(\psi)^2(1-\psi^5)}\left(\frac{d\psi}{dt}\right)^3 = 5+\sum_{k=0}^{\infty} n_k \frac{k^3 q^k}{1-q^k}\,, \qquad q = e^{2\pi i t}$$

mirror map: $\lambda = (5\psi)^{-5}$

why integer coeffs?

$$\begin{aligned}\lambda \;=\; & q + 154\,q^2 + 179139\,q^3 \\ & + 313195944\,q^4 + 657313805125\,q^5 + 1531113959577750\,q^6 \\ & + 3815672803541261385\,q^7 + 9970002717955633142112\,q^8 + \dots\end{aligned}$$

(Rigorous proof: Givental 96; Lian, Liu & Yau 97)

Schubert (84) →

Katz (86) →

Ellingsrud & Strømme (92) →

Candelas & XD

⋮

k	n_k
1	2875
2	6 09250
3	3172 06375
4	24 24675 30000
5	22930 58888 87625
6	248 24974 21180 22000
7	2 95091 05057 08456 59250
8	3756 32160 93747 66035 50000
9	50 38405 10416 98524 36451 06250
10	70428 81649 78454 68611 34882 49750

⋮

Table 1 The numbers of rational curves of degree k for $1 \leq k \leq 10$.

CONCLUSIONS

- There are many generalisations: the generating function presented today was the first example of a more general class of identities involving Gromov-Witten invariants
- MS is still a conjecture: much concerted effort has gone into its deep mathematical structure
- Mirror symmetry is but one example of a duality symmetry in string theory.

In each case, these give a deep relationship between different string theories and invariably involve very interesting connections to mathematics.

▶ Given a CY X, how do you find its mirror Y?

1994 Batyrev: mirror symmetric class of CY manifolds which are hypersurfaces in a toric variety

1996 Strominger, Yau & Zaslow ⇝ "T-duality"

- 1994 Kontsevich.
  homological mirror symmetry conjecture
  (D-branes: objects in categories)

Fukaya A_∞-category of Lagrangian submanifolds of X $\longleftrightarrow$ Bounded category of coherent sheaves on Y

- Mark Gross & Bernd Seibert programme
  -
  -
  -
- Other dualities!

THANKS!