Quantum Field Theory in Curved Space-Time

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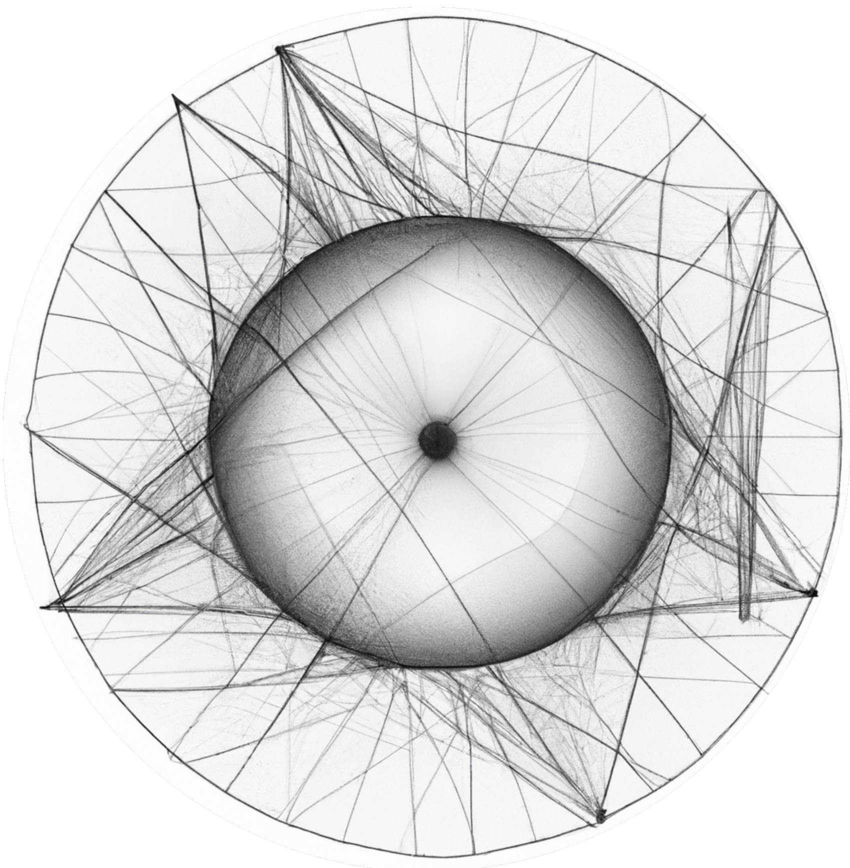

These notes are meant to complement the lectures and may be updated over the course of the term. Feedback is very welcome, especially if there are typos or places where the text lacks clarity. Please send any comments, corrections or questions to

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About this course

Welcome to the 2024-2025 edition of **Quantum Field Theory in Curved Space-Time** of the Oxford Mathematical and Theoretical Physics master course. This course builds on the courses Quantum Field Theory, Advanced Quantum Field Theory and General Relativity I and II. In these courses, the fundamental concepts of quantum field theory and general relativity were introduced. This course aims to develop these subjects more broadly by applying the tools of quantum field theory in some of the curved space-times encountered in general relativity. Along the way several concepts from differential geometry, representation theory as well as special functions will arise. Familiarity with these topics is not required but will be helpful.

Synopsis

The course will consist of the following topics:

- Lorentzian geometry and causality
- Quantum field theory in flat space-time
- Quantum field theory in curved space-time
- Quantum field theory in cosmological backgrounds
- Thermal quantum field theory
- The Unruh effect
- Hawking Radiation
- Quantum fields in Anti-de Sitter space and holography

References

There are a variety of excellent textbooks and lecture notes available in the literature. This course borrows from a number of them, the most relevant ones are listed below.

Standard textbooks on quantum field theory in curved spaces are:

- N.D. Birrell and P. Davies, *Quantum Fields in Curved Space*, (Cambridge University Press 1982)
- R. Wald, *QFT in Curved Space-time and Black Hole Thermodynamics*, (Chicago University Press 1994)
- V. Mukhanov and S. Winitzki, *Introduction to Quantum Effects in Gravity*, (Cambridge University Press 2007)
- L. Parker and D. Toms, *Quantum Field Theory in Curved Spacetime*, (Cambridge University Press 2009)

The following lecture notes are also recommended for reference:

- L. Mason, *Quantum Field Theory in Curved Space-Time*, Lecture notes.
- M. Mariño, *QFT in curved space*, Lecture notes.
- L.H. Ford, *Quantum Field theory in curved spacetime*, arXiv:gr-gc/9707062.
- P.K. Townsend, *Black Holes*, arXiv:gr-gc/9707012.
- T. Jacobson, *Introduction to Quantum Fields in Curved Spacetime and the Hawking Effect*, arXiv:gr-qc/0308048.
- E. Pajer, *Field Theory in Cosmology*, Lecture notes .
- N. Arkani-Hamed and Y. Kats, *Lecture Notes on Quantum Mechanics and Spacetime*. (PDF version available upon request)
- C. Bär and K. Fredenhagen, *Quantum Field Theory on Curved Spacetimes*, (Springer-Verlag Heidelberg 2009), DOI.
- D. Anninos, De Sitter Musings, Int. J. Mod. Phys. A 27 (2012), arXiv:hep-th/1205.3855.
- J. Penedones, TASI lectures on AdS/CFT, TASI 2015, 75-136, arXiv:hep-th/1608.04948.
- D. Simmons-Duffin, TASI Lectures on Conformal Field Theory in Lorentzian Signature, TASI 2019, Lecture notes.

Contents

Part I

FUNDAMENTAL ASPECTS

Chapter 1

Why you should take this course

One of the most pressing open questions in modern theoretical physics is the reconciliation of gravity with quantum mechanics. On one hand, Einstein's general relativity — rooted in the geometry of Lorentzian manifolds — provides a stunningly accurate description of spacetime and gravitational dynamics. On the other, the standard model of particle physics, built on quantum field theory (QFT), successfully explains the fundamental forces and interactions of elementary particles. Yet, these two frameworks are inherently incompatible. This inconsistency becomes particularly glaring in extreme regimes, such as the early universe or the vicinity of black holes, where both quantum effects and gravity must be taken into account.

Early attempts to quantize gravity by treating it as just another quantum field faced deep conceptual and technical difficulties, most notably its intrinsic non-renormalisability.[1] As a result, new paradigms emerged —– string theory being the most prominent, alongside alternative approaches like loop quantum gravity and causal dynamical triangulations. While mathematically compelling, these frameworks remain dauntingly complex, and a fully satisfactory understanding of quantum gravity remains elusive. Given this, any new tools that allow us to probe regimes where quantum mechanics and gravity coexist are of immense value.

1.1 When do we need quantum gravity?

To appreciate when quantum gravity becomes essential, consider a physical system characterised by a mass M and size L, and compare these scales with the relevant fundamental scales:[2]

- the Planck length, $\ell_p = \sqrt{\frac{\hbar G_N}{c^3}}$,
- the Planck mass, $m_p = \sqrt{\frac{\hbar c}{G_N}}$,
- and the Schwarzschild radius, $R_s = \frac{2GM}{c^2}$.

The importance of quantum effects is dictated by the dimensionless ratio:

$$q = \frac{m_p \ell_p}{ML} = \frac{\hbar}{MLc} \,.$$

[1] A somewhat oversimplified but instructive way to see this is through counting the dimension of the coupling constant. Recall that in $d+1$ dimensions Newton's gravitational constant has mass dimension $[G_N] = 2-d$. Hence, the Einstein-Hilbert action has dimension $d-2$ which suggests that gravity is non-renormalisable in $d > 2$. There are various subtleties that need to be taken into account to make this naive argument rigorous but in the context of gravity it turns out to be correct.

[2] In the remainder of this course we almost exclusively use natural units where $\hbar = c = G_N = k_B = 1$.

When this quantity is much smaller than one, quantum effects are negligible, and classical physics suffices. Conversely, when it approaches or exceeds unity, quantum mechanics becomes crucial. Meanwhile, the role of gravity is governed by another dimensionless parameter:

$$g = \frac{R_s}{L} = \frac{2G_N M}{c^2 L}\,.$$

If this quantity is small, gravitational effects are weak and can be treated using Newtonian gravity. However, when it approaches unity, general relativity is required to describe the system.

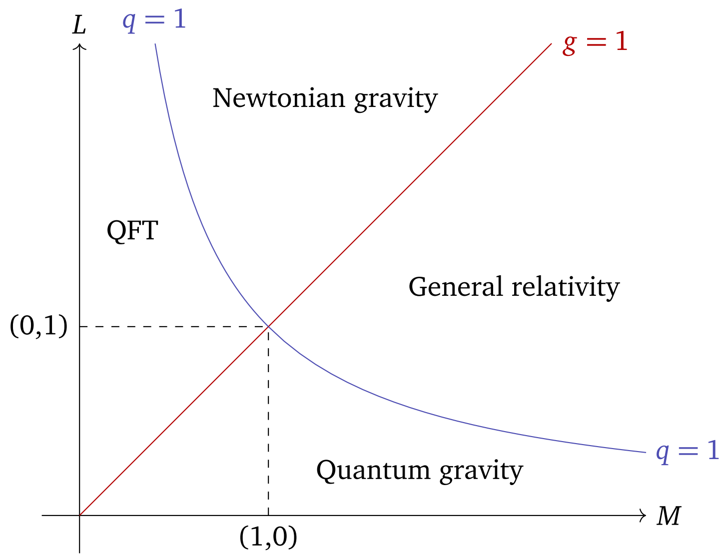


Figure 1.1: Domain of validity for different approximations. Above the blue hyperbola, quantum effects are suppressed, allowing for a classical description via Newtonian mechanics or general relativity. Below the hyperbola, quantum effects become significant and must be included.

These two dimensionless quantities define a parameter space that guides our understanding of different physical regimes. When $g, q \ll 1$, classical Newtonian physics provides a good approximation. When $g \gtrsim 1$ but $q \ll 1$, gravitational effects have to be taken into account but quantum effects remain negligible, making general relativity an appropriate framework. Conversely, when $g \ll 1$ and $q \gtrsim 1$, quantum effects are strong while gravitational effects can be ignored, allowing for a description using QFT in flat space. The truly enigmatic domain lies in the bottom-right region of the diagram in Figure 1.1, where both quantum and gravitational effects matter. Understanding this regime requires a genuine theory of quantum gravity, which remains one of the grand challenges of physics.

In this course, we will not attempt to construct a complete theory of quantum gravity. Instead, we will focus on an important stepping stone: the study of quantum field theory in curved space-time. This framework allows us to explore quantum effects in gravitational backgrounds without needing a full quantum theory of gravity.

This approach describes the transition from general relativity to quantum gravity as we approach the blue line in Figure 1.1. In other words, we are in the regime where $g \gtrsim 1$ and $q \ll 1$ but perturbatively include quantum effects. The key is to treat space-time as a classical manifold obeying Einstein's equations,[3]

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = 8\pi G_N \left\langle T_{\mu\nu} \right\rangle \,, \tag{1.1}$$

while introducing quantum fields that propagate in this curved background. This perspective enables us to study fascinating phenomena, such as particle creation in expanding universes and the Hawking radiation of black holes, providing glimpses into the intricate interplay between quantum mechanics and gravity.

1.2 Broader implications

Studying QFT in curved spacetime is not only of practical importance for understanding black holes and cosmology but also provides deeper insights into the nature of quantum field theory itself. Traditional QFT in Minkowski space relies on concepts like a unique vacuum, well-defined particle states, and S-matrix scattering. However, in curved spacetime, these notions break down. We will see, for instance, that the very concept of a "particle" depends on the observer, a revelation with profound implications for our understanding of quantum physics.

Moreover, curved space techniques have proven to be powerful tools in unexpected areas of theoretical physics. In certain cases, computations in curved space are more tractable than in flat space. Many quantum field theories suffer from infrared divergences in Minkowski space, making semi-classical methods such as instantons ineffective. However, placing the theory on a compact manifold (such as a sphere) naturally regulates the infrared behavior, enabling powerful techniques like supersymmetric localization to yield exact non-perturbative results.

If you are drawn to fundamental physics, this course offers an essential toolkit for exploring quantum field theory beyond the confines of flat spacetime. It is an active and evolving research area with many open questions, making it an exciting field to engage with.

But even if you are more mathematically inclined and less motivated by direct physical applications, do not dismiss this subject just yet. Quantum field theory in curved space has inspired deep mathematical developments, such as Donaldson-Witten theory and Chern-Simons theory, which provide powerful invariants for classifying manifolds. The interplay between geometry and gauge theories is one of the hallmarks of modern mathematical physics, and understanding QFT in curved backgrounds is an excellent stepping stone into this rich territory.

This course will equip you with the essential concepts and techniques to explore quantum field theory in curved spacetime. Whether you are interested in its physical applications to black hole physics and cosmology, or its mathematical connections to topology and geometry, this field offers a wealth of exciting ideas waiting to be explored. By the end of the course, you will be well-prepared to dive into current research and contribute to one of the most fundamental frontiers of modern theoretical physics.

[3] In this equation $\left\langle T_{\mu\nu} \right\rangle$ is the expectation value of the stress tensor of the (quantum) matter fields, which in the regime we are considering can be understood as a classical source for Einstein's equations.

1.3 What this course covers

This course offers an introduction to quantum field theory (QFT) in curved space-times. We begin in Chapter 2 with a study of Lorentzian geometry, emphasising both the local and global aspects of the causal structure. This foundational material is essential for understanding how classical spacetime geometry influences quantum phenomena.

Chapter 3 provides a concise review of QFT in flat Minkowski space, focusing on the structures and techniques most relevant for generalisation to curved backgrounds. Particular attention is given to the role of symmetries, the concept of the vacuum, and the construction of Green's functions.

With this groundwork established, Chapter 4 introduces canonical quantisation in curved space-time. Here we encounter some of the key conceptual challenges unique to the subject, such as the non-uniqueness of the vacuum state and the phenomenon of particle creation in a time-dependent or gravitational background.

Chapter 5 illustrates these ideas through concrete examples, including maximally symmetric space-times (such as de Sitter and Anti-de Sitter), cosmological space-times relevant for inflationary cosmology, and aspects of thermal field theory. The chapter concludes with a discussion of adiabatic expansion techniques, which allow for a perturbative treatment of quantum fields in slowly varying backgrounds.

Chapter 6 is devoted to the Unruh effect, wherein uniformly accelerating observers in flat space perceive the vacuum as a thermal state. This phenomenon provides an important bridge between quantum field theory and thermodynamics in non-inertial frames.

Building on this, Chapter 7 explores Hawking radiation — perhaps the most striking prediction of QFT in curved space-time — demonstrating how black holes emit thermal radiation due to quantum effects near the event horizon.

The final chapter, Chapter 8, turns to quantum field theory in Anti-de Sitter (AdS) space. While geometrically similar to de Sitter space, AdS gives rise to fundamentally different physical insights. This chapter serves as a gateway to the holographic principle, introducing the AdS/CFT correspondence – a conjectured duality between quantum gravity in AdS and a conformal field theory on its boundary.

To preserve the clarity of the main narrative, technical derivations and supplementary material are presented in the appendices. These include a summary of our conventions, a review of differential forms and general relativity essentials, and mathematical tools such as hypergeometric functions. We also include brief introduction to conformal field theory.

Chapter 2

Lorentzian geometry

In a classical field theory, one obtains the physical field configurations through variational principles, i.e. Euler-Lagrange equations. Once a solution has been found its stability can be studied through a local analysis in field space. Quantisation on the other hand is a global procedure where we need to take into account the full phase space. Indeed, already in quantum mechanics there is the possibility for a particle to jump over any potential barrier allowing it to probe the full phase space.

In order to quantise a theory (using canonical quantisation) we need a complete set of solutions to certain (linear) wave equations. Before moving on to quantum fields in curved space, we will therefore need some basic global notions from Lorentzian geometry. In particular, we focus on carefully defining causality and various related concepts, such as Cauchy hypersurfaces and global hyperbolic manifolds. On such manifolds, the above mentioned wave functions behave particularly nicely. More details can be found in for example [Wal84, O'N83].

2.1 Lorentzian manifolds

Before stating the definition of a Lorentzian manifold let us start to define what we mean by a Lorentzian scalar product on a vector space V.

Definition 2.1. Let V be a $(d+1)$-dimensional real vector space. A Lorentzian scalar product on V is a non-degenerate symmetric bilinear form, $\langle\cdot,\cdot\rangle$, of signature $(1,d)$.

This means that we can find a basis $\{e_\mu\}$, $\mu = 1,\ldots,d+1$, of V such that

$$\langle e_a, e_b\rangle = \eta_{ab}\,. \tag{2.1}$$

where η_{ab} is the usual Minkowski metric, $\eta_{ab} = \mathrm{diag}(-1,1,\cdots,1)$. With this definition at hand we can now give a precise definition of a Lorentzian manifold.

Definition 2.2. A Lorentzian manifold is a pair $(\mathcal{M},g)$, where $\mathcal{M}$ is a smooth $(d+1)$-dimensional manifold, and g is a Lorentzian metric, i.e. g associates with each point $p\in\mathcal{M}$ a Lorentzian scalar product g_p on the tangent space $T_p\mathcal{M}$.

As usual in differential geometry we require that g_p depends smoothly on p. For a choice of local coordinates, $(x_0, x_1, \cdots, x_d) : U \to V$, where $U \subset \mathcal{M}$ and $V \subset \mathbb{R}^{1,d}$ are open subsets, and for any $\mu,\nu = 0,\ldots,d$, the functions $g_{\mu\nu} : V \to \mathbb{R}$, defined by $g(\partial_\mu,\partial_\nu)$, are smooth. Here $\partial_\mu = \frac{\partial}{\partial x_\mu}$ denote the usual coordinate vector fields. With respect to these coordinates we write the line element $\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu$. Note that often we choose to call the time coordinate $t = x_0$.

Before we continue let us give a few examples.

Example 2.1 (Minkowski space)**.** *Minkowski space* $(\mathbb{R}^{1,d}, \eta)$ *is clearly a Lorentzian manifold.*

Example 2.2 (Warped product spaces)**.** *Let* $(\mathcal{N}, h)$ *be a connected Riemannian manifold, and* $I \subset \mathbb{R}$ *an open interval. For any smooth positive function* $f : I \to (0, \infty)$*, we can define a metric* $g_{\mu\nu}$ *with line element* $\mathrm{d}s^2 = -\mathrm{d}t^2 + f(t)^2 h$ *on* $\mathcal{M} = I \times \mathcal{N}$*. For any two vectors* $X_i = (a_i \partial_t \oplus Y_i) \in T_{(t,p)}(\mathcal{M})$*, with* $Y_i \in T_p\mathcal{N}$ *we have* $g(X_1, X_2) = -a_1 a_2 + f(t)^2 h(Y_1, Y_2)$*.*[1] *This type of Lorentzian metric is called a warped product metric.*

Many familiar Lorentzian manifolds are of the form of this second example. Friedman-Lemaître-Robertson-Walker space-times [Fri22, Fri24, Lem31, Lem33, Rob35, Rob36a, Rob36b, Wal37] are obtained by requiring $(\mathcal{N}, h)$ to be a maximally symmetric Riemannian manifold with a constant curvature metric. This type of metric is of particular relevance when studying cosmological models describing the big bang or the expansion of the universe. A special case is the de Sitter (dS) space-time in global coordinates, where $I = \mathbb{R}$, $\mathcal{N} = S^{n-1}$, with h the canonical metric on the $(n-1)$-sphere with unit radius, and $f(t) = \cosh(t)$.

As a final example, consider the four-dimensional Schwarzschild black hole.

Example 2.3 (Schwarzschild black hole)**.** *For a fixed mass* $M > 0$*, consider the function*

$$h : \mathbb{R}_+ \to \mathbb{R} : r \mapsto 1 - \frac{2M}{r} . \tag{2.2}$$

This function has a pole at $r = 0$ *and a root at* $r = 2M$*. On both patches* $P_I = \left\{(r,t) \in \mathbb{R}^2 | r > 2M\right\}$ *and* $P_{II} = \left\{(r,t) \in \mathbb{R}^2 | 0 < r < 2M\right\}$ *we define the Lorentzian line element as*

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu = -h(r)\mathrm{d}t^2 + \frac{1}{h(r)}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right) . \tag{2.3}$$

The singularity of the metric g at $r = 2M$ might seem problematic, but one can easily show (by going to Kruskal coordinates for example) that this is simply a coordinate singularity. For more details on the Schwarzschild black hole and its rotating and electromagnetically charged cousins we refer the reader to the course GR II.

2.2 Causal structure

Given a Lorentzian manifold with associated metric g, we can associate to each point $p \in \mathcal{M}$ the quadratic form

$$\gamma_p : T_p\mathcal{M} \to \mathbb{R} : \gamma_p(X) = g_p(X, X) . \tag{2.4}$$

A vector $X \in T_p\mathcal{M}$ is called time-like, light-like or space-like respectively if

$$\begin{cases} \gamma_p(X) < 0\,, & \text{time-like}\,, \\ \gamma_p(X) = 0\,, & \text{light-like}\,, \\ \gamma_p(X) > 0\,, & \text{space-like}\,. \end{cases} \tag{2.5}$$

[1] For any $t \in I$ and $p \in \mathcal{N}$ we identify the tangent space at (t,p) as follows, $T_{(t,p)}\mathcal{M} = T_t I \oplus T_p\mathcal{N}$.

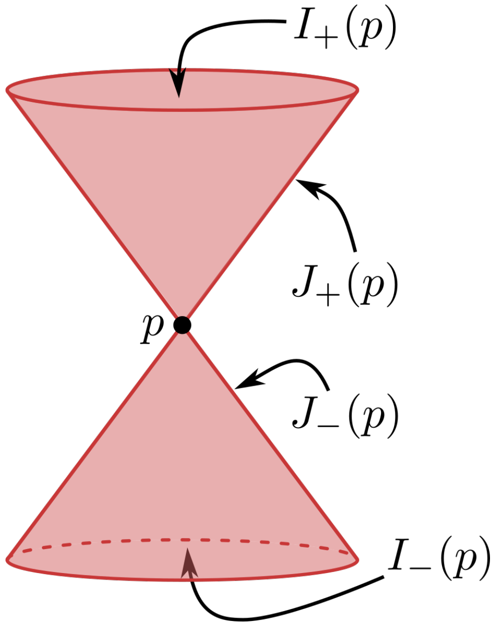


Figure 2.1: The lightcone associated to the point p.

A vector is called causal if it is time-like or light-like. For $d \geq 1$, the set of time-like vectors consists of two connected components, assigning a time-orientation consists of choosing one of these components which we denote by $I_+(p)$ and call future-directed. The closure $J_+(p) = \overline{I_+(p)}$ consists of the set of future-directed causal vectors. Analogously we call $I_-(p)$ resp. $J_-(p)$ the past directed light-like and causal vectors. The double cone formed as such is called the lightcone at the point p, see Figure 2.1. Similarly, we call a (piecewise) differentiable curve $s \in \mathcal{C}^1(\mathcal{M})$ in $\mathcal{M}$ time-like, light-like or space-like, if all of its tangent vectors are respectively time-like, light-like or space-like.

Having defined these concepts locally at each point p we now want to extend them to global properties of the manifold. A first step in this direction is the following definition.

Definition 2.3 (time-oriented manifold)**.** A Lorentzian manifold $(\mathcal{M}, g)$ is time-orientable if there exists a continuous time-like vector field τ on $\mathcal{M}$. A Lorentzian manifold together with such a vector is called time-oriented.

We will call time-oriented Lorentzian manifolds space-times. It should be noted that the concept of orientability depends only on the topology of $\mathcal{M}$, while the notion of time-orientability depends on the choice of Lorentzian metric. The question of whether a manifold can be equipped with some time-orientable metric is 'topological' however. Indeed, we have the following equivalent statements (see [O'N83]):

- $\mathcal{M}$ admits a smooth non-vanishing vector field.
- $\mathcal{M}$ can be equipped with a smooth Lorentzian metric.
- $\mathcal{M}$ can be equipped with a time-orientable Lorentzian metric.

Example 2.4. *Some (not) time-orientable manifolds:*

- *The standard Minkowski space is orientable and time-orientable.*
- *The two-sphere is orientable but cannot be equipped with a time-orientable metric. Indeed, famously S^2 does not admit a smooth non-vanishing vector field. Note that odd dimensional spheres do admit a non-vanishing vector field!*
- *The Mobius strip is not orientable but it can be equipped with Lorentzian metrics that are either time-orientable or not.*
- *The cylinder $\mathbb{R} \times S^1$ is orientable and can be equipped with a metric that is time-orientable,*

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 \,. \tag{2.6}$$

It can also be equipped with a metric that is not time-orientable,

$$\mathrm{d}s^2 = \cos\theta \mathrm{d}\theta^2 - 2\sin\theta \mathrm{d}x \mathrm{d}\theta - \cos\theta \mathrm{d}x^2 \,. \tag{2.7}$$

Exercise 2.1. *Show that the metrics on the cylinder written above are respectively time-orientable and not time-orientable.*

For future reference, we define the following causality relations. Let $p, q \in \mathcal{M}$, we have

$$\begin{cases} p \ll q & \longleftrightarrow \quad \exists \text{ a future directed time-like curve in } \mathcal{M} \text{ connecting } p \text{ and } q\,, \\ p < q & \longleftrightarrow \quad \exists \text{ a future directed causal curve in } \mathcal{M} \text{ connecting } p \text{ and } q\,, \\ p \leq q & \longleftrightarrow \quad p < q \text{ or } p = q\,. \end{cases} \tag{2.8}$$

Unless otherwise mentioned, we always consider both orientable and time-orientable manifolds. However, time-orientability is not quite strong enough to rule out all problematic cases. For linear operators to have well-defined unique, causal solutions we need something more.

In general relativity, world-lines of particles are modelled by causal curves. If the space-time is compact, something strange happens. Namely, in every compact space-time $\mathcal{M}$ there exists a closed time-like curve (CTC). When such curves exist, one easily runs into paradoxes as travel into the past is now a clear possibility. Therefore, when dealing with Lorentzian space-times we want to exclude such examples.

Definition 2.4 (Causal manifold)**.** A space-time $(\mathcal{M}, g)$ is causal if it does not contain any closed causal curve.

Definition 2.5 (Strongly causal manifold)**.** A space-time $(\mathcal{M}, g)$ is strongly causal if for any point $p \in \mathcal{M}$ and any neighbourhood U of p, there exists a causally convex neighbourhood V of p, contained in U. A neighbourhood is said to be causally convex if any causal curve with endpoints in V is entirely contained in V.

This property implies that there cannot be time-like curves that pass through V more then once. In other words, it is not possible to return to the same point in space-time by following a time-like curve, i.e. particles travelling slower then light cannot return to the same point in space-time. Strong causality obviously implies causality. For technical reasons we will always assume strong causality.

The wave equations that we will consider, such as the Klein-Gordon equation for a free scalar field $(\Box + m^2)\phi = 0$, are all hyperbolic (linear) partial differential equations. This means that an equation of order n has a well-posed initial value problem for the first $n-1$ derivatives.[2] More precisely, the Cauchy problem can be locally solved for arbitrary initial data along a non-characteristic hypersurface Σ.[3] In order to have well-defined global solutions on Lorentzian space-times we need some further technical definitions.

Definition 2.6 (Achronal hypersurface)**.** A hypersurface Σ is achronal if no pair of points $p, q \in \Sigma$ can be connected by a time-like curve.

Since solutions propagate along causal curves the data on the hypersurface Σ can only influence a restricted region. We can define the regions

$$\begin{aligned} J_+(\Sigma) &= \left\{p \in \mathcal{M} \,\middle|\, \exists \text{ a future directed causal curve starting on } \Sigma \text{ to } p\right\} , \\ J_-(\Sigma) &= \left\{p \in \mathcal{M} \,\middle|\, \exists \text{ a future directed causal curve from } p \text{ ending on } \Sigma\right\} . \end{aligned} \tag{2.9}$$

These sets are sometimes denoted as the future/past domain of influence. Analogously we can define $I_\pm(\Sigma)$ by restricting to the interior of $J_\pm(\Sigma)$.

Definition 2.7 (Domain of dependence)**.** The domain of dependence of a subset Σ is defined as the set of points, $D(\Sigma)$, in $\mathcal{M}$ through which every inextendable causal curve in $\mathcal{M}$ meets Σ, i.e.

$$D(\Sigma) = \{p \in \mathcal{M} |\text{ every inextendable causal curve passing through } p \text{ intersects } \Sigma\} . \tag{2.10}$$

Analogously we define the future/past domain of dependence $D_\pm(\Sigma)$ as the intersection $D(\Sigma) \cap J_\pm(\Sigma)$.

The domain of dependence is the region on which the initial value problem for wave equations can be proved to be well-posed by various PDE techniques. If p is a point lying on a causal curve that cannot be extended through Σ, then one can imagine waves coming in along that curve that are not determined by the data on Σ and therefore they would violate the uniqueness assumption. The interior of the domain of dependence is sometimes also called the Cauchy development of a set.

Definition 2.8 (Cauchy hypersurface)**.** A subset Σ of a connected space-time $\mathcal{M}$ is a Cauchy hypersurface if each inextendable causal curve in $\mathcal{M}$ meets Σ at exactly one point.

In other words a Cauchy hypersurface is a hypersurface for which $D(\Sigma) = \mathcal{M}$. Any two Cauchy hypersurfaces in $\mathcal{M}$ are homeomorphic. They are topological.

In analogy with the nomenclature for PDEs, we call a space-time globally hyperbolic when the future state of the system is entirely specified by initial conditions. There are several equivalent definitions of global hyperbolicity of which we note two.

[2] We say that the initial value problem on some region U with given data on Σ is well-posed if there exists a unique solution on U with given initial data on Σ.

[3] Consider a linear second order PDE of the form (2.11) and let Σ be a hypersurface defined implicitly by $\Phi(x) = q$ for some constant q. The hypersurface Σ is characteristic if the principal symbol of the PDE vanished when evaluated on the normal, i.e. when $g^{\mu\nu}(x)\partial_\mu\Phi\partial_\nu\Phi = 0$. This is known as the characteristic equation. For the wave equations we consider in this course, the characteristic surfaces are null hypersurfaces.

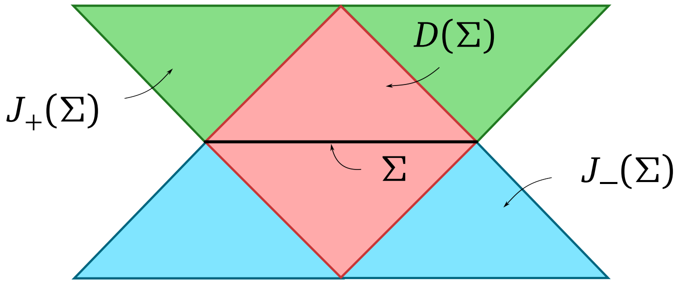


Figure 2.2: The domains of dependence and influence of a set Σ. The green area is the future domain of influence, the blue area the past domain of influence and the red area denotes the full future + past domain of dependence.

Definition 2.9 (globally hyperbolic manifold I)**.** A space-time $\mathcal{M}$ is a globally hyperbolic manifold if it is strongly causal and if for all $p, q \in \mathcal{M}$ the intersection $J_+(p) \cap J_-(q)$ is compact.

Definition 2.10 (globally hyperbolic manifold II)**.** A space-time $\mathcal{M}$ is a globally hyperbolic manifold if it admits a Cauchy surface.

A very useful theorem about globally hyperbolic manifolds goes as follows.

Theorem 2.1. *Let $\mathcal{M}$ be a connected time-oriented Lorentzian manifold. Then the following are equivalent:*

1. *$\mathcal{M}$ is globally hyperbolic.*
2. *$\mathcal{M}$ has a Cauchy hypersurface.*
3. *$\mathcal{M}$ is isometric to $\mathbb{R} \times \Sigma$ with metric $\mathrm{d}s^2 = -\beta \mathrm{d}t^2 + \mathrm{d}s^2_\Sigma(t)$, where $\mathrm{d}s^2_\Sigma(t)$ depends smoothly on t and each $\{t\} \times \Sigma$ is a Cauchy hypersurface in $\mathcal{M}$.*

The proof is rather technical so we do not state it here but the crucial step is proving that *1.* follows from *3.*. The proof can be found in [BS05] using a theorem by Geroch [Ger70]. Once this step has been proven the other implications follow straightforwardly.

Exercise 2.2. *Show that on a globally hyperbolic manifold $\mathcal{M}$ there always exists a smooth function $h : \mathcal{M} \to \mathbb{R}$ whose gradient is past-directed time-like at every point and all of whose level sets are space-like Cauchy hypersurfaces. Such a function is called a Cauchy time function.*

Globally hyperbolic manifolds are very useful since this property ensures that for a large class of operators the Cauchy problems are well-posed, with initial data on a Cauchy hypersurface in appropriate function spaces. In particular, most of the operators we will encounter are generalized d'Alembertians P, of the form

$$P = \sum_{\mu,\nu=0}^{d} g^{\mu\nu}(x)\partial_{x^\mu}\partial_{x^\nu} + \sum_{\mu=0}^{d} a^\mu(x)\partial_{x^\mu} + b(x)\,, \tag{2.11}$$

where the inverse metric $g^{\mu\nu}$, and the functions a^μ and b are smooth functions of x. On globally hyperbolic manifolds, one can prove a whole range of global existence and uniqueness theorems for the solutions of the homogeneous equation $P\phi = 0$ as well as for the Green's functions. For a more detailed treatment of linear wave equations on Lorentzian manifolds we refer the reader to the lecture notes [BBB+09].

For these reasons, we will mostly consider globally hyperbolic manifolds in the subsequent chapters. Minkowski space, de Sitter space, the Schwarzschild black hole as well as the FLRW solutions are all globally hyperbolic space-times. However, a notable exception is Anti-de Sitter (AdS) space. In this case we will see that we can overcome the issues accompanying the absence of global hyperbolicity by providing appropriate boundary conditions at the conformal boundary. In Chapter 8 we will come back to this example in more detail.

2.3 Conformal infinity and Penrose diagrams

To get a good grip on the global structure, a very useful tool is to study the asymptotics. A neat way to do so is via conformal compactifications. This consist of introducing a conformal boundary which corresponds to infinity in the physical space-time. Various aspects of this procedure were discussed in GRII and we refer the reader to that course for more details and references. Here we restrict ourselves to those aspects relevant for the remainder of this course.

Definition 2.11 (Conformal transformation)**.** A conformal transformation is a map from a space-time $(\mathcal{M}, g)$ to another space-time $(\widetilde{\mathcal{M}}, \tilde{g})$, such that

$$\tilde{g}_{\mu\nu}(x) = \Omega(x)^2 g_{\mu\nu}(x)\,, \tag{2.12}$$

where Ω is a non-vanishing smooth function of the coordinates, $\Omega(x) \neq 0$ for all $x \in \mathcal{M}$.

One reason why conformal transformations are useful is because they preserve the causal structure of space-time. Indeed, two space-times whose metrics are related by a conformal transformation have the same null geodesics.[4] We can use this fact to our advantage to study the causal structure of space-time by using suitably chosen conformal transformations to bring infinity to a finite coordinate distance allowing us to represent the causal structure on a finite sized diagram called a Penrose diagram.The process of mapping space-time to a compact domain is called conformal compactification.

Definition 2.12. A conformal compactification of a space-time $(\mathcal{M}, g)$ is a manifold $\widetilde{\mathcal{M}}$ with boundary $\mathscr{I} = \partial\widetilde{\mathcal{M}}$ and metric $\tilde{g}$ such that

1. $\tilde{g}$ is smooth on $\widetilde{\mathcal{M}}$.
2. $\mathcal{M}$ is diffeomorphic to the interior of $\widetilde{\mathcal{M}}$.
3. On $\mathcal{M}$ we have that $\tilde{g} = \Omega^2 g$ with Ω a smooth function on $\widetilde{\mathcal{M}}$.
4. In the interior, $\widetilde{\mathcal{M}} \backslash \partial\widetilde{\mathcal{M}} \simeq \mathcal{M}$ we have $\Omega \neq 0$.
5. On the boundary $\partial\widetilde{\mathcal{M}} = \mathscr{I}$, we have $\Omega = 0$, and $d\Omega \neq 0$.

The boundary $\mathscr{I}$ [5] is called conformal infinity. In addition to the hypersurface $\mathscr{I}$ the conformally

[4] Space-like and time-like geodesics on the other hand are not necessarily preserved under conformal transformations.
[5] Pronounced as scri as a shorthand for script I.

extended space-time might contain loci of higher co-dimension, these have to be considered separately and are denoted by i_0 or $i_\pm$ depending on their (time-like/space-like/null) causal properties.

To illustrate these concepts, let us consider in detail the example of Minkowski space.

Example 2.5 (Conformal compactification of Minkowski space-time). *Consider Minkowski space-time in $d+1$ dimensions with line element*

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \sum_{m=1}^{d} \mathrm{d}x_m^2\,, \qquad\qquad t, x_m \in (-\infty, \infty)\,. \tag{2.13}$$

In spherical coordinates the metric becomes

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2 \mathrm{d}s^2_{S^{d-1}}\,. \tag{2.14}$$

Defining the light-cone coordinates $u = t - r$ and $v = t + r$ and performing a further coordinate transformation[6]

$$u = \tan\tilde{u}\,, \qquad v = \tan\tilde{v}\,, \qquad \tilde{u}, \tilde{v} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \tag{2.15}$$

with $\tilde{u} \leq \tilde{v}$, the metric becomes

$$\mathrm{d}s^2 = -\frac{1}{4\cos^2\tilde{u}\cos^2\tilde{v}}\left[4\mathrm{d}\tilde{u}\mathrm{d}\tilde{v} - \sin^2(\tilde{v}-\tilde{u})\mathrm{d}s_{S^{d-1}}\right]. \tag{2.16}$$

We can now use a conformal transformation to remove the prefactor. Since the metric is now regular at the points at infinity we can now compactify the space by including the points $\tilde{u}, \tilde{v} = \pm\frac{\pi}{2}$. The Penrose diagram for Minkowski space is illustrated in Figure 2.3.

Alternatively, changing coordinates $\tilde{u} = \frac{1}{2}(T - \rho)$ and $\tilde{v} = \frac{1}{2}(T + \rho)$ we obtain the following metric for the conformal compactification

$$\widetilde{\mathrm{d}s}^2 = \mathrm{d}T^2 - \mathrm{d}s^2_{S^3}\,, \tag{2.17}$$

known as the Einstein cylinder metric. Consequently, the rescaling procedure described above maps Minkowski space into a (compact) region of the Einstein cylinder, see Figure 2.3.

Having constructed the conformal compactification of Minkowski space let us discuss the structure of conformal infinity.

- Future and past null infinity are defined as the hypersurfaces,

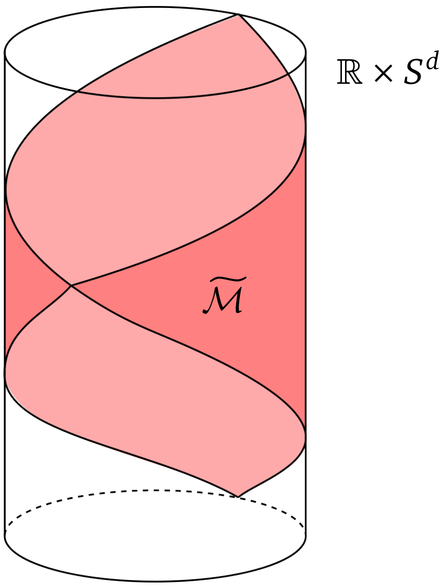
  $$\mathscr{I}^\pm = \left\{ p \in \widetilde{\mathcal{M}} \;\middle|\; 0 < \rho(p) < \pi, T(p) = \pm(\pi - \rho(p)) \right\}. \tag{2.18}$$

  By definition, on this hypersurface we have $\Omega = 0$ and $\mathrm{d}\Omega \neq 0$. Moreover, at this locus we have

  $$\tilde{g}(\mathrm{d}\Omega, \mathrm{d}\Omega) = 0, \tag{2.19}$$

  so that $\mathscr{I}$ are null hypersurfaces.

[6]Note that in order for r to be positive we need to restrict both u, v and $\tilde{u}$, $\tilde{v}$ to the domain $u < v$ and $\tilde{u} < \tilde{v}$.



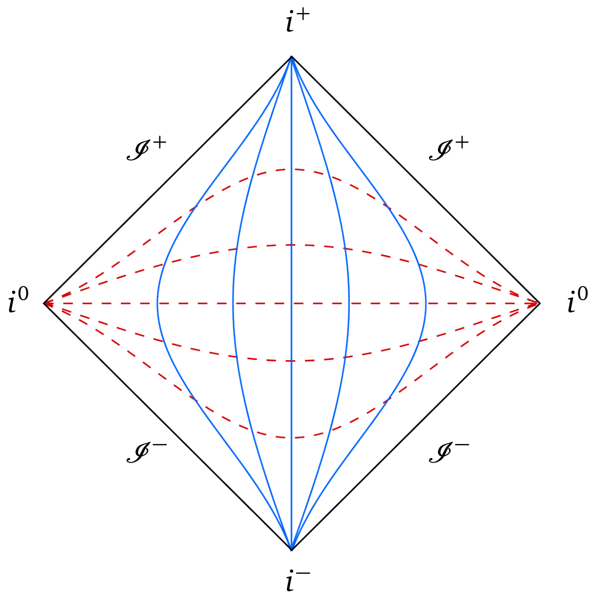


Figure 2.3: The Penrose diagram of Minkowski space. On the left we show the conformal compactification wrapped on the Einstein cylinder. In the Penrose diagram on the right, the time-like geodesics, i.e. lines with constant r, are illustrated as blue lines, while the space-like geodesics, lines with constant t, are illustrated as the dashed red lines.

- Spatial infinity is defined as

$$i^0 = \left\{ p \in \widetilde{\mathcal{M}} \;\middle|\; \rho(p) = \pi \,, T(p) = 0 \right\} . \tag{2.20}$$

At these points, the radius of the $(d-1)$-sphere vanishes in the usual (d)-sphere degeneration. At this point, both $\Omega = \mathrm{d}\Omega = 0$.

- Future and past time-like infinity are defined as

$$i^\pm = \left\{ p \in \widetilde{\mathcal{M}} \;\middle|\; \rho(p) = 0 \,, T(p) = \pm\pi \right\} . \tag{2.21}$$

Again, the $(d-1)$-spheres at these points have vanishing radius and $\Omega = \mathrm{d}\Omega = 0$.

The motivations for this nomenclature follows from the analysis of (inextensible) geodesics in the conformally compactified space. Indeed, one can show that space-like geodesics all end and start at i^0, while time-like geodesics start at i^- and end at i_+. Finally, null geodesics start at $\mathscr{I}^-$ and end at $\mathscr{I}^+$.

Next, let us consider the maximally symmetric space-times: Minkowski space, de Sitter space and anti-de Sitter space. These are solutions to the vacuum Einstein equations with resp. vanishing, negative or positive cosmological constant. See Appendix E for more details and a variety of metrics on (A)dS space-times. We leave the explicit construction of the conformal compactification in these cases as an exercise, and restrict ourselves here to a qualitative discussion.

The hypersurface at infinity, $\mathscr{I}$ has a rather different flavour depending on the value of the cosmological constant. In particular it is null for flat space, space-like for de Sitter, and time-like for anti de Sitter.

Moreover, as all of them are conformally flat, they can be mapped to a portion of the Einstein cylinder. de Sitter space occupies a horizontal strip, while Anti de Sitter maps to a vertical strip, in line with the nature of conformal infinity. In Figure 2.4 we show the Penrose diagram for (A)dS space-time.

As a final comment, note that Minkowski space and de Sitter are clearly globally hyperbolic, but that AdS is not. For AdS, we need to present, not just data on an initial $t = const.$ hypersurface, but also data, or at least boundary conditions at time-like infinity. Otherwise, one can imagine incoming radiation from infinity which in turn might cause very problematic instabilities.

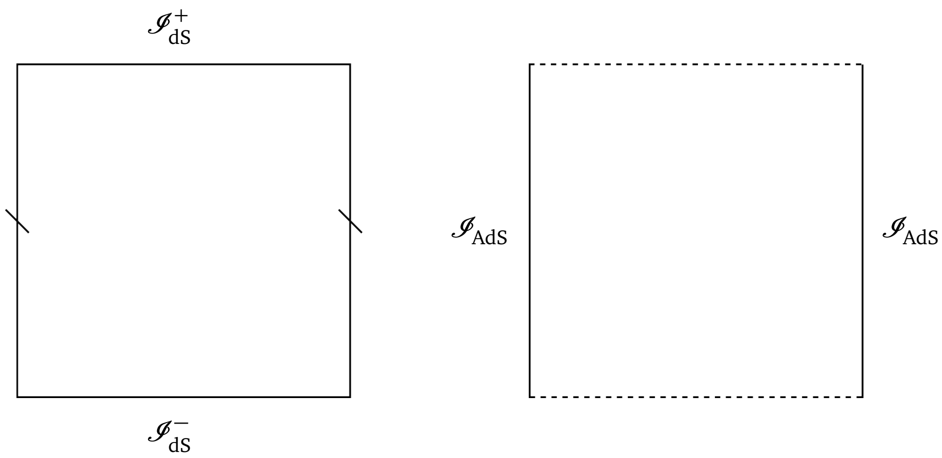


Figure 2.4: The Penrose diagrams for de Sitter (left) and Anti de Sitter (right) space-times. In the de Sitter case, the left and right vertical line can be identified. The topology of $\mathscr{I}^{\pm}_{\mathrm{dS}}$ is $\mathbb{R} \times S^2$, while for AdS conformal infinity is conformal to $\mathrm{Mink}_{1,d-1}$.

For the maximally symmetric spaces there is an elegant alternative construction for their conformal compactification though their embedding in $\mathbb{R}^{2,d+1}$. To see this let us consider the Lorentzian conformal group in $(d+1)$ dimensions, $\mathrm{SO}(2, d+1)$. This group only acts on the compactification, as it interchanges points at finite distance with points at infinity. The conformal group acts on $\mathbb{R}^{d+3}$ by orthogonal transformations preserving the quadratic form

$$X^2 = \eta_{IJ} X^I X^J = -s^2 + w^2 + \eta_{\mu\nu} x^\mu x^\nu , \tag{2.22}$$

where the coordinates on $\mathbb{R}^{d+3}$ are given by $X^I = (s, w, x^\mu)$ where $I = 0, \ldots, d+2$ and $\mu = 0, \ldots, d$.

To obtain the conformal compactification of the maximally symmetric space-times, we choose a non-zero constant vector $K^I \in \mathbb{R}^{(d+3)}$ and define the metric,

$$\mathrm{d}s^2_{K^2} = \left. \frac{\eta_{IJ} \mathrm{d}X^I \mathrm{d}X^J}{(K \cdot X)^2} \right|_{X^2=0} \tag{2.23}$$

where we defined the product between to vectors $X \cdot Y = X_I Y^I$. Up to $\mathrm{SO}(2, d+1)$ transformations, K^I is distinguished only by its norm K^2 so there are only 3 cases $K^2 = -1, 0, 1$. Since we divided by a quadratic function, the metric is invariant under constant rescalings of the embedding coordinates X^I. Moreover, on $X^2 = 0$ the form $X_I \mathrm{d}X^I = \frac{1}{2}\mathrm{d}X^2$ vanishes so it is easy to see that under $X^I \to f(X) X^I$, the metric $\mathrm{d}s^2_I$ is invariant for any (non-vanishing) function $f(X)$. Thus we can scale X so that $K \cdot X = 1$ on the interior of the conformal compactification. At conformal infinity we find that $K \cdot X = 0$. The isometry group of the metric $\mathrm{d}s^2_{K^2}$ can then be found as the subgroup of $SO(2, d+1)$ that preserves the vector K^I.

Exercise 2.3. *Show that by taking the vectors:*

$$K = (1, -1, 0, 0, 0, 0)\,, \tag{2.24}$$

$$K = (1, 0, 0, 0, 0, 0)\,, \tag{2.25}$$

$$K = (0, 1, 0, 0, 0, 0)\,, \tag{2.26}$$

the metric (2.23) reduces resp. to the metric on Minkowksi space, de Sitter space or anti-de Sitter space. Use this to construct the conformal compactifications. Show that the resp. preserved isometry groups are $SO(1, d)$*,* $SO(1, d+1)$ *and* $SO(2, d)$*.*

Exercise 2.4. *As a final example consider the FLRW backgrounds with spatial sections of constant curvature. The metric is given by*

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2 \mathrm{d}s^2_{d,k}\,, \tag{2.27}$$

where for $k = 0, 1, -1$ the spatial manifold is respectively flat Euclidean space, the sphere or hyperbolic space.

Show that these metrics are all conformally flat. Hint: consider the conformal time $\tau = \int_0^t \frac{\mathrm{d}t}{a(t)}$.

So far all our examples were conformally flat, allowing us to use a variety of tricks to easily construct their conformal compactification. For non-conformally flat space-times many of our tricks fail making the task of finding the conformal compactification more involved. Conceptually the procedure remains identical, as described in Definition 2.12 but has to be discussed on a case by case basis. However, if M is globally hyperbolic we can see that $\mathscr{I} = \mathscr{I}^+ \cup \mathscr{I}^-$ where future infinity $\mathscr{I}^+$ is to the future of a Cauchy hypersurface and past infinity $\mathscr{I}^-$ to the past.

2.4 Asymptotics and peeling

All the examples discussed above had an important feature in common, namely that they all admit a smooth conformal extension which attaches a conformal boundary to the space-time. A natural question is to what extend this property is shared by more generic manifolds.

Definition 2.13 (Asymptotically simple space-times)**.** A space-time $(\mathcal{M}, g)$ is asymptotically simple if there exists a smooth, oriented, time-oriented causal conformal compactification $(\widetilde{\mathcal{M}}, \tilde{g})$ such that each null geodesic of $(\widetilde{\mathcal{M}}, \tilde{g})$ acquires two distinct endpoints on $\mathscr{I}$.

Note that the completeness requirement in this definition excludes singular space-times such as the Schwarzschild black hole in which there exist null geodesics which do not reach $\mathscr{I}$. Not only those falling into the black hole but also those lying in the photon sphere are incomplete in this sense. Moreover, even without singularities, the fact that a space-time is smooth and geodesically complete does not guarantee that it admits a smooth conformal compactification.

Exercise 2.5. *Consider the Nariai space-time $\mathcal{M} = \mathbb{R} \times S^1 \times S^2$ with metric*

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \cosh^2 t\, \mathrm{d}\psi^2 + \mathrm{d}s^2_{S^2}\,. \tag{2.28}$$

The Nariai space-time is both geodesically complete and globally hyperbolic. Show that it does not allow for a smooth conformal extension.

To do so note that under a conformal transformation the squared Weyl tensor transforms as

$$C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} = \Omega^4 \widetilde{C}_{\mu\nu\rho\sigma}\widetilde{C}^{\mu\nu\rho\sigma}\,. \tag{2.29}$$

Use this fact to show that a smooth conformal extension cannot exist for this space-time.

The requirement of asymptotic simplicity is very natural from a physical point of view and can be thought of as stating that the matter density has to die off quickly enough at infinity such that the asymptotic geometry is purely described by the cosmological constant. More precisely, we only allow conformally invariant matter in the neighbourhood of $\mathscr{I}$.[7] Furthermore, we can prove the following theorem

Theorem 2.2. *Let $(\mathcal{M}, g)$ have conformal compactification $(\widetilde{\mathcal{M}}, \tilde{g})$, and suppose that the space-time asymptotically satisfies the Einstein equations with conformally invariant matter (so that the trace of the stress-energy tensor vanishes) with cosmological constant λ. Then $\mathscr{I}$ is space-like when $\lambda > 0$, time-like for $\lambda < 0$ and null when $\lambda = 0$.*

Proof: From Einstein's equations in $d+1$ dimensions we immediately obtain

$$R \simeq \frac{2(d+1)}{d-1}\lambda\,, \tag{2.30}$$

in a neighbourhood of $\mathscr{I}$. Define the Schouten tensor as

$$P_{\mu\nu} = -\frac{1}{d-1}\left(R_{\mu\nu} - \frac{1}{2d}Rg_{\mu\nu}\right). \tag{2.31}$$

Near $\mathscr{I}$ we have that $P^{\mu}_{\mu} \simeq -\frac{d+1}{d(d-1)}\lambda$ and note that near $\mathscr{I}$ it transforms as

$$P_{\mu\nu} = \tilde{P}_{\mu\nu} + \Omega^{-1}\tilde{\nabla}_\mu\tilde{\nabla}_\nu\Omega - \frac{1}{2}\Omega^{-2}\tilde{g}_{\mu\nu}\tilde{\nabla}_\rho\Omega\tilde{\nabla}^\rho\Omega\,, \tag{2.32}$$

under a conformal rescaling of the metric. Remember that on $\mathscr{I}$ we have $\Omega = 0$ so that,

$$\widetilde{g}^{\mu\nu}N_\mu N^\nu = \frac{\lambda}{24}\,, \tag{2.33}$$

on $\mathscr{I}$, where we defined the normal vector $N_\mu = -\tilde{\nabla}_\mu\Omega$. From this the proposition immediately follows.

Asymptotically simple space-times satisfying the conditions in the theorem above are called asymptotically flat, asymptotically AdS or asymptotically dS, depending on the value of the cosmological constant. Moreover, using the results above one can analyse the topology of conformal infinity in each case, the de Sitter and anti de Sitter cases are rather straightforward. For de Sitter one can show that topologically $\mathscr{I} \approx S^{d-1}$, while for anti de Sitter it is a compact time-like hypersurface. The case

[7] Conformally invariant matter comes with a traceless stress tensor and includes massless matter fields such as for example a massless scalar field or massless gauge fields. Therefore this criterion still allows for radiation to reach infinity.

of vanishing cosmological constant on the other hand is harder to analyse. However, one can prove that in any asymptotically simple space-time for which $\mathscr{I}$ is everywhere null, the topology of each component $\mathscr{I}^{\pm}$ is given topologically by $\mathscr{I}^{\pm} \approx \mathbb{R} \times S^2$, where the $\mathbb{R}$ factor can be thought of as the rays generating null infinity.

To finish this chapter we briefly discuss one of the most important results of the theory of asymptotics of the gravitational field, the so-called peeling theorem. The peeling theorem is based on the observation that in asymptotically simple space-times the Weyl tensor must vanish on $\mathscr{I}$ and quantises the allowed decay. A precise statement of the peeling theorem requires the introduction of a variety of new concepts and goes beyond the scope of these lectures. For this reason we only present a simplified sketch and refer the interested reader to the textbooks [PR84, Kro23] for more information.

As already mentioned before, it is reasonable to expect conformally invariant and massless fields to continue smoothly to $\mathscr{I}$ in the conformal compactification. Consider a (conformally coupled) scalar field ϕ solving the (conformally invariant) wave equation. Then $\tilde{\phi} = \phi/\Omega^{\frac{d-1}{2}}$ should be smooth on $\mathscr{I}$ (at least if it is in the domain of dependence of $\mathcal{M}$). In an asymtotically de Sitter space, where $\lambda > 0$, we can deduce that the massless scalar will evolve past $\mathscr{I}$ as if it wasn't there and so $\tilde{\phi}$ will be smooth and generically non-vanishing near $\mathscr{I}$ in the conformally extended space-time. Translating this back to the physical space-time this gives a sharp asymptotic fall-off of the physical scalar ϕ. It is instructive to compare this fall-off in terms of the affine parameter r along an outward going geodesic terminating on $\mathscr{I}$. When $\mathscr{I}$ is null we find

$$\Omega \sim \frac{1}{r} \qquad \to \qquad \phi \sim \frac{1}{r^{\frac{d-1}{2}}} \,. \tag{2.34}$$

On the other hand, in the de Sitter case, it is easily seen that $\Omega \sim \exp(-t)$, where t is the proper time along a time-like geodesic ending on $\mathscr{I}^+$.[8] Hence, for massive fields we find an exponential fall-off for the physical fields. In the asymptotically de Sitter case these conclusions can straightforwardly be extended to higher spin/helicity fields.

These considerations can straightforwardly be generalised to study the fall-off of scalars in asymptotically AdS or flat spaces. When $\lambda = 0$ however, the analysis is more involved for higher spin fields as they can scale differently according to whether they are aligned with $\mathscr{I}$ or transverse to it. Carefully doing so for the Weyl tensor results in a proper statement of the peeling theorem in asymptotically flat spaces, see for example [PR84, Kro23] for a careful statement.

Remark. As a final remark, note that in the peeling theorem one is usually focused on asymptotically flat space-times and the decay of gravitational waves at null infinity. In asymptotically AdS space-times, in particular in the context of holography, a more common way of analysing asymptotic expansions proceed through the use of the Fefferman-Graham expansion [FG85]. See Chapter 8 for more details in the context of holography.

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[8] This can be easily see by noting that $\mathrm{d}t = \frac{\mathrm{d}T}{T}$, where T is the time coordinate on the Einstein cylinder.

Chapter 3

Quantum fields in flat space

Quantum field theory in curved space-time is a generalisation of quantum field theory in flat space. It is not surprising that in many respects the behaviour of quantum fields in curved space-time can be directly inferred from the flat space theory. Local entities, such as commutation relations or field equations are determined by the principle of general covariance and the equivalence principle and will therefore remain unchanged. On the other hand, there are various global entities which will behave radically different in curved space. For example, in Minkowski space the vacuum is unambiguously determined by Poincaré invariance. However, as we will see in Chapter 4, the concept of a vacuum becomes ambiguous in curved space.

For this reason we will take a moment to review certain aspects of quantum field theory in flat space, fix our conventions and notation and highlight certain aspects which carry over to curved space as well as those which lose their meaning. To avoid having to deal carefully with gauge invariance etc. we mostly focus on scalar fields as they suffice to illustrate the properties we are interested in. For more details on spinors, vector or higher spin fields in Minkowski space we refer the reader to their favourite textbook on quantum field theory, for example [Wei95, PS95, Sre07].

3.1 Canonical quantisation in Minkowski space

In this chapter we exclusively deal with $(d+1)$-dimensional Minkowski space $\mathbb{R}^{1,d}$ where as before we use mostly plus signature. Sometimes it will useful to separate the time direction t and collect the spatial coordinates in the vector $\mathbf{x} = (x_1, \cdots, x_d)$.

Consider a classical scalar field $\phi(x)$ in $(d+1)$-dimensional Minkowski space $\mathbb{R}^{1,d}$, satisfying the Klein-Gordon field equation,

$$\left(\Box - m^2\right)\phi = 0\,, \tag{3.1}$$

where $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$ is the d'Alembertian, and m is the mass of the scalar field. This field equation can be derived from the action,

$$S = \int_{\mathbb{R}^{1,d}} \mathcal{L}\left(\phi, \partial_\mu\phi\right) \mathrm{d}^{d+1}x\,, \qquad \text{with} \qquad \mathcal{L}\left(\phi, \partial_\mu\phi\right) = -\frac{1}{2}\left(\eta^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi + m^2\phi^2\right), \tag{3.2}$$

by demanding that the variation δS with respect to ϕ vanishes. The conjugate momentum $\pi(x)$ is defined through the following definition,

$$\pi(x) = \frac{\delta\mathcal{L}(\phi, \partial\phi)}{\delta(\partial_t\phi)}\,, \tag{3.3}$$

such that for the free scalar we find $\pi(x) = \partial_t \phi$.

We proceed by defining the (classical) phase space V_ϕ as the vector space of fields satisfying the linear field equations, i.e. the Klein-Gordon equation in the case of the free scalar. There are some caveats regarding the asymptotic fall-off of these fields at infinity for the various formulae that we use to make sense. Without going into too much detail we assume that our fields have been appropriately restricted.[1] For the scalar field we denote the phase space as

$$V_\phi = \left\{ \phi \in C^\infty(\mathbb{R}^{1,d}) \,\middle|\, (\Box - m^2)\phi = 0 \right\} = \left\{ (\phi, \partial_t \phi) \in C^\infty(\mathbb{R}^d) \times C^\infty(\mathbb{R}^d) \right\}, \tag{3.4}$$

where in the second equality we identified the phase space with the initial data along a time slice $t = t_0$. A similar phase space can be defined for fermions and gauge fields but we leave the precise definition as an exercise to the reader.[2] The phase space comes with a natural symplectic structure, or skew symmetric form,[3]

$$\Omega(\phi_1, \phi_2) = \int_{\mathbb{R}^d} \phi_1 \star \mathrm{d}\phi_2 - \phi_2 \star \mathrm{d}\phi_1 = \int_{\mathbb{R}^d} (\phi_1 \partial_t \phi_2 - \phi_2 \partial_t \phi_1)\, \mathrm{d}^d x \tag{3.5}$$

More concretely, consider a basis $\{u_{\mathbf{k}}(x)\}$ of the phase space, and denote their conjugate momenta by $\{\pi_{\mathbf{k}}(x)\}$. In this basis the symplectic form takes the canonical form $\Omega = \sum_{\mathbf{k}} \mathrm{d}u_{\mathbf{k}} \wedge \mathrm{d}\pi_{\mathbf{k}}$ and the expression above can be obtained straightforwardly by expressing the fields ϕ_i in this basis. This skew symmetric form is dual to the Poisson bracket on V_ϕ which can be expressed as,

$$\left\{ \phi(x), \pi(x') \right\} = \delta^{(d)}(x - x'), \qquad \left\{ \phi(x), \phi(x') \right\} = \left\{ \pi(x), \pi(x') \right\} = 0. \tag{3.6}$$

Canonical quantisation then proceeds by promoting the fields and canonical momenta to operators on an appropriate Hilbert space and imposing the canonical (equal time) commutation relations, [4]

$$\begin{aligned} \left[\phi(t, \mathbf{x}), \phi(t, \mathbf{x}') \right] &= 0, \\ \left[\pi(t, \mathbf{x}), \pi(t, \mathbf{x}') \right] &= 0, \\ \left[\phi(t, \mathbf{x}), \pi(t, \mathbf{x}') \right] &= \mathrm{i}\, \delta(\mathbf{x} - \mathbf{x}'). \end{aligned} \tag{3.7}$$

To develop the quantum theory we relate the phase space to a one-particle Hilbert space $\mathcal{H}$ and define the Fock space to be

$$\mathcal{F} = \mathbb{C} \oplus \mathcal{H} \oplus (\mathcal{H} \otimes \mathcal{H}) \oplus \cdots = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes n}. \tag{3.8}$$

We can make this more precise using the Fourier transform, by decomposing complex fields into plane waves. An appropriately normalised set of solutions to the wave equation (3.1) is given by the

[1]For the interested reader: The relevant space is often taken to be the Schwartz space, which consists of infinitely differentiable functions that at infinity fall off faster than any reciprocal power of x. Crucially, this space has the property that it allows for the Fourier transform to be applied.

[2]Note that the Dirac equation is first order so that the initial value problem only needs the value of the field as boundary condition. For vector fields one has to deal with gauge invariance and quotient out gauge equivalent field configurations.

[3]For fermionic fields the phase space comes with a symmetric form.

[4]Note that here and throughout the text we put $\hbar = 1$ explaining the absence of the characteristic factor of $\hbar$ on the right hand side of the equations in (3.7).

following plane waves,

$$u_{\mathbf{k}}(t,\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^d 2\omega}} \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-\omega t)}\,, \tag{3.9}$$

with $\omega = \sqrt{\mathbf{k}^2 + m^2}$. The plane waves $u_{\mathbf{k}}$ are called positive frequency or positive energy solutions with respect to t, while the the complex conjugate solutions, $u^*_{\mathbf{k}}$, are the negative frequency/energy solutions. They are eigenfunctions of the time translation operators with eigenvalues $\mp \mathrm{i}\omega$,

$$\partial_t u_{\mathbf{k}} = -\mathrm{i}\,\omega\, u_{\mathbf{k}}\,, \qquad \partial_t u^*_{\mathbf{k}} = \mathrm{i}\,\omega\, u^*_{\mathbf{k}}\,. \tag{3.10}$$

Such plane waves, together with their complex conjugates, form a complete set of solutions and therefore any solution to the Klein-Gordon equation can be Fourier expanded as

$$\phi(x) = \int \mathrm{d}^d\mathbf{k} \left[a_{\mathbf{k}} u_{\mathbf{k}}(x) + a^\dagger_{\mathbf{k}} u^*_{\mathbf{k}}(x) \right]. \tag{3.11}$$

Such that the phase space for a complex scalar decomposes as $\mathbb{C}\otimes V_\phi = V^+_\phi \oplus V^-_\phi$ and the choice of ω above is precisely made so that the fields satisfy the on-shell condition, $k^2 = -m^2$.

The skew symmetric form on the phase space translates to a (positive definite) inner product on the Hilbert space defined as,

$$\langle \phi_1, \phi_2 \rangle = \mathrm{i} \int \mathrm{d}^d\mathbf{x}\, \left(\phi_1(x)\partial_t \phi_2(x) - \partial_t \phi_1(x)\phi_2(x) \right). \tag{3.12}$$

The main property of this inner product is that for two solutions to the Klein-Gordon equation the product is conserved under time translation. The plane waves defined in (3.9) are orthonormal with respect to this product,

$$\langle u_{\mathbf{k}}, u_{\mathbf{k}'} \rangle = \delta(\mathbf{k}-\mathbf{k}')\,, \qquad \langle u_{\mathbf{k}}, u^*_{\mathbf{k}'} \rangle = 0\,. \tag{3.13}$$

Exercise 3.1. *Prove that the plane waves $u_{\mathbf{k}}$ together with their complex conjugates form an orthonormal basis of $L^2(\mathbb{R}^{1,d})$, i.e. square integrable functions on $\mathbf{R}^{1,d}$.*[5]

We can therefore express the Fourier coefficients as follows,

$$a_{\mathbf{k}} = \langle u_{\mathbf{k}}, \phi \rangle\,, \qquad a^\dagger_{\mathbf{k}} = \langle u^*_{\mathbf{k}}, \phi \rangle\,. \tag{3.14}$$

From the equal time commutation relations (3.7), it follows that

$$\left[a_{\mathbf{k}}, a^\dagger_{\mathbf{k}'} \right] = \delta(\mathbf{k}-\mathbf{k}')\,, \qquad [a_{\mathbf{k}}, a_{\mathbf{k}'}] = \left[a^\dagger_{\mathbf{k}}, a^\dagger_{\mathbf{k}'} \right] = 0\,, \tag{3.15}$$

Exercise 3.2. *Prove the commutation relations,* (3.15) *, for the creation and annihilation operators starting from* (3.7).

Remark. In the above, $\mathbf{k}$ is a continuous parameter. Sometimes it is useful to introduce periodic

[5]Note that strictly speaking plane waves do not belong to $L^2(\mathbf{R}^{1,d})$ but nonetheless they form a complete basis for it. Similar comments apply to the dual basis given by the delta functions, $\delta(x - x_*)$.

boundary conditions on a torus T^{d+1} of volume L^{d+1}, a.k.a. putting the theory in a box. In particular, this avoids all kinds of volume divergences. Doing so, the wave-vectors become quantised $\mathbf{k} = \frac{2\pi}{L}\mathbf{n}$ and all integrals in the above get replaced by infinite sums.

It is often easier to compute physical quantities using such a (IR) regulator and taking the limit $L \to \infty$ at the end of the computation to obtain the infinite volume result.

Remark. In the above we consider the annihilation and creation operators as elements of $End(\mathcal{H})$. However, more rigorously, these operators are only well-defined in a smeared sense. Indeed, we should think of these operators as distribution-valued operators

$$a(f) = \int \mathrm{d}^d k f(k) a_k\,, \tag{3.16}$$

where f is a test function with Fourier coefficients $f(k)$.

For ϕ, we have that the associated creation operator can be defined as

$$a^\dagger(f) = -a(f^*)\,. \tag{3.17}$$

From the canonical commutation relations introduced above we then have

$$\begin{aligned}\left[a(f), a^\dagger(g)\right] =& \mathrm{i}\int \mathrm{d}^d\mathbf{x}\,(f^*(t,\mathbf{x})\partial_t g(t,\mathbf{x}) - \partial_t f^*(t,\mathbf{x})g(t,\mathbf{x}))\\ =& \langle f, g\rangle\,.\end{aligned} \tag{3.18}$$

Similarly, one can easily show that

$$[a(f), a(g)] = -\langle f, g^*\rangle\,, \qquad \left[a^\dagger(f), a^\dagger(g)\right] = -\langle f^*, g\rangle\,. \tag{3.19}$$

For all practical purposes it is entirely appropriate to work formally and think of the annihilation and creation operators and their smeared versions as elements of $End(\mathcal{H})$, acting on a dense subspace.

3.2 Particle interpretation

We can interpret the operators $a_\mathbf{k}$ and $a_\mathbf{k}^\dagger$ as annihilation and creation operators for an infinite amount of harmonic oscillators labelled by their momentum. In the Heisenberg picture, the states span a Hilbert space. A convenient basis for this Hilbert space is given by the Fock representation introduced above.

The state space consists of the vacuum $|0\rangle$, which is annihilated by all the annihilation operators,

$$a_\mathbf{k}|0\rangle = 0\,. \tag{3.20}$$

and all the excited states in the Hilbert, which space can be constructed by acting on the vacuum with the creation operators $a^\dagger_{\mathbf{k}}$,

$$a^\dagger_{\mathbf{k}}\,|0\rangle = |1_{\mathbf{k}}\rangle\ . \tag{3.21}$$

Successively acting with the creation operators we can then construct the most general states,

$$\prod_{i=1}^{p}\frac{1}{\sqrt{n_i!}}\left(a^\dagger_{\mathbf{k}_i}\right)^{n_i}|0\rangle = \left|n_1^{(\mathbf{k}_1)}n_2^{(\mathbf{k}_2)}n_3^{(\mathbf{k}_3)}\cdots n_p^{(\mathbf{k}_p)}\right\rangle . \tag{3.22}$$

These basis vectors are normalised such that

$$\left\langle n_1^{(\mathbf{k}_1)}n_2^{(\mathbf{k}_2)}n_3^{(\mathbf{k}_3)}\cdots n_p^{(\mathbf{k}_p)}\middle|\, m_1^{(\mathbf{k}_1)}m_2^{(\mathbf{k}_2)}m_3^{(\mathbf{k}_3)}\cdots m_q^{(\mathbf{k}_q)}\right\rangle =$$
$$\delta_{pq}\sum_\sigma \delta_{n_1 m_{\sigma(1)}}\delta_{n_2 m_{\sigma(2)}}\cdots\delta_{n_p m_{\sigma(p)}}\,, \tag{3.23}$$

where the sum runs over all permutations σ of the integers $1,...,p$.

In analogy with the harmonic oscillator we can then introduce the number operator,

$$N=\sum N_{\mathbf{k}}\,,\qquad\qquad N_{\mathbf{k}} = a^\dagger_{\mathbf{k}}a_{\mathbf{k}}\,, \tag{3.24}$$

whose expectation value in a generic state is given by

$$\langle\psi|\,N\,|\psi\rangle = \sum_i n_i\,, \tag{3.25}$$

where $|\psi\rangle = \left|n_1^{(\mathbf{k}_1)}n_2^{(\mathbf{k}_2)}n_3^{(\mathbf{k}_3)}\cdots n_p^{(\mathbf{k}_p)}\right\rangle$. Hence, we can interpret $N_{\mathbf{k}}$ and N as counting the number of quanta with momentum $\mathbf{k}$ and the total number of quanta respectively.

Note that the vacuum, as defined in (3.20) is unique. Naively it may seem to depend on a choice of inertial frame but an easy argument shows otherwise. Indeed, consider a second inertial frame $\tilde{x}^\mu = \Lambda^\mu{}_\nu x^\nu$ with Λ a Lorentz transformation. Analogous to the above we can define the positive frequency functions

$$\tilde{u}_{\tilde{\mathbf{k}}}(\tilde{t},\tilde{\mathbf{x}}) = \frac{1}{\sqrt{(2\pi)^d 2\tilde{\omega}}}\mathrm{e}^{\mathrm{i}(\tilde{\mathbf{k}}\cdot\tilde{\mathbf{x}}-\tilde{\omega}\tilde{t})}\,. \tag{3.26}$$

and expand the field ϕ as

$$\phi(\tilde{x}) = \mathrm{d}^d\tilde{\mathbf{k}}\left[\tilde{a}_{\tilde{\mathbf{k}}}\tilde{u}_{\tilde{\mathbf{k}}}(\tilde{x}) + \tilde{a}^\dagger_{\tilde{\mathbf{k}}}\tilde{u}^*_{\tilde{\mathbf{k}}(x)}\right]. \tag{3.27}$$

The "new" vacuum is then defined by the condition that

$$\tilde{a}_{\mathbf{k}}\left|\tilde{0}\right\rangle = 0\,. \tag{3.28}$$

To show that this is nothing but the old vacuum consider the mode function and notice that

$$\tilde{u}_{\mathbf{k}} = \frac{1}{\sqrt{(2\pi)^d 2\omega}}\mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\tilde{\mathbf{x}}-\omega\tilde{t})} = \left(\frac{\tilde{\omega}}{\omega}\right)^{\frac{1}{2}}\frac{1}{\sqrt{(2\pi)^d 2\tilde{\omega}}}\mathrm{e}^{\mathrm{i}(\tilde{\mathbf{k}}\cdot\mathbf{x}-\tilde{\omega}t)} = \left(\frac{\tilde{\omega}}{\omega}\right)^{\frac{1}{2}}u_{\tilde{\mathbf{k}}}\,. \tag{3.29}$$

Since we restrict to the orthochronous subgroup of the Lorentz group, to preserve time orientation, we have that $\tilde{\omega} > 0$ implies $\omega > 0$ and thus we find that

$$a_{\tilde{\mathbf{k}}} \left|\tilde{0}\right\rangle = 0 \quad \forall \tilde{\mathbf{k}} \qquad \Rightarrow \qquad a_{\mathbf{k}} \left|0\right\rangle = 0 \quad \forall \mathbf{k}\,, \tag{3.30}$$

and the converse follows by symmetry. Hence the vacuum is indeed unique and independent of the choice of frame.

3.3 Vacuum energy

To further explore the meaning of the Fock states we can compute their energy and momentum, which can be obtained from the expectation value of the energy-momentum tensor which for a scalar field is given by

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2}\eta_{\mu\nu}\left(\partial^\rho \phi \partial_\rho \phi - m^2 \phi^2\right). \tag{3.31}$$

We can define the conserved momentum operator as

$$P_\mu = \int_{t=t_0} T_{\mu 0} \mathrm{d}^d \mathbf{x}\,, \tag{3.32}$$

with the time-like component $P_0 = H$, the Hamiltonian. The expection value in a state $|\psi\rangle$ can then be computed as $\langle\psi| H |\psi\rangle$.

Exercise 3.3. *Give an expression for the Hamiltonian and conserved momentum in terms of the creation and annihilation operators and show that they commute with the number operator N.*

Naively compute the expectation value of the Hamiltonian density H and momentum density P_μ and show that they are divergent.

Similarly, consider the expectation value of the energy-momentum density $T_{\mu\nu}$ in the vacuum as well as in a generic state and find an expression in terms of the mode functions.

Computing the energy we encounter our first infinite result. Such troubling results are well-known to plague the subject of quantum field theory but can be cured through renormalisation. In this setup it will suffice to define the normal ordering operation, $: \bullet :$, which is understood to act on products of annihilation and creation operators such that it puts all the annihilation operators on the right of the creation operators.

Exercise 3.4 (Vacuum energy divergence)**.** *Compute the momentum and energy of the vacuum and show that it naively diverges. Use the normal ordering prescription to regularise this result and find the resulting vacuum energy.*

In the above we regularised the vacuum energy by passing through a normal ordered prescription which simply throws away the infinite contribution. However, as already mentioned before, in curved space, especially when gravity is included, the energy of the vacuum is physical since it gravitates. For this reason it is instructive to take a closer look at the vacuum expectation value of the energy in a situation where it becomes important. Consider a massless neutral scalar in $(3+1)$-dimensional Minkowski space in the presence of two parallel plates at $x_3 = 0$ and $x_3 = a$ with $a \ll 1$ and impose boundary conditions $\phi(0) = \phi(a)$ on the plates. In addition we impose periodic boundary conditions in the two other spatial directions with $x_{1,2} \sim x_{1,2} + L$, with $L \gg a$. This setup represents a modified version of the usual Casimir effect for two neutral conducting plates in a vacuum electric field vanishing on the plates.

Exercise 3.5. *Quantise the scalar field in the presence of the two plates and show that the average energy density is given by*

$$\rho(a) = a^{-1}L^{-2}\langle 0|\, T_{00}\, |0\rangle_a = \frac{1}{2aL^2}\sum_{\mathbf{k}} \omega_k\,, \tag{3.33}$$

and give an expression for ω_k.

We can regularise this sum by writing

$$\rho(a) = -\frac{1}{2aL^2}\lim_{\epsilon\to 0}\frac{\mathrm{d}}{\mathrm{d}\epsilon}\sum_{\mathbf{k}} \mathrm{e}^{-\epsilon\omega_k} \tag{3.34}$$

Remark. The parameter ϵ here is dimensionful so it is good practice to introduce an explicit length scale $\epsilon \to \epsilon/\Lambda$ so that ϵ becomes dimensionless. In the final result all the Λ dependence should drop out as can be easily verified.

Exercise 3.6. *Compute the regularised vacuum energy and show that the sum*

$$S(\epsilon, a) = L^{-2}\sum_{\mathbf{k}} \exp(-\epsilon\omega_k) \tag{3.35}$$

takes the form

$$\pi S(\epsilon, a) = aG(\alpha) + \frac{\pi^3}{45a^3}\epsilon + \mathcal{O}(\epsilon^2)\,. \tag{3.36}$$

Before taking the limit in (3.34), subtract the infinite part to arrive at the renormalised vacuum energy. Doing so compute $\rho(a)$.

What would have changed if we instead imposed vanishing boundary conditions at the plates?

Remark. In the last exercise we proceeded in a rather cavalier way. In order to obtain the same result in a mathematically more satisfying manner one can employ ζ-function regularisation. We invite the interested reader to explore this method and repeat this exercise in a more rigorous manner.

3.4 Symmetries, charges and topological operators

When the quantum theory enjoys some global symmetries, there will be associated conserved currents. The theories we consider in this course are all covariant under space-time diffeomorphisms. In particular, they are invariant under space and time translations. The associated conserved current is given by the stress tensor,

$$\partial^\mu T_{\mu\nu}\,, \qquad \text{where} \qquad T_{\mu\nu} = \frac{\delta \mathcal{L}}{\delta \partial^\mu \phi} \partial_\nu \phi - \delta_{\mu\nu} \mathcal{L}\,. \tag{3.37}$$

Note that in order for this to serve as a source for the Einstein equations we need to improve it to be symmetric under interchanging the indices. We will present a manifestly symmetric stress tensor formulation of the stress tensor later when dealing with curved space-times. Similarly, if the theory has additional global symmetries we can write the conserved current as

$$\partial^\mu J_\mu = 0\,, \qquad \text{where} \qquad J^\mu = \frac{\delta \mathcal{L}}{\delta \partial^\mu \phi} \delta \phi - T_{\mu\nu} \delta x^\nu\,, \tag{3.38}$$

where $\delta\phi = \phi' - \phi$ denotes the change of the field under an infinitesimal symmetry transformation and similarly $\delta x^\nu = x^{\nu\prime} - x^\nu$.

In a more modern incarnation, symmetries are often rephrased as the presence of topological operators. To illustrate this viewpoint, let us consider a general QFT with a continuous U(1) global symmetry. We define the associated charge operator Q as

$$Q = \oint \mathrm{d}^d x J_0\,. \tag{3.39}$$

The charge Q is conserved because of the conservation equation

$$\partial_t Q = \int \mathrm{d}^d x \partial_t J_t = -\int \mathrm{d}^d x \nabla^m J_m = -\int_{\partial \mathbb{R}^d} \mathrm{d}S_m J_m = 0\,, \tag{3.40}$$

where we assume appropriate fall-off conditions of the fields at infinity. The conserved, unitary operator U_θ implementing a U(1) symmetry transformation with angle θ is

$$U_\theta = \mathrm{e}^{\mathrm{i}\theta Q} = \exp\left[\mathrm{i}\theta \oint \mathrm{d}^d x J_t\right]. \tag{3.41}$$

The unitary symmetry operator can be generalised in a covariant way as follows. Define

$$U_\theta(\mathcal{M}_d) = \exp\left[\mathrm{i}\theta \oint_{\mathcal{M}_d} J_\mu \,\mathrm{d}n^\mu\right] = \exp\left[\mathrm{i}\theta \oint_{\mathcal{M}_d} \star J\right], \tag{3.42}$$

where n is a normal vector to the co-dimension one submanifold $\mathcal{M}_d$ and $\star$ denotes the Hodge dual of a differential form (see Appendix B for a definition). Since $\star J$ is a closed form, i.e. $\mathrm{d} \star J = 0$, it is easy to see that correlation functions involving $U_\theta(\mathcal{M}_d)$ are independent of small deformations of $\mathcal{M}_d$, as long as they do not cross any operators charged under the U(1) symmetry. Therefore, we see that a conserved current operator J_μ gives rise to a topological object $U_\theta(\mathcal{M}_d)$ supported on a codimension one submanifold in space-time.

So what are these operators $U_\theta(\mathcal{M}_d)$? When $\mathcal{M}_d$ spans the spatial directions at a fixed time $U_\theta(\mathcal{M}_d)$ is the conserved, unitary operator that acts on the Hilbert space $\mathcal{H}(\mathcal{M}_d)$. On the other hand, when $\mathcal{M}_d$ is extended along the time direction and localised in one of the spatial directions, say $x_1 = 0$,

$U_\theta(\mathcal{M}_d)$ is a defect that modifies the quantisation. Indeed, the quantisation in the presence of such a defect gives rise to a twisted Hilbert space $\mathcal{H}_\theta(\mathcal{M}_d)$.

Example 3.1 (Complex scalar in 1+1 d)**.** *To make these concepts more concrete, let us consider a free complex scalar in two dimensions. The Lagrangian for this theory is given by*

$$\mathcal{L} = \partial_\mu \Phi \partial^\mu \Phi^\dagger , \tag{3.43}$$

and enjoys a global U(1) *symmetry acting by a phase* $\Phi \to e^{i\theta}\Phi$ *with Noether current*

$$J_\mu = i\left(\partial_\mu \Phi^\dagger\right)\Phi - i\Phi^\dagger\left(\partial_\mu \Phi\right) . \tag{3.44}$$

Let the space be a circle S^1 *parametrized by the coordinate* $x \sim x+2\pi$. *The Hilbert space* $\mathcal{H}(S^1)$ *on a circle is obtained by the canonical quantization of the free scalar field subject to the periodic boundary condition* $\Phi(t, x+2\pi) = \Phi(t,x)$. *The conserved current leads to a unitary operator* $U_\theta(S^1) = \exp\left(i\theta \oint dx J_t\right)$ *acting on this Hilbert space.*

Alternatively, we can insert a defect $U_\theta(\mathbb{R}) = \exp\left(i\theta \oint dt J_x\right)$ *along the time direction at* $x = 0$. *This defect changes the boundary condition of the scalar field to* $\Phi(t, x+2\pi) = e^{i\theta}\Phi(t,x)$. *Canonical quantization subject to the above twisted boundary condition leads to a twisted Hilbert space* $\mathcal{H}_\theta(S^1)$ *labelled by the* U(1) *group element* $\theta \in [0, 2\pi)$.

This modern perspective in terms of topological defects not only neatly unifies defects and symmetry operators but generalises in many interesting ways. First of all, it immediately generalises to discrete symmetries G, like $\mathbb{Z}_N$. In this case we do not have a conserved Noether current or charge operator but we can still construct the topological operators.

Another interesting generalisation are higher form symmetries [GKSW15]. One can consider conserved/topological operators supported on higher codimension submanifolds. These generate so-called higher form symmetries which act on extended objects. I.e. 0-form symmetries act on particles, 1-form symmetries act on line operators, and so on. A q-form global symmetry $G^{(q)}$ acts on a q dimensional object W as

$$U_g(\mathcal{M}_{d-q})W(\mathcal{N}_q) = g(W)W(\mathcal{N}_q) , \tag{3.45}$$

where $\mathcal{M}_{d-q}$ and $\mathcal{N}_q$ are two submanifolds linking in space-time and $g(W)$ is a representation of $G^{(q)}$. Note that $G^{(q)}$ for $q > 0$ is necessarily abelian, since one can always move two higher-codimension objects around each other in a topological manner so that the order in which then act can not make a difference.

Example 3.2. *One of the simplest examples of a higher form symmetry arises in Maxwell theory with no charged matter. In this case one can define the topological operators*

$$U_\theta(\mathcal{M}_{d-1}) = \exp\left[-\frac{\theta}{e^2}\oint_{\mathcal{M}_{d-1}} \star F\right] , \tag{3.46}$$

where e is the electric coupling in the Maxwell action and the exponent in this expression is nothing but the electric flux which is topological thanks to Gauss' law $d \star F = 0$. *This topological operator implements* *a* $\mathrm{U}(1)^{(1)}$ *1-form global symmetry. The charged objects in this case are (non-topological) Wilson lines*

$$W(\mathcal{N}_1) = \exp\left[in \oint_{\mathcal{N}_1} A \right], \tag{3.47}$$

which are acted upon as follows,

$$U_\theta(\mathcal{M}_{d-1}) \cdot W_{\mathcal{N}_1} = \mathrm{e}^{\mathrm{i}n\theta} W_{\mathcal{N}_1} . \tag{3.48}$$

Due to the Bianchi identities $\mathrm{d}F = 0$ *there is a second* $\mathrm{U}(1)^{(1)}$ *symmetry which instead acts on 't Hooft lines.*

Remark. One can further generalise the concept of a symmetry by considering not just group-like symmetries but ring-like symmetries, where not every element has an inverse. We will not go into details of such non-invertible symmetries but refer the interested reader to the lecture notes [Sha23].

3.5 Correlation functions

On top of the vacuum energy and Hilbert space, much information about a quantum field theory is encoded in its n-point functions. Euclidean and Lorentzian correlation functions are related by analytic continuation. However, due to the more interesting causal structure in Lorentzian signature there are various subtleties related to this continuation.

These subtleties are most easily illustrated in a quantum mechanical setup which straightforwardly generalises to quantum field theory. Let us there fore consider a quantum mechanical system with Hermitian Hamiltonian H, whose spectrum is bounded from below. For energy eigenstates we have $H|\psi\rangle = E_\psi |\psi\rangle$ and we assume that there exists a unique vacuum state $|0\rangle$ with $H|0\rangle = 0$, and all other states have $E_\psi > 0$.

Let us consider first the Euclidean system with Euclidean time τ, and time evolution operator $\mathrm{e}^{-\tau H}$. Given a local operator $\mathcal{O}(0)$ we define the Euclidean Heisenberg picture operator as

$$\mathcal{O}_E(\tau) = \mathrm{e}^{\tau H} \mathcal{O}(0) \mathrm{e}^{-\tau H} . \tag{3.49}$$

A general correlation function in the vacuum state is then given by[6]

$$\begin{aligned} \langle 0| \mathcal{O}_{1,E}(\tau_1) \cdots \mathcal{O}_{n,E}(\tau_n) |0\rangle &= \big\langle \mathcal{O}_{1,E}(\tau_1) \cdots \mathcal{O}_{n,E}(\tau_n) \big\rangle \\ &= \big\langle \mathcal{O}_{1,E}(0) \mathrm{e}^{-(\tau_1-\tau_2)H} \mathcal{O}_{2,E}(0) \cdots \mathcal{O}_{n-1,E}(0) \mathrm{e}^{-(\tau_{n-1}-\tau_n)H} \mathcal{O}_{n,E}(0) \big\rangle \\ &= \sum_{\psi_i} \langle 0| \mathcal{O}_{1,E}(0) |\psi_1\rangle \langle \psi_1| \mathcal{O}_{2,E}(0) |\psi_2\rangle \cdots \langle \psi_{n-1}| \mathcal{O}_{n,E}(0) |0\rangle \\ &\quad \times \mathrm{e}^{-(\tau_1-\tau_2)E_1} \cdots \mathrm{e}^{-(\tau_{n-1}-\tau_n)E_{n-1}} , \end{aligned} \tag{3.50}$$

where we inserted complete sets of energy eigenstates ψ_i. This simple rewriting leads us to a first

[6] For ease of notation we often denote vacuum expectation values simply as $\langle \mathcal{O} \rangle$ omitting the explicit state $|0\rangle$.

important observation. In Euclidean signature, only time-ordered correlation functions of local operators make sense. Indeed, in the time-ordered case the evolution operators $e^{-(\tau_1-\tau_2)H}$ cause the high-energy states to be exponentially damped. Since Euclidean correlators can have at most power-law divergences as times become coincident, a similar statement holds for transitions between non-vacuum states $\langle\psi|\,\mathcal{O}(0)\,|\psi'\rangle$. For this reason we only consider time ordered Euclidean correlators

$$\begin{aligned}\left\langle\mathcal{O}_{1,E}(\tau_1)\cdots\mathcal{O}_{n,E}(\tau_n)\right\rangle &= \left\langle T\left\{\mathcal{O}_{1,E}(\tau_1)\cdots\mathcal{O}_{n,E}(\tau_n)\right\}\right\rangle \\ &= \left\langle\mathcal{O}_{1,E}(\tau_1)\cdots\mathcal{O}_{n,E}(\tau_n)\right\rangle\theta(\tau_1>\cdots>\tau_n) \\ &\quad + \text{permutations.}\end{aligned} \tag{3.51}$$

On the other hand, consider a Lorentzian system with time t. The Lorentzian time evolution operator is e^{-itH} and the Lorentzian Heisenberg picture operators are defined as

$$\mathcal{O}_L(t) = e^{itH}\mathcal{O}(0)e^{-itH} = \mathcal{O}_E(\tau = it)\,. \tag{3.52}$$

We can go through exactly the same manipulations as in (3.50) for a Lorentzian correlation function $\langle 0|\,\mathcal{O}_{1,L}(t_1)\cdots\mathcal{O}_{n,L}(t_n)\,|0\rangle$ where the only change is that the exponential factors are now oscillatory for any time ordering! Hence it makes sense to define any time ordering, or consider commutators $\left\langle\left[\mathcal{O}_{1,L}(t_1),\mathcal{O}_{2,L}(t_2)\right]\right\rangle$ which in Euclidean signature does not make sense because it always involves at least one unbounded operator. In Lorentzian QFTs, correlators with a fixed operator ordering are called Wightman functions. By contrast, a time-ordered Lorentzian correlator is a sum of Wightman functions with θ-functions enforcing the time ordering. The face that the e^{itH} insertions are oscillatory has an important consequence: Wightman functions are not actually functions. Instead they should be treated as distributions where we must smear the times t_i against smooth test functions to make the sum over high energy states converge,

$$\int dt_1\cdots dt_n f_1(t_1)\cdots f_n(t_n)\left\langle\mathcal{O}_{1,L}(t_1)\cdots\mathcal{O}_{n,L}(t_n)\right\rangle\,. \tag{3.53}$$

To observe this distributional character in more detail, let us explicitly construct the Lorentzian correlators as an analytic continuation of the Euclidean ones. However, note that the collection of θ-functions in the time ordered correlator cannot be continued in a natural way. Instead, we should think of the Euclidean correlator as a collection of different functions, one for each ordering of the τ_i. We can then separately continue each of these functions. For example,

$$F(\tau_1,\tau_2,\cdots,\tau_n) = \left\langle\mathcal{O}_{1,E}(\tau_1)\cdots\mathcal{O}_{n,E}(\tau_n)\right\rangle\,, \tag{3.54}$$

can be analytically continued to a holomorphic function of its arguments in the region

$$U:\quad \operatorname{Re}\tau_1>\cdots>\operatorname{Re}\tau_n\,. \tag{3.55}$$

Let us then write $\tau_i = \epsilon_i + it_i$ and change t_i away from zero. Upon this change, the time-evolution operator becomes

$$e^{-(\epsilon_1-\epsilon_2)H} \to e^{-(\epsilon_1-\epsilon_2)H-i(t_1-t_2)H}\,, \tag{3.56}$$

As long as we stay in the region U, the high-energy states remain exponentially damped. The imaginary parts t_i insert additional phases into the already convergent sums, thus the correlator is holomorphic in this region.

A Wightman function can be obtained as the boundary value of such a holomorphic function. Since F is holomorphic in the domain U we can safely continue t_i from 0 to any desired value. Afterwards we can then take the limit $\epsilon_i \to 0$, preserving the ordering. Formally, this produces the Wightman function

$$\lim_{\epsilon_i \to 0} \left\langle \mathcal{O}_{1,E}(\epsilon_1 + \mathrm{i}t_1) \cdots \mathcal{O}_{n,E}(\epsilon_n + \mathrm{i}t_n) \right\rangle = \left\langle \mathcal{O}_{1,L}(t_1) \cdots \mathcal{O}_{n,L}(t_n) \right\rangle . \tag{3.57}$$

However, this limit is not always well-defined because it requires approaching the boundary of the region of holomorphicity. We claim that if we smear against test functions $f_i(t_i)$, and subsequently take $\epsilon_i \to 0$, then the limit becomes well-defined. This is how the boundary value of F is defined as a distribution.

Remark.

In the above we have not defined the space of test functions relevant for QFT correlators. The technical statement is that Wightman functions are tempered distributions.

Given a space of functions $\mathcal{F}$, a distribution T on $\mathcal{F}$ is a continuous linear function $T : \mathcal{F} \to \mathbb{C}$. Formally we can write this as

$$T(f) = \int \mathrm{d}t f(t) T(t) \in \mathbb{C} , \tag{3.58}$$

Although the value $T(t)$ might not make sense.[7] The space of test functions relevant in QFT is the Schwartz space $\mathcal{S}$ of rapidly decreasing functions,

$$\mathcal{S} = \left\{ f \in C^\infty(t) \,|\, \sup |t^m \partial_t^n f(t)| < \infty \,, m, n \in \mathbb{Z}_{\geq 0} \right\} . \tag{3.59}$$

Distributions on $\mathcal{S}$ are called tempered distributions.

Exercise 3.7. *Consider a correlator $F(\tau)$ that is holomorphic in a single variable $\tau = \epsilon + \mathrm{i}t$ in the region $\epsilon > 0$. Assume that $F(\tau)$ has at most a power-law divergence as we approach the boundary of the regime of holomorphicity*

$$|F(\epsilon + it)| \leq \epsilon^{-k} P(t) , \tag{3.60}$$

where $P(t)$ is polynomially bounded for large t.

Show that $F(t)$ is a tempered distribution, i.e. show that $\lim_{\epsilon \to 0} \int \mathrm{d}t F(\epsilon + \mathrm{i}t) f(t)$ is finite for $f \in \mathcal{S}$.

The discussion above generalises straightforwardly to QFT by introducing additional spatial directions. Now, the Heisenberg picture operators are defined as

$$\mathcal{O}(x) = e^{-iP\cdot x} \mathcal{O}(0) e^{iP\cdot x} , \tag{3.61}$$

[7] A famous example is the Dirac δ-distribution $T(t) = \delta(t)$, which is only defined by its integral against test functions.

where P^μ are the energy-momentum generators. The signs above are fixed by requiring that the t-dependence of the right-hand operator is e^{-itH}, where $H = P^0$. We can go through exactly the same manipulations where now all positions can be analytically continued, $x_k = y_k + i\zeta_k$ where $y_k, \zeta_k \in \mathbb{R}^{1,d}$. The generalisation of the statement that H is bounded from below is that the spectrum of P is contained inside the future null cone. Consequently, the the real part of the exponential $e^{iP\cdot x_{12}}$ is negative provided that ζ_{12} is past-directed, which we write as $\zeta_{12} < 0$ or $\zeta_1 < \zeta_2$. When this condition holds, high energy states are exponentially damped. Thus, positivity of energy implies that the Wightman function is holomorphic in the region

$$\zeta_1 < \zeta_2 < \cdots < \zeta_n. \tag{3.62}$$

Wightman distributions in real space are defined as boundary values of holomorphic functions in the region (3.62). We can compute them with the following recipe:

- Start with x_i real and mutually space-like. For example, place them at times $t_i = 0$.
- Give the x_i small imaginary parts $\zeta_i = \operatorname{Im} x_i$ satisfying (3.62). For example, if all points start at Lorentzian time $t_i = 0$, we can assign them times $x_i^0 = -i\epsilon_i$ with $\epsilon_1 > \cdots > \epsilon_n$.
- Continue the real parts $y_i = \operatorname{Re} x_i$ to the desired values.
- Take the imaginary parts to ζ_i zero, treating the result a distribution.

The Osterwalder-Schrader reconstruction theorem states that correlators in a reflection positive Euclidean QFT can be analytically continued to give Wightman functions that are tempered distributions on Minkowski space $\mathbb{R}^{1,d-1}$.

3.6 Two-point functions

Among the correlation functions a particularly important role is played by two-point functions. In fact, in many applications we will effectively only work with free fields for which all non-trivial information is encoded in the two-point functions, i.e. the Green's functions for the relevant wave equation.

While in Euclidean signature only the "time"-ordered correlator makes sense, in Lorentzian signature there are various types of Green's functions depending on the choice of integration contour in the complex plane.

A useful set of Green functions are given by the following expectation values,

$$\begin{aligned} G^+(x_1, x_2) &= \langle 0| \mathcal{O}(x_1)\mathcal{O}(x_2) |0\rangle \, , \\ G^-(x_1, x_2) &= \langle 0| \mathcal{O}(x_2)\mathcal{O}(x_1) |0\rangle \, , \\ iG(x_1, x_2) &= \langle 0| [\mathcal{O}(x_1), \mathcal{O}(x_2)] |0\rangle \, , \\ G^{(1)}(x_1, x_2) &= \langle 0| \{\mathcal{O}(x_1), \mathcal{O}(x_2)\} |0\rangle \, . \end{aligned} \tag{3.63}$$

$G^\pm$ are Wightman functions as discussed in the previous section, while the other combinations can be built up from combinations of the Wightman functions. G is known as the Pauli-Jordan, or Schwinger function, while $G^{(1)}$ is called the Hadamard elementary function. These Green functions can be split into their positive and negative part,

$$\mathrm{i}G = G^+ - G^- \,, \qquad \text{and} \qquad G^{(1)} = G^+ + G^- \,, \tag{3.64}$$

Finally, the Feynman propagator G_F and retarded/advanced Green functions $G_{R/A}$ are defined as

$$G_F(x_1, x_2) = \langle 0| \, T\left[\mathcal{O}(x_2)\mathcal{O}(x_1)\right] |0\rangle = \theta(t_1 - t_2)G^+(x_1, x_2) + \theta(t_2 - t_1)G^-(x_1, x_2)\,, \tag{3.65}$$

where T denotes time ordering, and

$$G_R(x_1, x_2) = \theta(t_1 - t_2)G(x_1, x_2)\,, \qquad G_A(x_1, x_2) = -\theta(t_2 - t_1)G(x_1, x_2)\,. \tag{3.66}$$

All these two-point functions have the same form in momentum space but they have different iϵ prescriptions and are used in different contexts. Time ordered products, such as the Feynman or retarded/advanced Green's functions are relevant for S-matrix calculations. The Green's functions involving (anti-)commutators on the other hand are useful to describe how the field responds to a source. Finally, the Wightman functions are useful since they describe the effect of the field on a moving detector, as will be described in detail in Chapter 6.

Exercise 3.8. *Show that the average of the retarded and advanced Green functions $\overline{G} = \frac{1}{2}(G_R + G_A)$ is given by*

$$G_F(x_1, x_2) = \mathrm{i}\overline{G}(x_1, x_2) + \frac{1}{2}G^{(1)}(x_1, x_2)\,. \tag{3.67}$$

In the above we kept the operator $\mathcal{O}$ completely generic, but let us now specify to scalar fields and see how the two point functions can be obtained respectively as an analytic continuation and next by carefully applying the various iϵ prescriptions.

To illustrate the first perspective, let us start by considering a two-point function of a massless scalar $\mathcal{O}$ in a $d+1$ dimensional CFT. The Euclidean correlator is

$$\langle \mathcal{O}_E(\tau_1, \mathbf{x}_1)\mathcal{O}_E(\tau_2, \mathbf{x}_2)\rangle = \frac{1}{(\tau_{21}^2 + \mathbf{x}_{21}^2)^\Delta}, \tag{3.68}$$

where Δ is the dimension of $\mathcal{O}$. Let us compute the Wightman function

$$\langle \mathcal{O}(t_2, \mathbf{x}_2)\mathcal{O}(t_1, \mathbf{x}_1)\rangle\,. \tag{3.69}$$

We start with the operators $\mathcal{O}_E$ at Euclidean times $\tau_2 = \epsilon_2 > \tau_1 = \epsilon_1$ and then continue $\epsilon_i \to \epsilon_i + it_i$, staying in the region $\epsilon_2 > \epsilon_1$,

$$\langle \mathcal{O}_E(\epsilon_2 + it_2, \mathbf{x}_2)\mathcal{O}_E(\epsilon_1 + it_1, \mathbf{x}_1)\rangle = \frac{1}{(-t_{21}^2 + \mathbf{x}_{21}^2 + 2i\epsilon_{21}t_{21} + \epsilon_{21}^2)^\Delta}. \tag{3.70}$$

Finally, we take $\epsilon_{21} = \epsilon \to 0$,

$$\langle \mathcal{O}(t_2, \mathbf{x}_2)\mathcal{O}(t_1, \mathbf{x}_1)\rangle = \lim_{\epsilon\to 0} \frac{1}{(x_{21}^2 + i\epsilon t_{21})^\Delta}, \tag{3.71}$$

where we have used the Minkowski norm $x^2 = -t^2 + \mathbf{x}^2$.

The denominator involves a fractional power of a complex number. However, its phase is fixed by our prescription for analytic continuation.

Exercise 3.9. *Show that the phases in different causal configurations are*

$$\langle \mathcal{O}(t_2, \mathbf{x}_2) \mathcal{O}(t_1, \mathbf{x}_1) \rangle = \frac{1}{|x_{21}^2|^\Delta} \times \begin{cases} e^{-i\pi\Delta} & x_2 > x_1 \\ 1 & x_1 \approx x_2 \\ e^{i\pi\Delta} & x_1 > x_2 \end{cases} . \tag{3.72}$$

Here, we write $x_i > x_j$ to denote that x_i is in the future of x_j, and $x_i \approx x_j$ to denote that x_i is spacelike from x_j.

To compute the other ordering $\langle \mathcal{O}(t_1, \mathbf{x}_1) \mathcal{O}(t_2, \mathbf{x}_2) \rangle$, we choose $\epsilon_{12} = \epsilon > 0$, which leads to

$$\langle \mathcal{O}(t_1, \mathbf{x}_1) \mathcal{O}(t_2, \mathbf{x}_2) \rangle = \lim_{\epsilon \to 0} \frac{1}{(x_{12}^2 + i\epsilon t_{12})^\Delta} , \tag{3.73}$$

which differs from (3.71) only in its $i\epsilon$ prescription. In particular, the expectation value of the commutator $\langle [\mathcal{O}(t_1, \mathbf{x}_1), \mathcal{O}(t_2, \mathbf{x}_2)] \rangle$ vanishes at space-like separation, as it should by microcausality. Henceforth, when writing Wightman distributions, we often leave $\lim_{\epsilon \to 0}$ implicit.

Next, let us consider the free scalar field in $d+1$ dimensions. Using the field equations (3.1) one can show that the Green functions $\mathcal{G} \in \{G, G^{(1)}, G^{\pm}\}$ all satisfy the homogeneous equation

$$(\Box_{x_1} + m^2) \mathcal{G}(x_1, x_2) = 0 , \tag{3.74}$$

while the Feynman and retarded/advanced Green functions satisfy

$$\begin{aligned} (\Box_{x_1} + m^2) G_F(x_1, x_2) &= -\delta^{(d+1)}(x_1 - x_2) , \\ (\Box_{x_1} + m^2) G_{A/R}(x_1, x_2) &= \delta^{(d+1)}(x_1 - x_2) . \end{aligned} \tag{3.75}$$

We can use translation invariance to translate one of the points, say x_2 to the origin in which case we can denote the Green functions by $\mathcal{G}(x) = \mathcal{G}(x, 0)$. It si straightforward to solve (3.74) in momentum space where the solution is simply given by

$$\widetilde{G}(k) = \frac{-\mathrm{i}}{k^2 + m^2} , \tag{3.76}$$

Having done so, an integral representations for the Green functions can be obtained by substituting the mode decomposition (3.11) in the definitions above. All the Green functions can then be represented as

$$\mathcal{G}(x) = \frac{1}{(2\pi)^{d+1}} \int \mathrm{d}^{d+1}k \frac{\mathrm{e}^{-\mathrm{i}k \cdot x}}{k^2 - m^2} . \tag{3.77}$$

This integral has poles at $\omega = \pm\sqrt{|\mathbf{k}^2 + m^2|}$ and the various Green functions correspond to various contour prescriptions for the integration. In Figure 3.1, the contours are shown for the various Green functions.

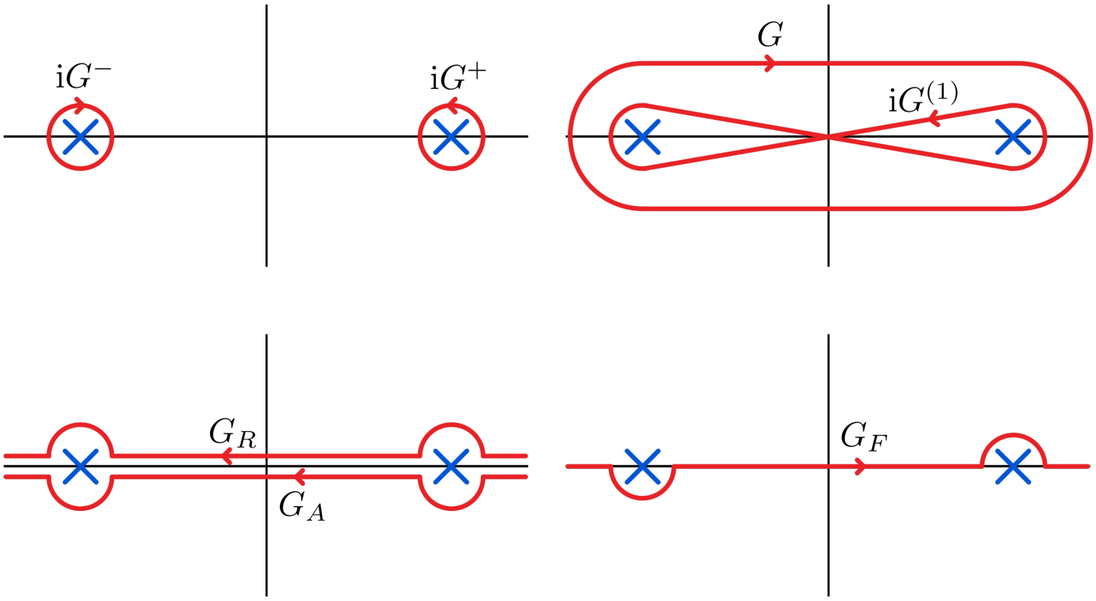

Figure 3.1: The various Green functions are associated with the above contours for the integral (3.77). The open contours should be interpreted as closed by an infinitely large semicirle in the upper/lower plane.

From (3.64)–(3.66) we can see that we can obtain all the Green functions from the Wightman functions $G^{\pm}$ so we mostly focus on those.

Exercise 3.10. *Use the mode expansion of the scalar field to show that the Green functions take the manifestly Lorentz invariant form* (3.77).

Example 3.3 (Massless scalar)**.** *To illustrate the formalism let us consider a massless scalar in four-dimensional Minkowski space. To compute the Wightman functions perform the relevant contour integration as indicated in Figure 3.1. We focus on G^+ but the computation for G^- is entirely analogous. Near the pole the integrand reduces to*

$$I_+ = \frac{1}{2\omega_{\mathbf{k}}} \mathrm{e}^{-ik\cdot x}\,, \tag{3.78}$$

where $\omega_{\mathbf{k}} = |\mathbf{k}|$. Using Cauchy's residue theorem we obtain

$$G^+(x) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 2|\mathbf{k}|} \mathrm{e}^{-\mathrm{i}(|\mathbf{k}|t - \mathbf{k}\cdot\mathbf{x})}\,. \tag{3.79}$$

We can use spherical coordinates to rewrite this as

$$G^+(x) = \int \frac{|\mathbf{k}|\mathrm{d}|\mathbf{k}| \sin\theta \mathrm{d}\theta \mathrm{d}\varphi}{2(2\pi)^3} \mathrm{e}^{\mathrm{i}|\mathbf{k}|(r\cos\theta - t)} = -\frac{\mathrm{i}}{2(2\pi)^2 r} \int_0^\infty \mathrm{d}|\mathbf{k}| \left(\mathrm{e}^{-\mathrm{i}|\mathbf{k}|(t-r)} - \mathrm{e}^{-\mathrm{i}|\mathbf{k}|(t+r)}\right)\,. \tag{3.80}$$

This integral is divergent and needs to be regularised. We do so by shifting $t \to t - \mathrm{i}\epsilon$, with $\epsilon > 0$ so that

$$\int_0^\infty \mathrm{d}|\mathbf{k}| \mathrm{e}^{-\mathrm{i}|\mathbf{k}|(t - \mathrm{i}\epsilon \pm r)} = -\frac{\mathrm{i}}{t \pm r - \mathrm{i}\epsilon}\,. \tag{3.81}$$

This has to be understood as a distribution, since we have (Sokhotski-Plemelj theorem)

$$\lim_{\epsilon\to 0}\frac{1}{x\pm \mathrm{i}\epsilon}=\mathcal{P}\frac{1}{x}\mp \mathrm{i}\pi\delta(x)\,, \tag{3.82}$$

where $\mathcal{P}$ denotes the Cauchy principal value. Using this we find the Wightman function

$$G^{+}(x)=-\frac{1}{4\pi^2}\mathcal{P}\frac{1}{t^2-r^2}+\frac{\mathrm{i}}{8\pi r}\left(\delta(t+r)-\delta(t-r)\right). \tag{3.83}$$

The massless scalar field is conformal with conformal dimension $\Delta=\frac{d-1}{2}$. Show that the result obtained here agrees with the general discussion for conformal scalar fields above.

Exercise 3.11. *Use this result to compute the Feynman propagator and Hadamard's elementary function,*

$$G_F(x)=\frac{\mathrm{i}}{4\pi^2}\mathcal{P}\frac{1}{x^2}-\frac{1}{4\pi}\delta(x^2)\,,\qquad G^{(1)}=-\frac{1}{2\pi^2}\mathcal{P}\frac{1}{x^2}\,. \tag{3.84}$$

Exercise 3.12 (Massive scalar)**.** *Use spherical coordinates in the spatial directions to simplify the integral and find an explicit expression for the Wightman functions for a massive scalar.*

Show that for space-like separated points, the Wightman function takes the form

$$G^{+}(x_1,x_2)=\frac{m}{4\pi^2 s}K_1(ms)\,,\qquad s=\sqrt{-(x_1-x_2)^2}\,, \tag{3.85}$$

where s is the proper distance and K_1 is a Bessel function of the second kind. Show that for timelike separated points it becomes

$$G^{+}(x_1,x_2)=\frac{\mathrm{i}m}{8\pi\tau}H_1^{(2)}(m\tau)\,,\qquad \tau=\sqrt{(x-y)^2}\,, \tag{3.86}$$

where τ is the proper time and $H_1^{(2)}$ is a Hankel function of the second kind.

What is the behaviour for large space-like separation $r\gg 1$? Give a physical explanation of this behaviour.

What happens for light-like separated points? You can study this by taking the limit $s\to 0$ in (3.85) or $\tau\to 0$ in (3.86). Show that there is a branch point singularity in the Wightman distribution. This essential singularity is known as the lightcone singularity of Wightman functions at null separation.

In order to perform the calculations of the (Feynman) Green's function it is often useful to rotate the contour by 90 degrees to obtain the Euclidean Green's function G_E. The integration variables are changed to $\kappa=-\mathrm{i}\omega$ and similarly we replace $\tau=-\mathrm{i}t$, such that we have the relation

$$G_F(t,\mathbf{x})=-\mathrm{i}G_E(\mathrm{i}\tau,\mathbf{x})\,. \tag{3.87}$$

where

$$G_E(\tau,\mathbf{x})=\frac{1}{(2\pi)^{d+1}}\int\frac{e^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x}+\omega\tau)}}{\omega^2+|\mathbf{k}|^2+m^2}\mathrm{d}\kappa\,\mathrm{d}^d\mathbf{k}\,,\qquad (\Box-m^2)G_E=-\delta^{(d+1)}(x)\,, \tag{3.88}$$

where $\Box$ now denotes the d'Alembertian on $(d+1)$-dimensional Euclidean space. The advantage of the Euclidean theory is that the operator $\Box-m^2$ has a unique well-defined inverse because the poles now lie on the imaginary rather than the real axis. Hence it is often much easier to work in Euclidean space and Wick rotate the result to obtain the Lorentzian Feynman Green's function. Note that this only works for the Feynman propagator, as all of the other contours in Figure 3.1 can not be rotated without crossing poles. A useful way to compute Euclidean Green's functions is using heat kernel regularisation, as demonstrated in the following exercise.

Exercise 3.13. *An alternative and often useful way to compute Green's functions is by analytically continuing the expressions to Euclidean signature and using heat kernel regularisation.*

Let us consider the Laplacian operator $A = -\Box$ in Euclidean space $\mathbb{R}^d$. The heat kernel of an operator A is defined as

$$K(x,x';\tau) = \langle x|\, \mathrm{e}^{-\tau A} \left|x'\right\rangle . \tag{3.89}$$

Show that, in the case of the Laplacian, the heat kernel solves the heat equation

$$\Box_x K(x,x';\tau) = \frac{\partial K(x,x';\tau)}{\partial \tau} , \tag{3.90}$$

with boundary condition

$$K(x,x';0) = \delta(x-x') . \tag{3.91}$$

Show that the Euclidean Green's function of A is then given by

$$G(x,x') = \int_0^\infty K(x,x';\tau)\mathrm{d}\tau , \tag{3.92}$$

and use the explicit expression for the heat kernel to derive the Euclidean Green's function for a massless scalar in $\mathbb{R}^d$. Compare the result with a direct calculation of $G^+(x,x')$ in four-dimensional Minkowski space-time.

The last Green's function we want to introduce is the thermal Green's function about which we will have much more to say in Chapter 6. Instead of looking at the Green's functions in the vacuum state, the thermal Green's functions $G_\beta(x)$ are obtained by considering a thermal state at temperature $T = \frac{1}{\beta}$. These Green's functions have the important property that they are periodic in imaginary time,

$$G_\beta(t,\mathbf{x}) = G_\beta(t+\mathrm{i}\beta,\mathbf{x}) . \tag{3.93}$$

3.7 Charged scalars, gauge fields and spinors

To finish this chapter, we briefly consider charged scalars, gauge fields and Dirac spinors. Many details are omitted as this section mainly serves to set our notation and conventions.

So far we considered real, neutral scalar fields. A charged scalar on the other hand can be described by a pair of Hermitian scalar fields ϕ_1 and ϕ_2 which we can collect in a single complex field

$$\phi(x) = \phi_1(x) + \mathrm{i}\phi_2(x) . \tag{3.94}$$

The Lagrangian density is given by the slightest generalisation of (3.2) as

$$\mathcal{L} = \eta^{\mu\nu}\partial_\mu \phi^\dagger \partial_\nu \phi - m^2 \phi^\dagger \phi \,, \tag{3.95}$$

which is invariant under the global symmetry transformation

$$\phi(x) \to \phi'(x) = \mathrm{e}^{\mathrm{i}\alpha}\phi(x)\,, \qquad \alpha \in \mathbb{R}\,. \tag{3.96}$$

Exercise 3.14. *What is the conserved current for this symmetry J_μ? And what is the symmetry generator $G(\bullet) = \int \mathrm{d}^d\, J_0 \bullet$?*

Show that the generator can be written as

$$G = \alpha \sum_{\mathbf{k}} \left(N^- - N^+\right)\,, \tag{3.97}$$

where $N^\pm$ are respectively the number operators for positively and negatively charged particles.

We can couple the charged scalar field to an external electromagnetic field with field strength $F = \mathrm{d}A$. The minimal coupling prescription is defined by replacing the derivatives ∂ in $\mathcal{L}$ with covariant derivatives $D = \partial + igA$, where g is the coupling constant. The Lagrangian is then invariant under the local symmetry transformation

$$\begin{aligned} A_\mu(x) \to& A'_\mu(x) = A_\mu(x) - \frac{1}{g}\partial_\mu \theta(x)\,, \\ \phi(x) \to& \phi'(x) = \mathrm{e}^{\mathrm{i}\theta(x)}\phi(x)\,. \end{aligned} \tag{3.98}$$

If the gauge field A_μ is dynamic we must add to the Lagrangian a Maxwell term, gauging the original global symmetry,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}D^\mu \phi^\dagger D_\mu \phi - m^2 \phi^\dagger \phi\,. \tag{3.99}$$

Remark. In the above we considered a U(1) gauge field. If ϕ is instead charged under a G symmetry we can proceed analogously. The Lagrangian remains unchanged but the gauge field A now transforms in the adjoint representation of the gauge group G,

$$A_\mu(x) = A^a_\mu(x) T^a\,, \tag{3.100}$$

where T^a are the generators of the gauge group which satisfy

$$\left[T^a, T^b\right] = \mathrm{i} f^{ab}{}_c T^c\,, \tag{3.101}$$

with the structure constants $f^{ab}{}_c$. The (covariant) field strength in turn is given by

$$F = -\mathrm{i}g^{-1}\,[D,D] = \mathrm{d}A + \mathrm{i}g\,[A,A]\,. \tag{3.102}$$

In this case the kinetic term for the gauge fields, also known as the Yang-Mills action contains a quartic interaction term, making this case significantly harder to analyse.

Finally, the Dirac spinor ψ is governed by the action

$$S = \overline{\psi}(\mathrm{i}\partial\!\!\!/ - m)\psi\,, \qquad \partial\!\!\!/ = \gamma^\mu \partial_\mu\,. \tag{3.103}$$

where the gamma matrices γ^μ satisfy the following commutation relations

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\,. \tag{3.104}$$

The Dirac adjoint is defined as $\overline{\psi} = \psi^\dagger \gamma^0$ and transforms such that the Lagrangian density is a scalar.

For more information on the quantisation of Maxwell theory, Yang-Mills theory or the (charged) Dirac spinor we refer the reader to any textbook on quantum field theory cited above. For additional information on representations of gamma matrices and minimal spinors in various dimensions see [VP99, FVP12].

3.8 Interacting theories and generalised free fields

The free field theories discussed so far are very special: while their spectrum can be determined exactly, the particle excitations do not interact with one another. To add interactions one can add higher-order terms to the Lagrangian. A generic such interaction in for a single scalar field can thus be written as

$$\mathcal{L} = \frac{1}{2}\partial_m u\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \sum_{n\geq 3}\frac{\lambda_n}{n!}\phi^n\,, \tag{3.105}$$

where the coefficients λ_n are known as coupling constants. An central question is which of these interaction terms can be treated perturbatively. At first glance, one might think that this simply requires choosing $\lambda_n \ll 1$ but the situation is subtler.

To begin with, observe that the action $S = \int \mathrm{d}^{d+1}x\mathcal{L}$ is a scalar quantity and therefore dimensionless, $[S] = 0$, so the Lagrangian must have dimension $[\mathcal{L}] = d + 1$. From this we can deduce the mass dimensions of the fields and couplings,

$$[\partial] = 1\,, \quad [\phi] = \frac{d-1}{2}\,, \quad [m] = 1\,, \quad [\lambda_n] = d + 1 - \frac{n(d-1)}{2} \tag{3.106}$$

This dimensional analysis immediately clarifies why it is insufficient to simply demand that λ_n be small: only dimensionless quantities can meaningfully be said to be small or large. Depending on their dimensions, the interaction terms fall into three qualitatively different categories:

- Relevant couplings: These correspond to $[\lambda_n] > 0$, i.e. $n < 2 + \frac{4}{d-1}$. The dimensionless parameter is $\lambda_n/E^{[\lambda_n]}$ where E is the characteristic energy scale in the problem. This implies that such perturbations are small at high energies but large at low energies. Terms of this type are called relevant, because they are most relevant at low energies. In a relativistic theory, $E > m$ so we can always make these perturbations small by taking $\lambda_n \ll m$.
- Marginal couplings: $[\lambda_n] = 0$, i.e. $n = 2 + \frac{4}{d-1}$. The coupling is already dimensionless, and

perturbation theory is valid simply when $\lambda_n \ll 1$.

- Irrelevant couplings: These couplings are characterised by $[\lambda_n] < 0$, i.e. $n = 2 + \frac{4}{d-1}$. Here, the dimensionless parameter is small at low energies and grows with energy. Such perturbations are called irrelevant as these perturbations are suppressed at low energies.

In practice, it is typically impossible to avoid high energy processes in quantum field theory. We have already seen a hints of this when discussing the vacuum energy. This means that we might expect problems with irrelevant operators. Indeed, these lead to "non-renormalizable" field theories in which one cannot make sense of the infinities at arbitrarily high energies. This does not necessarily mean that such theories are useless; just that they are incomplete at some energy scale and have to be completed by some more fundamental UV theory.

It is also important to note that the classification into relevant, marginal, and irrelevant operators based on classical dimensions is not absolute. Quantum corrections can modify the scaling behaviour of operators, potentially changing their classification.[8] For more details we refer the reader to the course on the renormalisation group. For this course we simply note that we will always restrict ourselves to theories with only renormalisable interactions.

In a free theory, all local dynamics are encoded in the two-point functions; all connected higher-point functions vanish. As a result, higher-point correlators decompose entirely into disconnected products of two-point functions. By contrast, in interacting theories, the connected higher-point functions contain genuinely new dynamical information. These contributions can be systematically computed in perturbation theory using the interaction picture. For a detailed discussion of perturbative expansions in flat spacetime, we refer the reader to standard quantum field theory textbooks.

So far, we have always taken the Lagrangian as the starting point, deriving the two- and higher-point correlation functions from it. However, it is important to stress that the Lagrangian is not a fundamental object in quantum field theory. The basic observables are the n-point correlation functions themselves. In particular, for intrinsically strongly coupled theories, it may not even be meaningful to write down a Lagrangian. This viewpoint is especially prominent in conformal field theories (CFTs), where the local dynamics are fully specified by the scaling dimensions Δ_i and three-point coefficients C_{ijk}. Higher-point functions can then be constructed via successive applications of the operator product expansion (OPE).

An instructive intermediate case is provided by generalized free fields (GFFs). These are characterized by two-point functions of the form

$$\langle \varphi(x)\varphi(0)\rangle = \int_{\mathbb{R}_+} \mathrm{d}\rho(m^2)\,\mathcal{G}_m(x), \tag{3.107}$$

where $\mathcal{G}_m(x)$ denotes the two-point function of a free Klein-Gordon field with mass m, and $\mathrm{d}\rho(m^2)$ is a positive, polynomially bounded weight known as the Källén-Lehmann spectral density [Kal52, Leh54]. Remarkably, this is the most general form for the two-point function of any scalar quantum field

[8] Classically irrelevant couplings which become relevant at the classical level are called dangerously irrelevant. Operators which remain marginal at the quantum level are called exactly marginal. In a CFT such operators characterise the conformal manifold.

theory. A general theorem asserts that any field whose two-point function can be written as in (3.107) with supp(ρ) contained in a finite mass interval must be a generalized free field [Gre62].

Despite the name, a generalized free field is not an interacting theory. Like in ordinary free field theories, all connected higher-point functions vanish. However, GFFs are more flexible than free fields and can mimic certain features of interacting theories—making them valuable as analytically tractable toy models.

Chapter 4

Quantum fields in curved space

Having reviewed quantum field theory in flat spacetime and established the essential global properties of Lorentzian manifolds, we are now prepared to extend our discussion to quantum fields in curved spacetime. This chapter lays out the necessary conceptual and technical ingredients, and explores the general features of quantisation in a curved background. The subsequent chapters will focus on applying this framework to a number of particularly instructive and physically relevant examples.

The most profound new aspect introduced by curved spacetime is the loss of a clear, invariant notion of a particle. Closely related to this is the ambiguity in defining the vacuum state: unlike in Minkowski space, there is no canonical choice. These issues lie at the heart of many of the conceptual and practical challenges in quantum field theory on curved backgrounds, and they will guide much of our discussion going forward.

4.1 Classical fields in curved space-time

To formulate a classical field theory in curved spacetime we need to know how the various fields couple to the background metric. Let us once more emphasize that the metric will not be a dynamic field and we only wish to consider fixed background metrics. This situation is very similar to the charged scalar we considered in Chapter 3.

Analogous to that case, we can couple the theory to a non-trivial background metric simply by changing all partial derivatives into covariant derivatives, $\partial_\mu \to \nabla_\mu$. However, this is not quite enough. In order to guarantee that the action transforms as a scalar under Lorentz transformations we need to simultaneously change the measure $\mathrm{d}^{d+1}x$ to the Lorentz invariant measure $\sqrt{|g|}\mathrm{d}^{d+1}x$. This procedure is in a sense the minimal consistent way to couple the system to gravity and for that reason goes under the name minimal coupling.[1]

Taking for example the free massive scalar field we find the action

$$S = \frac{1}{2}\int_{\mathcal{M}} \mathrm{d}^{d+1}x\sqrt{|g|}\left(-g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - m^2\phi^2\right). \tag{4.1}$$

This leads to the equation of motion,

$$\left(\Box - m^2\right)\phi = 0\,, \tag{4.2}$$

[1]Moreover, this coupling is consistent with Einstein's principle of equivalence, according to which local gravitational effects are not present in a neighbourhood of the spacetime origin of a locally inertial frame of reference. A similar comment applies to the theory with conformal coupling.

where now $\Box$ represents the d'Alembertian on the curved space,[2]

$$\Box\phi = g^{\mu\nu}\nabla_\mu\nabla_\nu\phi = |g|^{-1/2}\partial_\mu\left[|g|^{1/2}g^{\mu\nu}\partial_\nu\phi\right]. \tag{4.3}$$

This is not the most general way of coupling the scalar field to a background metric. A slightly more general action one can consider is,

$$S = -\frac{1}{2}\int_{\mathcal{M}} \mathrm{d}^{d+1}x\,\sqrt{|g|}\left(g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + m^2\phi^2 + \xi R\phi^2\right), \tag{4.4}$$

where R is the Ricci scalar of the curved manifold $\mathcal{M}$. The term proportional to R disappears in flat space so in this sense it is a generalization of the usual flat space action. The equation of motion after including the curvature coupling is given by

$$\left(\Box - m^2 - \xi R\right)\phi = 0, \tag{4.5}$$

Remark. In principle one can consider more general higher derivative interactions in the Lagrangian. Such terms could be of the form $R^n\phi^2$ but equally well terms like $(\partial^\mu\phi\partial_\mu\phi)^n$ could be included. In many context such terms are indeed present, but in this course we will mostly ignore them. However, there are various arguments why to a first approximation this is a reasonable thing to do. First of all, under the renormalisation group flow such terms will always be suppressed with respect to the two-derivative couplings. In many cases these terms will be "irrelevant" and can therefore be ignored at low energies. See the course on renormalisation for more details on this first point. Secondly, when including higher degree terms in the Lagrangian a multitude of issues with causality arise which need a careful treatment which goes beyond the scope of this course.

One reason why the curvature coupling $R\phi$ is useful is that for a specific values of ξ its addition to the Lagrangian of a massless scalar makes the action conformally invariant. For this value $\xi = \xi_{\text{conf.}}$ this coupling to the background metric is called conformal coupling.

Exercise 4.1. *Consider a free massless scalar field in $d+1$ dimensions. Show how the action transforms under a conformal transformation of the background metric,*

$$g_{\mu\nu} \to \Omega(x)^2 g_{\mu\nu}, \qquad \phi \to \Omega(x)^\Delta\phi, \tag{4.6}$$

for any function $\Omega(x) \neq 0$.

For which values of ξ and Δ is the action invariant under conformal transformations? In conformally invariant theories Δ is called the scaling dimension of the field ϕ.

Next, let us briefly consider Maxwell theory, i.e. the Abelian gauge theory with gauge field $A_\mu(x)$ and field strength $F_{\mu\nu} = \nabla_{[\mu}A_{\nu]}$. For gauge fields, the differential form language really starts to show it's

[2]Written as a differential form, the kinetic term in the action takes the form $\int \mathrm{d}\phi \wedge \star\mathrm{d}\phi$, where $\star$ is the Hodge star operator. In this language, the d'Alembertian takes the form $\Box\phi = \star(\mathrm{d}\star\mathrm{d}\phi)$.

elegance. In this language, we write the gauge field as a one form

$$A = A_\mu \mathrm{d}x^\mu \in \Omega^1(\mathcal{M}) \,. \tag{4.7}$$

In this language the field strength can be written as the two-form $F = \mathrm{d}A \in \Omega^2(\mathcal{M})$. The action for Maxwell theory coupled to a background metric is given by

$$S = -\frac{1}{4}\int_{\mathcal{M}} F \wedge \star F = -\frac{1}{2}\int_{\mathcal{M}} \sqrt{|g|}\mathrm{d}^{d+1}x F_{\mu\nu}F^{\mu\nu} \,, \tag{4.8}$$

which is invariant under the gauge transformations, $A_\mu \to A_\mu + \partial_\mu f$, or equivalently $A \to A + \mathrm{d}f$. The Bianchi identity for F can be written as $\mathrm{d}F = \nabla_{[\mu}F_{\nu\rho]} = 0$ while the equations of motion are given by

$$\mathrm{d}\star F = 0 \,, \qquad \text{or} \qquad \nabla^\mu F_{\mu\nu} = 0 \,. \tag{4.9}$$

The gauge invariance noted above adds a redundancy to Maxwell theory. In order to obtain a deterministic equation, one should first fix a gauge. A common choice of gauge in this situation is provided by the Lorenz gauge defined by $\nabla^a A_a = 0$. Upon this gauge fixing, the equations of motion reduce to the wave equation $\Box A_a = 0$. Note however that there is nevertheless still a residual gauge freedom parameterised by $A \to A + \mathrm{d}f$ provided that $\Box f = 0$. Finally, one can also consider spinors in curved spacetime. But we will not discuss this in this course.

Exercise 4.2. *Show how the Maxwell action transforms under conformal transformations. Demonstrate that in $d + 1 = 4$ dimensions the action and equations of motion (and Bianchi identities) are conformally invariant.*

Just as in general relativity (and its generalisations such as e.g. Einstein-Maxwell theory) we can define the energy momentum tensor as

$$T^{\mu\nu} = \frac{2}{|g|^{1/2}}\frac{\delta S}{\delta g_{\mu\nu}} \,, \tag{4.10}$$

which in general relativity this provides a source to the Einstein equation. In this course we consider quantum fields in fixed background metric and do not consider fluctuations of the metric. One can think of this as solving Einsteins equations with a source given by the vacuum expectation value of the stress tensor $\langle T_{\mu\nu}\rangle$. In a second step of our semi-classical treatment we can then consider fluctuations of the dynamical fields around this background metric. Note that up to possible improvement terms this reduces to the definitions in flat space presented in Chapter 3.

Conformal invariance of a theory is manifested in the stress tensor as the property $T^\mu_\mu = 0$. This is a general consequence of the invariance of the action under variations of the form $\delta g_{\mu\nu} = \sigma g_{\mu\nu}$.

Exercise 4.3. *Show that the stress tensor of a conformally invariant theory is necessarily traceless.*

Exercise 4.4. *Compute the stress tensor for a minimal and conformally coupled scalar as well as Maxwell theory in curved space.*

Check that for the latter two the stress tensor is indeed traceless.

4.2 Canonical quantisation in curved spacetime

To avoid subtleties, in this chapter we always assume the spacetime to be globally hyperbolic with a foliation by Cauchy hypersurfaces Σ_t where $t \in \mathbb{R}$ is a time-like coordinate. Having discussed how various fields couple to a background metric we repeat and modify where necessary the same steps as in Chapter 3 to quantise the theory.

Concepts such as the phase space, the associated symplectic form and the commutation relations generalise straightforwardly to curved space. The phase space for a scalar field on a spacetime $\mathcal{M}$ can be defined as

$$V_\phi(\mathcal{M}) = \left\{\phi \in C^\infty(\mathcal{M}) \,\middle|\, (\Box - m^2 - \xi R)\phi = 0\right\} = \left\{(\phi, \dot{\phi}) \in C^\infty(\Sigma_t) \times C^\infty(\Sigma_t)\right\} \tag{4.11}$$

where we allow for an arbitrary coupling to the Ricci scalar. Note that here we defined the initial data on the Cauchy hypersurface at time t and defined $\dot{\phi} = n^\mu \nabla_\mu \phi$ with n^μ a unit normal vector to the hypersurface at time t. Due to the global hyperbolicity of spacetime any choice of Cauchy hypersurface results in an isomorphic phase space. Similarly, one can define the phase space for fermions or gauge fields entirely analogous to the flat space case. The symplectic form is given by

$$\Omega(\phi_1, \phi_2) = \int_\Sigma \phi_1 \star \mathrm{d}\phi_2 - \phi_2 \star \mathrm{d}\phi_1 = \int_\Sigma (\phi_1 \dot{\phi}_2 - \phi_2 \dot{\phi}_1)\mathrm{d}\Sigma \tag{4.12}$$

where $\mathrm{d}\Sigma$ is the volume form on the hypersurface Σ. For more details on the geometry of hypersurfaces see Appendix C.

Remark. Note that for the complex scalar field we can write the inner product more invariantly as

$$\langle \phi_1, \phi_2 \rangle = \int_\Sigma \mathrm{d}\Sigma^\mu J_\mu\,, \tag{4.13}$$

where $\mathrm{d}\Sigma^\mu = \mathrm{d}\Sigma n^\mu$ and J_μ is the conserved current for the U(1) global symmetry,

$$J_\mu = \mathrm{i}\left(\phi_1^* \nabla_\mu \phi_2 - \phi_2^* \nabla_\mu \phi_1\right)\,. \tag{4.14}$$

In order to proceed with the quantisation we need a Hamiltonian description of the theory, and therefore a preferred time coordinate. Assuming there is a splitting (t, x_m) we will quantise the theory using the hypersurfaces Σ_t defined by $t =$ constant. Let us be more explicit and write the metric as

$$\mathrm{d}s^2 = -g_{00}\mathrm{d}t^2 + 2g_{0m}\mathrm{d}t\mathrm{d}x^m + h_{mn}\mathrm{d}x^m\mathrm{d}x^n\,, \tag{4.15}$$

where h_{mn} is the induced metric on the Cauchy hypersurface Σ. In terms of these coordinates, we can write the momentum conjugate to ϕ as

$$\pi = \frac{\delta \mathcal{L}}{\delta(\partial_t \phi)} = |g|^{1/2} g^{0\mu} \nabla_\mu \phi = |h|^{1/2} n^\mu \nabla_\mu \phi \,. \tag{4.16}$$

Exercise 4.5. *Prove the second equality in Equation (4.16). The most straightforward way to do so is by explicit computation.*

Analogous to flat space we can now quantise the theory by promoting all fields to operators on a Hilbert space and imposing the canonical commutation relations

$$\begin{aligned} \left[\phi(t,\mathbf{x}), \phi(t,\mathbf{x}')\right] &= 0\,, \\ \left[\pi(t,\mathbf{x}), \pi(t,\mathbf{x}')\right] &= 0\,, \\ \left[\phi(t,\mathbf{x}), \pi(t,\mathbf{x}')\right] &= \mathrm{i}\,\delta(\mathbf{x}-\mathbf{x}')\,. \end{aligned} \tag{4.17}$$

where now the δ-function is normalised against the volume form on the hypersurface,

$$f(\mathbf{x}') = \int_\Sigma \delta(\mathbf{x}-\mathbf{x}') f(\mathbf{x}') \mathrm{d}\Sigma \,. \tag{4.18}$$

Similarly, this Hilbert space now comes with a generalised inner product,

$$\langle \phi_1, \phi_2 \rangle = \mathrm{i}\,\Omega(\phi_1, \phi_2)\,. \tag{4.19}$$

Similar to flat space it follows immediately that this inner product is independent of the choice of Cauchy hypersurface in the given foliation. For this reason it was justified not to specify the precise time t in the above.

To further develop the quantum field theory we again want to relate the above phase space to a one-particle Hilbert space $\mathcal{H}$. However, in curved space there is no canonical analogue of the Fourier transform so this step will need some more care. In flat space, the plane waves represent a complete orthonormal set of solutions for the Klein-Gordon equation and allowed us to construct the one-particle Hilbert space. Successively applying the associated creation operators then allowed us to construct the full Hilbert space in the Fock representation. In addition, we showed that the vacuum state constructed as such is unique.

In order to have a complete understanding of the Hilbert space in curved space, the first step is to find a complete orthonormal basis of the phase space, analogous to the plane waves in flat space. In other words we have to find a complete set of functions u_i, solving the wave equation and satisfying,

$$\left\langle u_i, u_j \right\rangle = \delta_{ij}\,, \qquad \left\langle u_i^*, u_j \right\rangle = 0\,, \qquad \left\langle u_i^*, u_j^* \right\rangle = -\delta_{ij}\,. \tag{4.20}$$

Having found such a basis, we can expand the field ϕ as follows,

$$\phi(x) = \sum_i \left(a_i u_i + a_i^\dagger u_i^* \right)\,, \tag{4.21}$$

where the quantum annihilation and creation operators, a_i and $a_i^\dagger$, satisfy the standard commutation relations,

$$\left[a_i, a_j^\dagger\right] = \delta_{ij} \,. \tag{4.22}$$

A very important point is that on curved space, there is no natural way to perform this mode decomposition. This property is key to many odd but interesting phenomena in the theory of quantum field theory on curved space and will lie at the origin of many of the observations that will follow.

A special scenario where there exists a "natural" choice of decomposition is when the spacetime admits a globally well-defined, non-vanishing Killing vector $K = \partial_t$, which is irrotational, i.e. when the spacetime is globally static. In this case one can write the metric as

$$\mathrm{d}s^2(t, \mathbf{x}) = -g_{00}(\mathbf{x})\mathrm{d}t^2 + h_{mn}(\mathbf{x})\mathrm{d}x^m \mathrm{d}x^n \,, \tag{4.23}$$

where h and g_{00} are respectively the metric and a positive function on the transverse (space-like) manifold Σ, both independent of t. In this case the quantisation follows very closely in the steps of quantisation in flat space. We can separate variables and analogous to flat space define the positive modes V_ϕ^+ as the set of functions P_n of the form

$$P_n(x) = \frac{\chi_n(\mathbf{x})}{\sqrt{2\omega_n}} \mathrm{e}^{-\mathrm{i}\omega_n t} \,. \tag{4.24}$$

where the functions $\chi_n(\mathbf{x})$ collectively give a basis of functions on the manifold Σ that satisfy the reduced wave equation,

$$\left(\Delta + \omega_n^2\right)\chi_n = 0 \,, \tag{4.25}$$

for some second order operator Δ on Σ. The functions χ_n should satisfy the completeness relation,

$$\sum_n \bar{\chi}_n(x)\chi_n(x') = \delta^{(d)}(x, x') \,, \tag{4.26}$$

where the δ-function is understood to be normalised against the measure $\mathrm{d}^d x \sqrt{g_{00}\, h}$. Hence, it follows that the inner product reduces to

$$\langle P_m, P_n \rangle = \int \mathrm{d}^d x \sqrt{g_{00}\, h}\, \bar{\chi}_m \chi_n = \delta_{mn} \,. \tag{4.27}$$

Similarly, one can introduce negative frequency modes $N_n \in V_\phi^-$ and complete the quantisation exactly like in flat space. However, this limited approach does not generalise to generic curved spacetimes and moreover, even though the Schwarzschild black hole is static we will see that this approach is insufficient to fully understand quantum field theory on a black hole background.

Remark. In the above we silently assumed that Σ is compact such that the labels n take value in a countably infinite set. More generally, for a non-compact spatial slicing such as $\mathbb{R}^3$, m and n must be replaced by continuous variables such as $\mathbf{k}, \mathbf{k}'$. Similarly, the Kronecker delta, δ_{mn}, should then be replaced by $\delta^{(3)}(\mathbf{k} - \mathbf{k}')$ and $\sum_n$ by $\int_{\mathbb{R}^3} d^3\mathbf{k}/\omega_{\mathbf{k}}$. For the more formal development we will mostly stick to the discrete modes and introduce continuous alternatives when needed.

Another example of a natural choice of vacuum arises for conformally invariant theories in conformally flat spacetimes. Consider a conformally flat metric, i.e. $g_{\mu\nu}(x) = \Omega^2(x)\eta_{\mu\nu}$. A solution to Klein-Gordon equation for a conformally coupled massless scalar can then be obtained from the Minkowski solution $u^M_{\mathbf{k}}(x)$ simply by using the conformal transformation of the scalar field. Therefore, we have a natural choice of solutions of the wave equation given by

$$u_{\mathbf{k}}(x) = \Omega(x)^{\frac{1-d}{2}} u^M_{\mathbf{k}}(x)\,. \tag{4.28}$$

The vacuum associated to this choice of modes is called the conformal vacuum. In this case the analysis of the two-point functions becomes particularly easy, as one can obtain the Wightman functions simply as,

$$G_{\pm}(x,x') = \Omega(x)^{\frac{1-d}{2}} G^M_{\pm}(x,x')\Omega(x')^{\frac{1-d}{2}}\,, \tag{4.29}$$

where $G^M_{\pm}(x,x')$ is the Wightman function in Minkowski space introduced in Chapter 3. The Wightman functions transform as bi-scalars under conformal transformations.

4.3 Quantisation in generic curved spacetimes

In general there does not exist such a natural choice of orthonormal basis. For example, consider a spacetime that is asymptotically static both in the future and past region, respectively $\mathcal{M}_+$ and $\mathcal{M}_-$. In both asymptotic regions one can write down a metric in the form (4.23) and define the natural basis $P^{\pm}_n$ and $N^{\pm}_n$ resp. in the past and future region. This will lead to two notions of positive and negative modes and in general we do not expect these to agree.

We can quantise the theory using a Cauchy hypersurface in either of the asymptotic regions. In order to construct the transition between them we can impose that the quantum field ϕ is the same in both regions, i.e.

$$\phi(x) = \sum_n a^-_n P^-_n + a^{-\dagger}_n N^-_n = \sum_n a^+_n P^+_n + a^{+\dagger}_n N^+_n\,,. \tag{4.30}$$

The main question we'll try to answer in this section is how to identify the Fock spaces $\mathcal{F}^+$ and $\mathcal{F}^-$ of the different asymptotic regions. Similarly, in spacetimes with no asymptotically static regions, we have to learn how to identify the Fock spaces at different times.

To do so, consider two different orthonormal bases of the phase space $\{u_n\}_{n\in\mathbb{N}}$ and $\{v_n\}_{n\in\mathbb{N}}$ such that

$$\phi(x) = \sum_n a_n u_n + a^\dagger_n u^*_n = \sum_n b_n v_n + b^\dagger_n v^*_n,\,, \tag{4.31}$$

where a_n, $a^\dagger_n$ and b_n, $b^\dagger_n$ are the creation and annihilation operators defined with respect to either the u basis or v basis. Since both sets of functions are independent bases of $\mathbb{C}\otimes V_\phi$ we can relate them through linear maps

$$\begin{pmatrix} v_n \\ v^*_n \end{pmatrix} = \sum_m \begin{pmatrix} \alpha_{nm} & \beta_{nm} \\ \beta^*_{nm} & \alpha^*_{nm} \end{pmatrix} \begin{pmatrix} u_m \\ u^*_m \end{pmatrix} = \sum_m S_{nm} \begin{pmatrix} u_m \\ u^*_m \end{pmatrix} \tag{4.32}$$

where S is the classical S-matrix whose entries are called the Bogoliubov coefficients,

$$\alpha_{mn} = \langle u_n, v_m \rangle \,, \qquad \beta_{mn} = -\left\langle u_n^*, v_m \right\rangle \,, \tag{4.33}$$

and analogous for their complex conjugates. Since the u and v basis were both assumed to be orthonormal, the Bogoliubov coefficients have to satisfy the normalisation relation,

$$\alpha\alpha^\dagger - \beta\beta^\dagger = 1 \,, \qquad \alpha\beta^T - \beta\alpha^T = 0 \,. \tag{4.34}$$

Going back to (4.31) we find that the (transposed) Bogoliubov coefficients similarly encode the relation between the a and b creation and annihilation operators,

$$\begin{pmatrix} b_n \\ b_n^\dagger \end{pmatrix} = \sum_m \begin{pmatrix} \alpha_{nm}^* & -\beta_{nm}^* \\ -\beta_{nm} & \alpha_{nm} \end{pmatrix} \begin{pmatrix} a_m \\ a_m^\dagger \end{pmatrix} \tag{4.35}$$

Exercise 4.6. *Starting from* (4.31) *and* (4.32)*, show that the Bogoliubov coefficients encode the relation* (4.35) *between the a and b operators.*

Exercise 4.7. *Show that in order for the commutation relations of the b operators to be properly normalised, we have to impose the condition* (4.34).

Define the vacua with respect to the u or v expansion, $|0\rangle_a$ and $|0\rangle_b$ in the usual way as the states annihilated by all annihilation operators,

$$a_n \,|0\rangle_a = 0 \,, \qquad b_n \,|0\rangle_b = 0 \,, \qquad \forall n \geq 0 \,. \tag{4.36}$$

When $\beta = 0$, the two vacua are equivalent and can be identified up to possibly a phase. However, if $\beta \neq 0$, the b vacuum will contain a particles and vice versa! To see this more explicitly, define the number operators,

$$N_n^{(a)} = a_n^\dagger a_n \,, \qquad N^{(a)} = \sum_n N_n^{(a)} \,, \tag{4.37}$$

and compute their expectation values in the b vacuum,

$$\begin{aligned}
\left\langle N_n^{(a)} \right\rangle_b &= \left\langle a_n^\dagger a_n \right\rangle_b && (4.38)\\
&= \left\langle \sum_m \left(\alpha_{mn}^* b_n^\dagger + \beta_{mn} b_m\right) \sum_k \left(\alpha_{mn} b_n + \beta_{mn}^* b_m^\dagger\right) \right\rangle_b && (4.39)\\
&= \sum_{m,k} \beta_{kn}^* \beta_{mn} \left\langle b_m b_k^\dagger \right\rangle_b && (4.40)\\
&= \sum_m |\beta_{mn}|^2 \,. && (4.41)
\end{aligned}$$

Hence, we see that the changing gravitational field creates particles (in pairs). In a sensible physical systems the number of quanta in any vacuum should be finite so we require that the Bogoliubov transformation is such that $\left\langle N_n^{(a)} \right\rangle_b < \infty$ for all n. Indeed, without this condition various problems regarding the convergence of all the expressions in this section arise.

To find an explicit relation between the a and b vacuum we would like to extend the map S from the classical phase space to the Fock space, $\mathcal{S}:\mathcal{F}_a\to\mathcal{F}_b$. As a first step, we can construct the b vacuum using the Bogoliubov coefficients. After some algebra, one can show that (up to phase) we have

$$|0\rangle_b=\frac{1}{|\det\alpha|^{\frac{1}{2}}}\exp\left\{\frac{1}{2}\sum_{m,n}M_{mn}a_m^\dagger a_n^\dagger\right\}|0\rangle_a\,,\qquad M_{mn}=\left((\alpha^*)^{-1}\beta^*\right)_{mn}\,,\tag{4.42}$$

Such states are sometimes called squeezed states, in analogy with the nomenclature for the ground state for an oscillator with a different frequency. The full quantum evolution operator translating between the two Fock spaces then takes the form

$$\mathcal{S}=\frac{1}{|\det\alpha|^{\frac{1}{2}}}\exp\left\{\frac{1}{2}\sum_{m,n}\left(M_{mn}a_m^\dagger a_n^\dagger-M_{mn}^*a_ma_n\right)\right\}.\tag{4.43}$$

This operator is now unitary as the exponent is skew Hermitian so it preserves norms.

Exercise 4.8. *Show that the a and b vacua are related as in Equation* (4.42). *Hint: use the Baker-Campbell-Hausdorff formula to expand expressions of the form as* $\mathrm{e}^{-F}a_n\mathrm{e}^F$.

4.4 Curved space and spin statistics

In curved space an interesting connection between statistics and dynamics appears that is not present in Minkowski space. The statistics is determined by the algebra of creation and annihilation operators. Commuting operators give rise to bosons, while anti-commuting operators give rise to fermions. This is nothing new, but as we will argue, in curved spacetime this is the only consistent option!

To see this, consider a spin-0 field, i.e. a scalar field in curved space and two possible vacua, the a and b vacuum among which we can interpolate using the Bogoliubov coefficients defined above, satisfying $|\alpha|^2-|\beta|^2=1$. Note that this relation follows purely from the field equations. Assume that the a creation and annihilation operators satisfy the (anti-)commutation relations

$$[a_n,a_m]_\pm=\left[a_m^\dagger,a_n^\dagger\right]_\pm=0\,,\qquad\left[a_m,a_n^\dagger\right]_\pm=\delta_{mn}\,,\tag{4.44}$$

where the plus sign denotes the anti-commutator and the minus the commutator. Following the above, one can show that in the b vacuum, the (anti-)commutation relations become

$$[b_m,b_n]_\pm=\left[b_m^\dagger,b_n^\dagger\right]_\pm=(\alpha_{mn}\beta_{mn}^*\pm\alpha_{mn}\beta_{mn}^*)\delta_{m,-n}\,,\tag{4.45}$$

and

$$\left[b_m,b_n^\dagger\right]_\pm=(|\alpha|_{mn}^2\pm|\beta|_{mn}^2)\delta_{m,n}\,,\tag{4.46}$$

Hence, we see that in order for the particles in the b vacuum to satisfy the same statistics as the particles in the a vacuum it is necessary to pick the minus sign such that (4.45) vanishes identically and (4.46) reduces to the standard commutation relations for bosons.

This is a purely curved space derivation, since in flat space $\beta=0$ and the connection derived above is absent and both statistics seem allowed. Similar conclusions can be reached for fermionic fields as well as higher spin fields.

Part II

APPLICATIONS

Chapter 5

First examples

In the previous part, we developed the main theory necessary to perform the canonical quantisation of a quantum field theory in curved space-times, focussing on free scalar fields. The extension of this formalism to fermionic or higher spin fields and interacting theories – at least within perturbation theory – follows closely the flat space case.

In this chapter, we shift our attention from general formalism to concrete applications. We will examine quantum field theory on several classes of physically relevant spacetimes, including maximally symmetric and cosmological backgrounds. We will also explore how the formalism can be used to describe quantum fields in thermal states. These settings allow for explicit computations and offer valuable insight into the behaviour of quantum fields in curved geometries. Beyond these examples, we introduce adiabatic expansions as a powerful method to investigate more general backgrounds. Together, these examples will highlight both the conceptual subtleties and the practical techniques involved in applying the general framework in specific physical situations.

5.1 Maximally symmetric spaces

As a first set of examples we will consider a free scalar field in a maximally symmetric space-time. The maximally symmetric space-times all have constant Ricci curvature so can be split in three cases. For zero curvature we have Euclidean or Minkowski space, for positive curvature we have the sphere S^d or de Sitter space dS_d and for negative curvature we have hyperbolic space or anti-de Sitter space AdS_d. More details about the maximally symmetric spacetimes as well as the various useful coordinate systems can be found in appendix E. In this section we will focus on scalar fields in de Sitter space, while the other maximally symmetric examples will be left as an exercise.

Geometry of de Sitter space

The de Sitter space dS_{d+1} can be described as a hyperboloid

$$-X_0^2 + \sum_{i=1}^{d+1} X_i^2 = L^2\,, \tag{5.1}$$

embedded in $(d+2)$-dimensional Minkowski space, with metric,

$$\mathrm{d}s_{d+2}^2 = -\mathrm{d}X_0^2 + \sum_{i=1}^{d+1} \mathrm{d}X_i^2\,. \tag{5.2}$$

As such, de Sitter space inherits an $\mathrm{O}(1,d+1)$ symmetry from the ambient Minkowski space (See also exercise 2.3). de Sitter space has constant scalar curvature

$$R = \frac{d(d+1)}{L^2}\,. \tag{5.3}$$

An important quantity in the following analysis will be the geodesic distance between two points. As in the case of a sphere embedded in $\mathbb{R}^3$, the distance between two points in de Sitter space is closely related to the distance in embedding space. Hence, let us define,

$$P(X,X') = L^{-2}\eta_{ab}X^a X'^b\,. \tag{5.4}$$

Notice that for two identical points $X = X'$ in dS_{d+1}, we have $P = 1$, while for antipodal points $X = -X'$, we have $P = -1$. One nice property of the quantity $P(X,X')$ is that it is manifestly $\mathrm{O}(1,d+1)$ invariant, since it is constructed out of the Lorentz product in $\mathbb{R}^{1,d+1}$. Depending on the causal relationship between the two points, X and X', we have the following behaviour for $P(X,X')$:

- Time-like separated points: the two points can be joined by a time-like geodesic, hence $P(X,X') > 1$ and the geodesic distance is given by

$$\zeta(X,X') = L\cosh^{-1}(P) \tag{5.5}$$

- Space-like separated points: the two points can be joined by a space-like geodesic, hence $|P(X,X')| < 1$ and the geodesic distance is given by

$$\zeta(X,X') = L\cos^{-1}(P) \tag{5.6}$$

- light-like separated points: the two points can be joined by a light-like geodesic, hence $P(X,X') = 1$ and the geodesic distance is given by

$$\zeta(X,X') = 0\,. \tag{5.7}$$

Notice that there are points in de Sitter space which cannot be joined by geodesics to a given point X. These are the points in the interior of the past and future light cones of the point $-X$, the antipodal point of X. For these points, we have that $P(X,X') < -1$.

Exercise 5.1. *Prove the above formulae for the geodesic distance between two points in de Sitter space.*

Hint: it might be useful to explicitly compute the geodesics in de Sitter space. See also Appendix E for more details.

Quantisation in de Sitter space

Let us for now consider the free scalar field $\phi(x)$ with mass m in a fixed de Sitter background. The action of such a scalar is given by:

$$S = -\frac{1}{2}\int \mathrm{d}^{d+1}x\,\sqrt{|g|}\left(g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi + m^2\phi^2 + \xi R\phi^2\right)\,. \tag{5.8}$$

Using the conformal time coordinate η, the de Sitter metric becomes a FLRW metric where the function $C(\eta)$ is given by

$$C(\eta) = \frac{L^2}{\eta^2} \,. \tag{5.9}$$

In de Sitter space, we have

$$C(\eta)R(\eta) = \frac{d(d+1)}{\eta^2} \,, \tag{5.10}$$

so that the Klein-Gordon equations reduces to

$$\chi_{\mathbf{k}}'' + \left[k^2 + \frac{m^2L^2}{\eta^2} + \frac{d(d+1)(\xi - \xi(d))}{\eta^2} \right] \chi_{\mathbf{k}} = 0 \,. \tag{5.11}$$

Performing the change of coordinates $s = -k\eta$ and defining

$$\chi_{\mathbf{k}} = \sqrt{s} f(s) \,, \tag{5.12}$$

it is easy to see that $f(s)$ satisfies the Bessel equation,

$$s^2 f''(s) + s f'(s) + (s^2 - \nu^2) f(s) = 0 \,, \tag{5.13}$$

where

$$\nu^2 = \frac{1}{4} + (\xi(d) - \xi)\, d(d+1) - m^2L^2 = \frac{d^2}{4} - d(d+1)\xi - m^2L^2 \,. \tag{5.14}$$

We can then write $\chi_{\mathbf{k}}$ in terms of Bessel functions, for $\eta < 0$, as

$$\chi_{\mathbf{k}}(\eta) = \sqrt{k|\eta|}\, [A_{\mathbf{k}} J_\nu(k|\eta|) + B_{\mathbf{k}} Y_\nu(k|\eta|)] \,. \tag{5.15}$$

and the normalisation condition leads to

$$-\mathrm{i}k^2\eta \left(A_k B_k^* - A_k^* B_k \right) W\left[J_\nu(k|\eta|), Y_\nu(k|\eta|) \right] = 1 \,. \tag{5.16}$$

Here $W[f,g] = f g' - f' g$ is the Wronskian, where the derivative is with respect to the full argument $k|\eta|$. The Wronskian can be computed to be

$$W\left[J_\nu(k|\eta|), Y_\nu(k|\eta|) \right] = \frac{2}{\pi x} \,, \tag{5.17}$$

so that we find

$$A_k B_k^* - A_k^* B_k = -\frac{\mathrm{i}\pi}{2k} \,. \tag{5.18}$$

To get a better feeling for the behaviour of these functions consider the asymptotic region near $\mathscr{I}_+$, i.e. $\eta \to 0$. In this region, the modes asymptotically behave as

$$\chi_k \propto (-k\eta)^{\frac{1}{2} \pm \nu} \,. \tag{5.19}$$

Notice that for small mass m and ξ, ν is real and the modes either blow up or vanish asymptotically while for large masses and/or ξ it becomes an oscillatory mode with positive frequency $|\nu|$.

de Sitter vacua

De Sitter space is a static space-time, and moreover maximally symmetric. Hence, guided by the discussion in the previous sections we look for the 'preferred' vacuum which preserves all said symmetries. For this reason, let's consider the behaviour in the asymptotic past $\mathscr{I}_-$, i.e. $k|\eta| \gg 1$, where we have $\omega_k \simeq k$. In this regime we want to consider modes which behave like Minkowski modes in conformal time η,

$$\chi_k \propto \frac{1}{\sqrt{2k}} e^{-ik\eta} . \tag{5.20}$$

In order to find solutions to the Klein-Gordon with this behaviour, consider the asymptotic behaviour of the Bessel functions. We have

$$\chi_k \sim \sqrt{\frac{2}{\pi}} \left(\frac{A+iB}{2} e^{-i\lambda} + \frac{A-iB}{2} e^{i\lambda} \right) , \qquad \eta \to -\infty , \tag{5.21}$$

where

$$\lambda = k|\eta| - \frac{\nu\pi}{2} - \frac{\pi}{4} . \tag{5.22}$$

Therefore, we require that $A + iB = 0$, which together with the condition (5.18) results in

$$|A|^2 = \frac{\pi}{4k} . \tag{5.23}$$

Inserting this in the expression for the modes we find,

$$\chi_k(\eta) = \frac{1}{2}(\pi|\eta|)^{\frac{1}{2}} \left(J_\nu(k|\eta|) + iY_\nu(k|\eta|)\right) = \frac{1}{2}(\pi|\eta|)^{\frac{1}{2}} H_\nu^{(1)}(k|\eta|) , \tag{5.24}$$

where $H_\nu^{(1)}$ is the Hankel function of the first kind. This vacuum is called the Euclidean vacuum or Bunch-Davies vacuum $|0\rangle_{BD}$, after [BD78][1]

Positive frequency modes in the Bunch-Davies vacuum state are those which become the positive frequency modes in Minkowski space upon taking the limit $k|\eta| \to \infty$ [BD78] (see also [STY95]). We can thus expand our quantum field in terms of the creation and annihilation operators associated to $|0\rangle$,

$$\phi(\eta, \vec{x}) = \sum_{\mathbf{k}} \left[a_{\mathbf{k}} u_{E,\mathbf{k}}(\eta, \mathbf{k}) + a_{\mathbf{k}}^\dagger u_{E,\mathbf{k}}^*(\eta, \mathbf{k}) \right] , \tag{5.25}$$

where the creation and annihilation operators satisfy the usual properties:

$$a_{\mathbf{k}} |0\rangle = 0 , \qquad \left[a_{\mathbf{k}}, a_{\mathbf{k}'}^\dagger \right] = \delta_{\mathbf{k}\mathbf{k}'} . \tag{5.26}$$

This vacuum has various interesting properties. Namely, it is invariant under the de Sitter isometry group SO$(1,4)$. Clearly, it is invariant under rotations of the spatial coordinates $\mathbf{x}$, since $\chi_k(\eta)$ only depends on the modulus of k. It is also invariant under the dilatation,

$$\eta \to \lambda\eta , \qquad \mathbf{x} \to \lambda\mathbf{x} , \qquad \lambda \in \mathbb{R}\backslash\{0\} \tag{5.27}$$

[1] History has its ways so that Bunch and Davies got the honour of naming this vacuum. However, it was already described earlier in various papers such as [CT68, SS76].

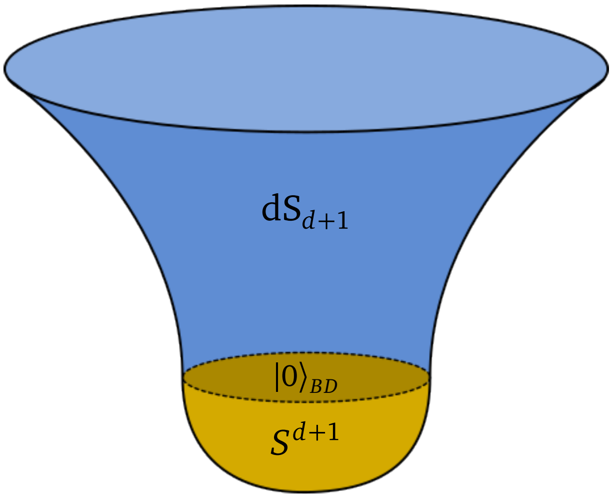

Figure 5.1: The Bunch-Davies state is prepared by gluing the de Sitter space to a sphere, i.e. Euclidean de Sitter space. This construction can be though of as a toy model for the universe.

Indeed, under this transformation the wave-vector transforms as $\mathbf{k} \to \frac{1}{\lambda}\mathbf{k}$ such that the argument of the Hankel function remains invariant. Collecting the overall factor $|\eta|^{\frac{1}{2}}$ in $\chi_k(\eta)$, together with the factor $C^{\frac{d-1}{4}}/4$, we get a total factor of $|\eta|^{\frac{d}{2}}$. However, this factor gets cancelled against the factor of $1/V^{\frac{1}{2}}$ in the wave-function u_k, where $V = L^d$ is the spatial volume. This factor combines with the η factor to produce $(|\eta|/L)^{d/2}$, showing that the modes are invariant under dilatations. Below we will see that the invariance of this vacuum manifests itself in the $\mathrm{O}(1,4)$ invariance of the corresponding Wightman functions.

Only demanding that the vacuum state is invariant under the de Sitter isometries does not uniquely pick out the Bunch-Davies vacuum. Indeed, the vacuum state of the quantum field could be rather different. A more general family of vacua is given by the so-called the α-vacua, $|\alpha\rangle$ [Mot85, All85], parameterised by a complex number α. These vacua and their properties are reviewed in [BMS02, SSV01] but as we'll see in a bit, these vacua have some funny properties for which reason we usually discard them. In this course we will mostly focus on the Bunch-Davies vacuum as defined above but surely at some point these funny extra vacua will turn out to have some purpose in life.

Remark. The Bunch-Davies vacuum is picked out for another reason. We can prepare a de Sitter vacuum by starting with a Euclidean sphere, cutting it in half and gluing it to 'half' of the Sitter space. The state prepared by this procedure is precisely the Bunch-Davies vacuum. This origin furthermore elucidates where the name 'Euclidean vacuum' comes from. This construction is motivated by the 'no boundary wave function' proposal of Hartle and Hawking [HH83], where a conjectural description for the wavefunction of the universe was given. According to their prescription, the universe has no origin. If we would travel back towards the beginning of the universe we would note that, similar to in the interior of the black hole, our notion of space and time changes. In this case they propose that close enough to the singularity time stops to exist and we are left with a Euclidean space which smoothly caps off.

Finally, note that when $m = 0$ and $\xi = \xi(d)$, the Bunch-Davies vacuum reduces to the conformal vacuum. Indeed, in this case the index of the Hankel function is $\nu^2 = \frac{1}{4}$ and we have

$$H_{\nu}^{(1)}(z) = -\mathrm{i}\left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \mathrm{e}^{\mathrm{i}z} \,. \tag{5.28}$$

so that the modes reduce up to a phase to the Minkowski modes, $\chi_k = \frac{\mathrm{i}}{\sqrt{2k}} \mathrm{e}^{-ik\eta}$, appropriate for a conformal vacuum.

Green's functions

Having defined the vacuum we would like to compute the Wightman functions. This can be done in two ways. One way is the brute-force approach, by explicitly plugging the equations for the modes in (5.140) and performing the integration over momenta. This has been done in [BD78, SS76, CT68], but it's rather tedious. Another, more elegant approach is to solve the Klein-Gordon equation for $G_+(x, x')$ while assuming $\mathrm{O}(1, d+1)$ invariance along the way. This in fact gives the answer not only for the Bunch-Davies vacuum but also for the more general family of α-vacua $|\alpha\rangle$.

If the Wightman function is computed in a state invariant under the de Sitter isometry group it should only depend on the $\mathrm{O}(1, d+1)$ invariant distance between the two points. This distance is nothing but the geodesic distance and can be expressed in terms of $P(X, X')$. Therefore, we have

$$G^+(X, X') = G^+(P(X, X')) \,. \tag{5.29}$$

The Wightman function then satisfies the homogeneous Klein-Gordon equation,

$$(\Box_x + m^2 + \xi R) G^+(x, x') \,, \tag{5.30}$$

where the Laplacian only acts on the first coordinate X. To solve this equation let us note two useful properties of the geodesic distance,

$$\nabla^{\mu} P \nabla_{\mu} P = \frac{P^2 - 1}{L^2} \,, \qquad \nabla_{\mu} \nabla_{\nu} P = g_{\mu\nu} \frac{P}{L^2} \,. \tag{5.31}$$

Exercise 5.2. *Prove the two properties above. You can do so abstractly or by explicit computation in your favourite coordinate system. Hint: in conformal coordinates $(\eta, \mathbf{x})$ these take a particularly simple form. To do so you will need to compute the Christoffel symbols.*

Consider now a function $F(P)$ depending on X only through $P(X, X')$. We then have,

$$\Box_x F(P) = \frac{P^2 - 1}{L^2} F''(P) + \frac{d+1}{L^2} P F'(P) \,. \tag{5.32}$$

where the primes denote derivatives with respect to P. It then follows that the Wightman function for a scalar field in de Sitter space satisfies,

$$(P^2 - 1) G^{+\prime\prime} + (d+1) P G^{+\prime} + \left(m^2 L^2 + \xi d(d+1)\right) G^+ = 0 \,. \tag{5.33}$$

After a change of variables, $z = \frac{1+P}{2}$ this becomes a hypergeometric equation,

$$z(1-z)\partial_z^2 G^+ + \left(\frac{d+1}{2} - (d+1)z\right)\partial_z G^+ - \left(m^2L^2 + \xi d(d+1)\right)G^+ = 0\,. \tag{5.34}$$

Comparing this to the standard form (see Appendix F), we find that

$$\alpha + \beta = d\,, \qquad \alpha\beta = m^2L^2 + \xi d(d+1)\,, \tag{5.35}$$

so that we find the general solution

$$G^+ = c_d\, {}_2F_1\left(h_+, h_-, \frac{d+1}{2}, z\right)\,, \tag{5.36}$$

where c_d is a normalisation constant to be determined shortly and we defined

$$h_\pm = \frac{1}{2}\left[d \pm \sqrt{d^2 - 4(m^2L^2 + \xi d(d+1))}\right]\,. \tag{5.37}$$

The hypergeometric function in (5.36) has a pole at $z = 1$, or $P = 1$ and a branch cut for $1 < P < \infty$. The pole occurs when the points x and x' are separated by a null geodesic. At short distances, the scalar field is insensitive to the fact that it lives in a de Sitter space and the form of the singularity should be the same as that of the propagator in flat Minkowski space. Indeed, in the UV all energy scales are much higher than the scale set by the curvature of the space-time (as well as the mass) so the UV/short distance behaviour should not depend on it. We can use this fact to fix the normalisation constant. Near $z = 1$, the hypergeometric function behaves as

$${}_2F_1\left(h_+, h_-, \frac{d+1}{2}, z\right) \approx \frac{\zeta(P(x,x'))^{1-d}}{2^{1-d}}\frac{\Gamma\left(\frac{d+1}{2}\right)\Gamma\left(\frac{d-1}{2}\right)}{\Gamma(h_+)\Gamma(h_-)}\,, \tag{5.38}$$

where $\zeta(P) = \cos^{-1} P \ll 1$ the geodesic separation between the two points. Comparing this expression with the usual short distance singularity,

$$G^{+,\text{flat}}(x,x') \sim (-1)^{\frac{d-1}{2}}\frac{\Gamma\left(\frac{d-1}{2}\right)}{4\pi^{\frac{d+1}{2}}}\frac{1}{(d^2)^{\frac{d-1}{2}}}\,, \qquad l^2 = (x-x')^2\,, \tag{5.39}$$

we find the normalisation constant

$$c_d = L^{1-d}\frac{\Gamma(h_+)\Gamma(h_-)}{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)}\,. \tag{5.40}$$

Exercise 5.3. *Use the properties of hypergeometric functions to show that near the pole the hypergeometric function behaves as in* (5.38)*. (Hint: to do so write* $P = 1 + \delta$*, where* $\delta = \frac{(\Delta\eta)^2 - (\Delta\mathbf{x})^2}{2\eta\eta'}$ *and expand for small* δ*.)*

Exercise 5.4. *Show that for* $m = 0$ *and* $\xi = \xi(d)$ *this expression reduces to the expected expression for a two-point function in a conformal vacuum.*

As noted above, the hypergeometic function (5.36) has a branch cut along the semi-infinite axis running from 1 to ∞. This corresponds to points where $P(X,X') \geq 1$, i.e. points inside the light-cone.

The prescription for avoiding the singularity at the light-cone is the same as in Minkowski space and simply consists of changing

$$(\eta-\eta')^2 \to (\eta-\eta'-\mathrm{i}\epsilon)^2\,. \tag{5.41}$$

Finally, note that the equation (5.33) is symmetric under interchanging $P \longleftrightarrow -P$. So if $G(P)$ is a solution, then so is $G(-P)$. We therefore find a second linearly independent solution,

$$G^{+(2)} = c_d\, {}_2F_1\left(h_+, h_-, \frac{d+1}{2}, \frac{1-P}{2}\right). \tag{5.42}$$

The singularity now lies at $P=-1$, which corresponds to X being null separated from the antipodal point to X'. This singularity sounds rather unphysical at first, but we should recall that antipodal points in de Sitter space are always separated by a cosmological horizon. The Green's function (5.42) can thus be thought of as arising from an image source behind the horizon, and is non-singular everywhere within an observer's horizon. Hence the unphysical singularity can never be detected by any experiment. The de Sitter space therefore has a one parameter family of de Sitter invariant Green's functions

$$G_+^{(\alpha)} = \langle\alpha|\,\phi(x)\phi(x')\,|\alpha\rangle = \sin\alpha\, G^+ + \cos\alpha\, G^{+(2)}\,, \tag{5.43}$$

corresponding to the α vacua $|\alpha\rangle$ discussed above. Putting $\alpha=0$ the antipodal singularity disappears and the vacuum reduces to the Bunch-Davies vacuum.

Other maximally symmetric spacetimes

The quantisation, analysis of vacua and the computation of the Green's functions is very similar in other maximally symmetric spacetimes. In the following exercises we will have a closer look at the sphere S^d, while a more in-depth analysis of quantum fields in AdS will be left for Chapter 8.

Exercise 5.5. *The d-dimensional sphere S^d is defined by the following embedding in $(d+1)$-dimensional Euclidean space,*

$$\sum_{I=1}^{d+1} X_i^2 = R^2\,. \tag{5.44}$$

we can parametrise this as

$$X_{d+1} = R\cos\theta\,, \qquad X_i = R\sin\theta\,\omega_i\,, \tag{5.45}$$

where $i=1,\ldots,d$ and ω_i are embedding coordinates for a $(d-1)$-dimensional unit sphere.

1. *Compute the geodesic distance $\zeta(x,x')$ between two points x and x'. (Hint: note that using the rotational symmetry we can always put one of the points at the north pole of the sphere, $\theta=0$.*

2. *Since the sphere is maximally symmetric, the Green's function is restricted to be a function of ζ alone. Show that the Laplacian of the Green's function takes the form*

$$\Box G(\zeta) = G''(\zeta) + \frac{d-1}{R}\cot\frac{\zeta}{R}G'(\zeta)\,. \tag{5.46}$$

3. *Find the Green's function for a scalar field with mass m and coupling ξ to the Ricci scalar. (Hint: consider the change of variables $z=\cos^2\left(\frac{\zeta}{2R}\right)$)*

4. *Do the same for all the maximally symmetric spaces in a uniform way.*

Exercise 5.6. *A topological theory on the equator Consider the Euclidean action*

$$S = \int \mathrm{d}^3x\,\sqrt{g}\tilde{q}^a(x)\mathcal{D}_a{}^b(x)q_b(x)\,, \tag{5.47}$$

defined on the round three-sphere S^3 with radius R, where the operator $\mathcal{D}_a{}^b$ is defined by

$$\mathcal{D}_a^b(x) = \begin{pmatrix} -\Box + \frac{3}{4R^2} + \frac{\sigma^2}{R^2} & -\frac{\sigma^2}{R^2} \\ \frac{\sigma^2}{R^2} & -\Box + \frac{3}{4R^2} + \frac{\sigma^2}{R^2} \end{pmatrix}. \tag{5.48}$$

Here the scalars q_a and $\tilde{q}^a$ are similar to a complex scalar in Lorentzian signature where after continuation to Lorentzian signature the tilde would become the complex conjugate. In Euclidean signature the precise reality condition is more subtle but for the sake of this exercise you can thing of it as the complex conjugate.

1. *How do you interpret the various terms in the differential operator $\mathcal{D}^q{}_b$?*
2. *Determine the two-point function $G_a{}^b(x,x') = \langle q_a(x)\tilde{q}^b(x')\rangle$ by solving the differential equation*

$$\mathcal{D}_a{}^c G_c{}^b(x,x') = \frac{\delta_a^b}{\sqrt{g}(x')}\delta^{(3)}(x-x')\,, \tag{5.49}$$

Either by using the previous exercise or by brute force computation. Show that the solution is given by

$$G_a{}^b(x,x') = \langle q_a(x)\tilde{q}^b(x')\rangle = \frac{1}{8\pi R\cosh\sigma\pi}\begin{pmatrix} \frac{\cosh(\sigma\pi-\sigma\zeta)}{\sin(\zeta/2)} & \frac{\sinh(\sigma\pi-\sigma\zeta)}{\cos(\zeta/2)} \\ -\frac{\sinh(\sigma\pi-\sigma\zeta)}{\cos(\zeta/2)} & \frac{\cosh(\sigma\pi-\sigma\zeta)}{\sin(\zeta/2)} \end{pmatrix}, \tag{5.50}$$

where ζ is the relative angle (or geodetic distance) between the points x and x'.

3. *Now consider putting both q_a and $\tilde{q}^b$ on the equator and define the operators*

$$Q(\varphi) = q_1(\varphi)\cos\frac{\varphi}{2} + q_2(\varphi)\sin\frac{\varphi}{2}\,, \qquad \tilde{Q}(\varphi) = \tilde{q}_1(\varphi)\cos\frac{\varphi}{2} + \tilde{q}_2(\varphi)\sin\frac{\varphi}{2}\,, \tag{5.51}$$

where we use the metric

$$\mathrm{d}s^2 = R^2\left(\mathrm{d}\theta^2 + \cos^2\theta\,\mathrm{d}\tau^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right). \tag{5.52}$$

Obtain the two-point function for these operators restricted to the equator.

Show that after taking a careful limit $\sigma \to 0$ the green's function becomes

$$G_Q(\varphi_1 - \varphi_2) = \langle Q(\varphi_1)\tilde{Q}(\varphi_2)\rangle = -\frac{\mathrm{sgn}(\varphi_1-\varphi_2)}{8\pi R}\,. \tag{5.53}$$

It becomes topological!

This might seem a weird coincidence but it is actually a deep consequence of $\mathcal{N}=4$ supersymmetric theories in three-dimensions. By going to the cohomology of a specific supercharge $\mathcal{Q}$, i.e. by only *considering operators that are annihilated by $\mathcal{Q}$, i.e. $[\mathcal{Q},\mathcal{O}]=0$ but are not $\mathcal{Q}$-exact, $\mathcal{O}\neq[\mathcal{Q},\mathcal{O}']$, the theory reduces to a one-dimensional topological theory (or mildly non-topological if we allow for non-zero σ). The theory we studied above corresponds to the $\mathcal{N}=4$ hypermultiplet! If you want to know more about these protected sectors have a look at the following papers:*

- *C. Beem, W. Peelaers, and L. Rastelli, "Deformation quantization and superconformal symmetry in three dimensions," Commun. Math. Phys. 354 (2017), no. 1 345–392, 1601.05378.*
- *M. Dedushenko, S. S. Pufu, and R. Yacoby, "A one-dimensional theory for Higgs branch operators," JHEP 03 (2018) 138, 1610.00740.*

5.2 Cosmological spacetimes

Maximally symmetric spacetimes such as Minkowski, de Sitter (dS), or anti-de Sitter (AdS) exhibit a high degree of symmetry, including time translation invariance. In such backgrounds, no moment in time is physically distinguished from another -— there is no notion of a "beginning" or "end" of time, and thus no meaningful concept of a cosmological history. While these eternal universes are often favoured for their mathematical elegance and philosophical appeal, they are fundamentally at odds with the cosmological data accumulated over the past century, which compellingly points to a dynamic, evolving universe.

To model this evolution, we turn to the class of Friedmann–Lemaître–Robertson–Walker (FLRW) spacetimes, which provide non-trivial generalization of maximally symmetric geometries. FLRW universes allow for time-dependent expansion or contraction, thereby capturing the essential features of cosmological evolution. Crucially, such time dependence leads to rich physical phenomena: quantum fields in an expanding background can undergo particle production, and, in the context of an early period of accelerated expansion, i.e. inflation, this same mechanism gives rise to primordial quantum fluctuations. These fluctuations have left an observable imprint in the cosmic microwave background (CMB) and are the seeds for the large-scale structure of the universe we observe today.

Cosmological models

FLRW models describe universes that are both homogeneous and isotropic from the perspective of a set of co-moving observers—that is, the universe looks the same at every point (homogeneity) and in every direction (isotropy). In Appendix E, we review the key geometric features of FLRW spacetimes and present the Friedmann equations in general spacetime dimensions. In this section we will primarily focus on the physically relevant case of three spatial dimensions ($d=3$). For a more comprehensive treatment, we refer the reader to standard cosmology textbooks, such as [Dod03].

An FLRW metric can be written as

$$\mathrm{d}s^2=-\mathrm{d}t^2+a(t)^2\mathrm{d}s^2_{\mathcal{M}_{k,3}}\,, \tag{5.54}$$

where t is the time measured by a co-moving observer, $a(t)$ is the scale factor encoding the expansion (or contraction) of the universe and $\mathcal{M}_{k,3}$ is a maximally symmetric spatial manifold with curvature $k=0,\pm1$, corresponding respectively to flat, spherical or hyperbolic spatial geometries. This ansatz is the most general form of the spacetime metric compatible with spatial homogeneity and isotropy, and serves as the starting point for most theoretical and observational analyses in cosmology. In our own universe, the spatial curvature is tantalisingly close to $k = 0$, but so far experiments are not yet able to determine whether the spatial curvature is exactly vanishing or not.

For any choice of spatial manifold, the FLRW metric is conformally flat. This becomes manifest upon introducing the conformal time coordinate η defined via

$$\eta(t) = \int^{t} \frac{\mathrm{d}t'}{a(t')}, \tag{5.55}$$

in terms of which the metric takes the form

$$\mathrm{d}s^2 = C(\eta)^2 \left(-\mathrm{d}\eta^2 + \mathrm{d}s^2_{\mathcal{M}_{k,3}}\right), \qquad C(\eta) = a(t(\eta)). \tag{5.56}$$

This conformally flat form proves particularly useful for analysing the causal structure of cosmological models, as light-cones and null geodesics are more transparent in these coordinates.

To determine the dynamics of the scale factor $a(t)$, or equivalently $C(\eta)$, one needs to specify the matter content of the universe. A natural and symmetry-compatible assumption is that the stress-energy tensor takes the form of a perfect fluid,

$$T_{\mu\nu} = \mathrm{diag}(-\rho, p, p, p), \tag{5.57}$$

where ρ is the energy density and p is the pressure. Inserting this ansatz into Einstein's equations for the FLRW background yields the Friedmann equations. Expressed in conformal time, they read

$$\left(\frac{\dot{C}}{C}\right)^2 = -\frac{8\pi G_N}{3}\rho - \frac{k}{C^2}, \qquad 6\frac{\ddot{C} + C}{C^3} = 8\pi G_N(\rho - 3p)C + \frac{k}{C}. \tag{5.58}$$

where dots denote derivatives with respect to η. These equations are supplemented by the continuity equation, which encodes local conservation of energy:

$$\dot{\rho} + 3\frac{\dot{C}}{C}(\rho + p) = 0, \tag{5.59}$$

Taken together, these equations govern the full evolution of an FLRW universe once an equation of state $p(\rho)$ is specified. While this relation is not determined by the Einstein equations themselves, standard cosmological models often employ the simple barotropic form $p = w\rho$, with $w \approx 0$ for non-relativistic matter (dust), $w = \frac{1}{3}$ for relativistic matter (radiation), and $w = -1$ for a cosmological constant.

A key quantity characterising the dynamics of FLRW spacetimes is the Hubble parameter, defined by

$$H(t) = \frac{\dot{a}(t)}{a(t)} = \frac{\dot{C}(\eta)}{C(\eta)^2}. \tag{5.60}$$

It quantifies the relative rate of expansion of the universe at a given time and plays a central role in nearly all cosmological phenomena we will encounter. In particular, the behaviour of $H(t)$ encodes important information about the energy content of the universe and its evolution.

As long as the null energy condition (NEC), which in this case reduces to $\rho + p \geq 0$, is satisfied, the expansion rate slows down over time, in other words, the Hubble parameter H decreases during the evolution. When the universe is dominated by a positive cosmological constant, the NEC is saturated and the universe expands at a constant rate. This corresponds precisely to de Sitter space, in which the Hubble parameter takes the fixed value

$$H_{\mathrm{dS}} = \frac{1}{L}, \tag{5.61}$$

with L the de Sitter radius.

Example 5.1. *As an example, let us consider dust in an FLRW universe with $k = 1$. After solving the Friedman equations we find,*

$$C(\eta) = c(1 - \cos\eta), \qquad t = c(\eta - \sin\eta). \tag{5.62}$$

Hence, we find a big bang ($C = 0$) at $\eta = 0$, followed by a big crunch at $\eta = \pi$.

Remark. Why inflation?

To finish the discussion of cosmological models, let us recall a particular problem for universes undergoing a decelerated expansion. In the current standard cosmological model (known as ΛCDM) an accelerated expansion is induced at late times by the cosmological constant. At earlier times, when the universe is radiation or matter dominated, we expect a period of decelerated expansion. We refer to such models as hot big bang models, where the adjective hot refers to the temperature of radiation.

In general, in order to justify the homogeneity of the spatial slices, we would like to have that the distance between regions of space that look the same is much smaller then the maximal distance travelled by light since the beginning of time. Otherwise it is hard to explain why the two region can look similar, since their causal past is disconnected. However, in hot big bang models, this desired inequality is dramatically violated. More precisely, cosmological observations of far away objects allow us to see regions in the past that are separated by much more that than the particle horizon at the time, which is the furthest a signal can travel. Any mechanism attempting to explain homogeneity across these regions then necessarily violates causality and/or locality, leading to the horizon problem.

This apparent tension is resolved by inflation. This scenario posits that there is a surface of last scattering at some time t_s soon after the big bang, before which we cannot clearly see what was going on. This surface is where radiation decouples from matter and so after this time, we can see what is going on, whereas before, we just have what we see from the cosmic microwave radiation. Inflationary models are then obtained by gluing in an exponentially expanding region of de Sitter space, before the surface of last scattering. This inflationary phase gives the past of distant regions time to mix and homogenise so as to explain the homogeneity and isotropy of the universe.

Guided by experimental evidence, it is by now firmly believed that the cosmological constant of our universe is positive. This implies that at large times the scale factor will diverge. At such time the contributions from the cosmological constant will dominate the Friedmann equations which in this regime (for $k = 1$) we can approximate by,

$$\dot{C} = \sqrt{\frac{\Lambda}{3}} C^2 \,. \tag{5.63}$$

Hence we find that

$$C \sim -\sqrt{\frac{\Lambda}{3}} \frac{1}{\eta - \eta_{\mathscr{I}}} + \cdots \,, \tag{5.64}$$

such that the scale factor has a pole at conformal infinity $\eta = \eta_{\mathscr{I}}$. This implies that near $\mathscr{I}^+$, conformal infinity looks like that of de Sitter, i.e. we have an asymptotically de Sitter universe. For $k = 0, -1$ the result is similar. This exponential expansion arises because a positive cosmological constant has a stress-energy tensor $T_{\mu\nu} = \mathrm{diag}(-\Lambda, \Lambda, \Lambda, \Lambda)$, so that although the effective energy density is positive, the pressure is negative. At the current age of the universe, the contribution of Λ to the energy density is thought to be of the same order as that of the matter including dark matter (visible matter being thought to be 3%, dark matter 30% and cosmological constant about 67% of the critical mass of the universe). Such a ratio of matter to cosmological constant is extremely high at early times, and extremely low at late times, and this sometimes leads to the 'why are we alive now?' question. The later periods are however, very cold and boring, and the early periods rather hot, and too early for structure to form, leading to the answer in the form of an anthropic principle.

In this section we introduced just one motivation for considering inflation, i.e. the horizon problem. However, there are various other problems arising in hot big bang models, such as

- The curvature related to the approximate spatial flatness of our current universe,
- The particle horizon problem related to the statistical isotropy of our universe,
- The phase coherence problem related to the homogeneity of the CMB,
- The scale invariance problem related to the scale invariance of the CMB.

Discussing all these problems in depth would take us too far, but importantly inflation offers a way out for each of them and provides us with a plausible explanation for the current state of the universe. For more details we refer to [Dod03, Paj20].

Canonical quantisation in an FLRW universe

Having set up the FLRW background and its dynamics, we now turn to the quantisation of a free massive scalar field propagating in such a spacetime. Let us for simplicity consider FLRW backgrounds with flat spatial slices.

Switching to conformal time η, the d'Alembertian operator in this background takes the form

$$\Box\phi = C(\eta)^{-(d+1)} \partial_\eta \left(C(\eta)^{(d-1)} \partial_\eta \phi \right) - C(\eta)^{-2} \nabla^2 \phi \,. \tag{5.65}$$

Thanks to the spatial translation invariance, we can decompose the scalar field into Fourier modes and focus on their time evolution:

$$u_{\mathbf{k}}(\eta,\mathbf{x}) = \frac{\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{\frac{d}{2}}} C(\eta)^{\frac{1-d}{2}} \chi_{\mathbf{k}}(\eta)\,. \tag{5.66}$$

Substituting this ansatz into the wave equation, one finds after a direct computation:

$$\begin{aligned} \Box u_{\mathbf{k}} =& \frac{C(\eta)^{-\frac{3+d}{2}}}{(2\pi)^{\frac{d}{2}}} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} \left[\ddot{\chi}_{\mathbf{k}}(\eta) + \mathbf{k}^2 \chi_{\mathbf{k}}(\eta) + \frac{1-d}{2} \left(\frac{2\ddot{C}(\eta)}{C(\eta)} + (7-d) \left(\frac{\dot{C}(\eta)}{C(\eta)} \right)^2 \right) \chi_{\mathbf{k}}(\eta) \right] \\ =& \frac{C(\eta)^{-\frac{3+d}{2}}}{(2\pi)^{\frac{d}{2}}} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} \left[\ddot{\chi}_{\mathbf{k}}(\eta) + \mathbf{k}^2 \chi_{\mathbf{k}}(\eta) - \xi(d) C^2(\eta) R(\eta)\, \chi_{\mathbf{k}}(\eta) \right]. \end{aligned} \tag{5.67}$$

where $R(\eta)$ is the Ricci scalar of the background and $\xi(d) = \frac{1}{4}\frac{1-d}{d}$. Including the mass and curvature coupling ξ, we find that $u_{\mathbf{k}}$ solves the Klein-Gordon equation if $\chi_{\mathbf{k}}$ solves

$$\ddot{\chi}_{\mathbf{k}} + \omega^2(\eta)\chi_k = 0\,, \tag{5.68}$$

where

$$\omega^2(\eta) = \mathbf{k}^2 + m^2\, C^2(\eta) + C^2(\eta) R(\eta) \left(\xi - \xi(d)\right). \tag{5.69}$$

To proceed with the canonical quantisation, we define an inner product on the space of solutions to the wave equation. The induced metric on a hypersurface at constant η is

$$h_{mn} \mathrm{d}x^m \mathrm{d}x^n = C^2(\eta) \delta_{mn} \mathrm{d}x^m \mathrm{d}x^n\,, \tag{5.70}$$

so that we can define the inner product

$$\begin{aligned} \langle u_{\mathbf{k}}, u_{\mathbf{k}'} \rangle =& \mathrm{i} \int \mathrm{d}^d\mathbf{x}\, C(\eta)^{(d-1)} \left(u^*_{\mathbf{k}} \partial_\eta u_{\mathbf{k}'} - \partial_\eta u^*_{\mathbf{k}} u_{\mathbf{k}'} \right) \\ =& \mathrm{i}\delta^{(d)}(\mathbf{k}-\mathbf{k}') \left(\chi^*_{\mathbf{k}} \partial_\eta \chi_{\mathbf{k}'} - \partial_\eta \chi^*_{\mathbf{k}} \chi_{\mathbf{k}'} \right). \end{aligned} \tag{5.71}$$

We therefore normalise the modes $\chi_{\mathbf{k}}$ by fixing the Wronskian:

$$\mathrm{i} \left(\chi^*_{\mathbf{k}} \partial_\eta \chi_{\mathbf{k}'} - \partial_\eta \chi^*_{\mathbf{k}}\, \chi_{\mathbf{k}'} \right) = 1\,. \tag{5.72}$$

The inner produce, or equivalently the symplectic form on the phase space, is conserved in time. However, now that the space-time is explicitly time-dependent, energy is not conserved and in the absence of asymptotically static patches, particles will be created at any time and there is no fixed or preferred vacuum.

However, at each time η there is some notion of vacuum, i.e. the state without particles, or equivalently the instantaneous lowest energy state. Expanding the quantum scalar field as

$$\phi = \int \frac{\mathrm{d}^3\mathbf{k}}{\sqrt{(2\pi)^3 C(\eta)}} \left(a_{\mathbf{k}} \chi_{\mathbf{k}} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} + a^\dagger_{\mathbf{k}} \chi^*_{\mathbf{k}} \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}} \right), \tag{5.73}$$

we find the Hamiltonian

$$H = \frac{1}{4}\int \mathrm{d}^3\mathbf{k}\left(F_{\mathbf{k}}a_{\mathbf{k}}a_{-\mathbf{k}} + F^*_{\mathbf{k}}a^\dagger_{\mathbf{k}}a^\dagger_{-\mathbf{k}} + (a^\dagger_{\mathbf{k}}a_{\mathbf{k}} + a_{\mathbf{k}}a^\dagger_{\mathbf{k}})E_k\right), \tag{5.74}$$

where

$$E_{\mathbf{k}} = |\dot{\chi}|^2 + \omega^2|\chi|^2, \qquad F_{\mathbf{k}} = \dot{\chi}^2_{\mathbf{k}} + \omega^2\chi^2. \tag{5.75}$$

With this implicit choice of vacuum we find

$$\frac{E}{V} = \langle 0|H|0\rangle = \frac{1}{4}\int \mathrm{d}^d\mathbf{k}\, E_{\mathbf{k}}, \tag{5.76}$$

where as usual we took out a diverging volume factor $V \sim \delta^{(d)}(0)$ to get the local energy density. Minimising $E_{\mathbf{k}}$ for each mode subject to the appropriate normalisation condition a brief computation shows that we obtain the following initial data to impose at the hypersurface $\eta = \eta_0$

$$(\chi_{\mathbf{k}}, \dot{\chi}_{\mathbf{k}})\big|_{\eta=\eta_0} = \frac{1}{\sqrt{2\omega(\eta_0)}}(1, -\mathrm{i}\omega(\eta_0)). \tag{5.77}$$

The particle content measured in this vacuum is that of the instantaneous 'static' observer.

Exercise 5.7. *Derive the initial conditions* (5.77) *starting from the reduced Klein-Gordon equation* (5.68) *by demanding that the energy is minimized for each mode.*

Remark. When the frequencies $\omega(\eta)$ are varying slowly enough, we can use an adiabatic approximation to define also an evolution of the vacuum. This arises as the WKB approximation at leading order; more generally, the adiabatic approximation takes the WKB approximation beyond leading order and then there will be particle creation. [2]

Having defined the instantaneous vacuum at each time η and ideally also have an evolution equation, we can define the Bogoliubov transformations between any two times η_1 and η_2 and compute the particle creation just like we did in the previous chapters. As we discussed above, the $E_{\mathbf{k}}$ determines the energy at some time η. The $F_{\mathbf{k}}$ on the other hand determines the instantaneous particle creation at each time. If we are in the lowest energy vacuum at some time η_0 the $F_{\mathbf{k}}$ at that time vanishes.

For more details on the adiabatic vacuum and WKB approximation we refer the reader to the textbook [BD84].

To make the above analysis more explicit, let us consider FLRW models in which the scale factor asymptotically approaches constant values in the far past and future:[3]

$$a(t) = \begin{cases} a_1 & \text{as} \quad t \to -\infty \\ a_2 & \text{as} \quad t \to \infty \end{cases}. \tag{5.78}$$

[2] This requires a mechanism for the dissipation of the particles created and otherwise we are working in the full quantum theory in the Heisenberg representation where the state is fixed and the evolution is carried by the operators.

[3] As usual we are not very careful with the appropriate fall-off and smoothness conditions but assume the function $a(t)$ is sufficiently well-behaved.

This assumption allows us to interpret the asymptotic regions as Minkowski spacetimes with different effective scales, making the identification of vacuum states more transparent.

Let us now consider the case in which the conformal scale factor $C(\eta)$ approaches constant values both in the distant past and future. Then the frequency $\omega(\eta)$ also asymptotes to constants:

$$\omega(\eta) = \begin{cases} \omega_{\text{in}} & \text{as} \quad \eta \to -\infty \\ \omega_{\text{out}} & \text{as} \quad \eta \to \infty \end{cases} . \tag{5.79}$$

In such situations, the theory admits natural vacuum states both in the asymptotic past and future. These are defined by choosing mode functions that reduce to positive-frequency solutions in the corresponding Minkowski limits:

$$\begin{aligned} \chi_{\mathbf{k}}^{\text{in}}(\eta) &\to \frac{1}{\sqrt{2\omega_{\text{in}}}} \exp\left(-\mathrm{i}\omega_{\text{in}}\eta\right), \qquad \text{as} \quad \eta \to -\infty\,, \\ \chi_{\mathbf{k}}^{\text{out}}(\eta) &\to \frac{1}{\sqrt{2\omega_{\text{out}}}} \exp\left(-\mathrm{i}\omega_{\text{out}}\eta\right), \qquad \text{as} \quad \eta \to \infty\,. \end{aligned} \tag{5.80}$$

Each of these choices defines its own Fock space with vacuum states $\left|0_{\text{in/out}}\right\rangle$ and creation and annihilation operators, $a_{\mathbf{k}}^{\text{in/out}}$ and $\left(a_{\mathbf{k}}^{\text{in/out}}\right)^{\dagger}$, satisfying

$$a_{\mathbf{k}}^{\text{in/out}} \left|0_{\text{in/out}}\right\rangle = 0\,. \tag{5.81}$$

To relate these two vacua we compute the Bogoliubov coefficients (4.33).The spatial homogeneity and isotropy imply that the Bogoliubov coefficients take the form

$$\alpha_{\mathbf{k},\mathbf{k}'} = \alpha_{\mathbf{k}}\delta(\mathbf{k}-\mathbf{k}')\,, \qquad \beta_{\mathbf{k},\mathbf{k}'} = \beta_{\mathbf{k}}\delta(\mathbf{k}+\mathbf{k}')\,. \tag{5.82}$$

i.e. they only mix modes with momenta $\mathbf{k}$ and $-\mathbf{k}$, and are functions of $|\mathbf{k}|$ only due to rotational symmetry. The Bogoliubov transformation therefore takes the form

$$u_{\mathbf{k}}^{\text{in}} = \alpha_{\mathbf{k}} u_{\mathbf{k}}^{\text{out}} + \beta_{\mathbf{k}} u_{-\mathbf{k}}^{*\text{out}} \tag{5.83}$$

The first condition in (4.34) is automatically satisfied, while the second gives

$$|\alpha_{\mathbf{k}}|^2 - |\beta_{\mathbf{k}}|^2 = 1 \tag{5.84}$$

The physical interpretation of this result is that the in-vacuum appears as a many-particle state from the perspective of the out-vacuum. The number of particles with momentum $\mathbf{k}$ in the out region is given by $|\beta_{\mathbf{k}}|^2$, so that the total particle production is finite only if

$$\int \mathrm{d}^d\mathbf{k} |\beta_{\mathbf{k}}|^2 < \infty\,. \tag{5.85}$$

Remark. Note that since FLRW metric is conformally flat, in the case of a massless, conformally coupled scalar we can always construct a conformal vacuum. Indeed, in this case we have

$$\omega_k^2(\eta) = k^2 \tag{5.86}$$

and the solution simplifies drastically to

$$\chi_{\mathbf{k}}(\eta) = \frac{1}{\sqrt{2\omega_k}} \mathrm{e}^{-ik\eta} \,. \tag{5.87}$$

In this case the function $u_{\mathbf{k}}$ can simply be obtained as a conformal transformation of the plane waves in standard Minkowski space.

Exercise 5.8 (An exactly solvable model)**.** *To get a better feeling for these types of properties, it is interesting to work out some non-trivial models where the equations for $\chi_{\mathbf{k}}$ can be solved explicitly. One such model is given by the two dimensional spacetime with $\xi = 0$ and $C(\eta) = A + B\tanh(\rho\eta)$ studied in [BD77].*

Verify that the equations of motion for χ_k can be solved by the following two sets of functions,

$$\chi_{\mathbf{k}}^{\mathrm{in}}(\eta) = \frac{1}{\sqrt{2\omega_{\mathrm{in}}}} \exp\left[-\mathrm{i}\omega_+\eta - \frac{\mathrm{i}\omega_-}{\rho}\log(2\cosh\rho\eta)\right] \tag{5.88}$$

$$\times\, {}_2F_1\left(\frac{\rho+\mathrm{i}\omega_-}{\rho}, \frac{\mathrm{i}\omega_-}{\rho}, \frac{\rho-\mathrm{i}\omega_{\mathrm{in}}}{\rho}, \frac{1}{2}(1+\tanh\rho\eta)\right), \tag{5.89}$$

$$\chi_{\mathbf{k}}^{\mathrm{out}}(\eta) = \frac{1}{\sqrt{2\omega_{\mathrm{out}}}} \exp\left[-\mathrm{i}\omega_+\eta - \frac{\mathrm{i}\omega_-}{\rho}\log(2\cosh\rho\eta)\right] \tag{5.90}$$

$$\times\, {}_2F_1\left(1+\frac{\mathrm{i}\omega_-}{\rho}, \frac{\mathrm{i}\omega_-}{\rho}, 1-\frac{\mathrm{i}\omega_{\mathrm{out}}}{\rho}, \frac{1}{2}(1+\tanh\rho\eta)\right), \tag{5.91}$$

where $\omega_\pm$ are defined as $\omega_\pm = \frac{1}{2}(\omega_{\mathrm{out}} \pm \omega_{\mathrm{in}})$.

Show that these functions have the appropriate plane wave limit as $\eta \to \pm\infty$ and show that the squared Bogoliubov coefficients are given by

$$|\alpha_{\mathbf{k}}|^2 = \frac{\sinh^2\left(\frac{\pi\omega_+}{\rho}\right)}{\sinh\left(\frac{\pi\omega_{\mathrm{in}}}{\rho}\right)\sinh\left(\frac{\pi\omega_{\mathrm{out}}}{\rho}\right)}, \quad |\beta_{\mathbf{k}}|^2 = \frac{\sinh^2\left(\frac{\pi\omega_-}{\rho}\right)}{\sinh\left(\frac{\pi\omega_{\mathrm{in}}}{\rho}\right)\sinh\left(\frac{\pi\omega_{\mathrm{out}}}{\rho}\right)} \,. \tag{5.92}$$

Hint: Use the identities for hypergeometric functions introduced in Appendix F.

A novel feature of having the time-dependent frequencies is that for large length scales and small enough k and m, the frequency ω_k^2 can become negative. This, as we will see now, leads to the corresponding modes ceasing to oscillate and essentially freezing out.

To see this in more detail, let us consider the simplest possible cosmological model, namely de Sitter. Looking at the planar coordinates (see Appendix E) we see that de Sitter is an FLRW space with scale factor $a = L\mathrm{e}^{t/L}$ and constant Hubble parameter $H = 1/L$. In particular, the Hubble parameter is proportional to the radius of the cosmological horizon.

The de Sitter mode functions, discussed in the previous section, behave very different from their

Minkowski counterparts when the wave-number $\mathbf{k}$ becomes smaller that the co-moving Hubble parameter

$$k < k_{\mathrm{HC}} = aH = \frac{1}{|\eta|}\,, \tag{5.93}$$

where HC stands for Hubble crossing, sometimes also called horizon crossing. In physical length scales this means that the physical wavelength $\lambda = a/k$ is stretched by the expansion to become larger than the Hubble radius $1/H$. Since k and H are constant, while $a = \mathrm{e}^{t/L}$ grows with time, all modes cross the Hubble radius as time proceeds and eventually become "super-Hubble" modes. Unlike "sub-Hubble" modes with $k \gg aH$, which oscillate, super-Hubble modes freeze out and asymptote to a constant.

Cosmological correlators

Having discussed general properties of quantum fields in cosmological backgrounds, a natural next question is: What are the physical observables in these theories? As in flat-space QFT, the primary observables are expectation values of operator products. However, in a cosmological context, we typically have access only to correlators evaluated at late times, $\eta \to 0$. In this limit, the observables tend to freeze out and become time-independent, which is why we focus on equal-time expectation values of local operators. These are known as cosmological correlators:

$$\lim_{\eta\to 0} \langle \mathcal{O}(\mathbf{x}_1,\eta)\mathcal{O}(\mathbf{x}_2,\eta)\cdots\mathcal{O}(\mathbf{x}_n,\eta)\rangle\ . \tag{5.94}$$

In the free theory, all information is encoded in the two-point functions of the field ϕ and its conjugate momentum π. Odd-point correlators vanish due to the $\mathbb{Z}_2$ symmetry $\phi \longleftrightarrow -\phi$, and higher even-point functions are reducible to two-point functions via Wick's theorem. We therefore focus on the two-point function which in momentum space take the following form:

$$\begin{aligned}\lim_{\eta\to 0} \left\langle \phi(\mathbf{k})\phi(\mathbf{k}')\right\rangle =&|u_{\mathbf{k}}|^2 \left\langle a_{\mathbf{k}} a^\dagger_{-\mathbf{k}'}\right\rangle \\ =&(2\pi)^3\delta(\mathbf{k}-\mathbf{k}')P(k)\,,\end{aligned} \tag{5.95}$$

where $P(k)$ is the power spectrum, which for a massless scalar field in de Sitter space takes the form

$$P(k) = \frac{H^2}{2k^3}\,, \tag{5.96}$$

The delta function reflects momentum conservation, and the isotropy of the background ensures that $P(k)$ depends only on the magnitude of $\mathbf{k}$, not its direction. The fact that $P(k)$ asymptotes to a constant as $\eta \to 0$ signals the absence of mass, while the k^{-3} scaling encodes the scale invariance of the massless scalar field. This is more transparent in position space, where the correlator becomes independent of the separation between the two points.

When a mass is introduced, the power spectrum becomes

$$P(k) = |u_{\mathbf{k}}|^2 = \frac{H^2}{\pi 2^{2(\nu-1)}\Gamma(\nu)^2}\frac{(-k\eta)^{3-2\nu}}{k^3}\,, \tag{5.97}$$

valid for $m^2 < \frac{9}{4}H^2$, with $\nu = \sqrt{\frac{9}{4} - \frac{m^2}{H^2}}$ for a minimally coupled scalar field. The mass breaks scale invariance, giving $P(k) \propto k^{-2\nu}$, and introduces time dependence. For $m^2 > 0$, we find $3 - 2\nu > 0$, so the power spectrum decays over time and vanishes in the far future. This reflects the restoring force of the quadratic potential, which drives the field back to $\phi = 0$. For $m^2 < 0$, an instability appears: the power spectrum grows with time and diverges as $\eta \to 0$.

When $m^2 > \frac{9}{4}H^2$, ν becomes complex and the power spectrum oscillates with decaying amplitude, scaling as η^3. In cosmological applications, we are primarily interested in (nearly) massless fields, as these give rise to long-lived perturbations that can be observed at late times without causing instabilities.

Fluctuating gravitons

The quantization of gravitational fluctuations proceeds analogously to the scalar field case. We decompose the full metric into a classical FLRW background and small quantum perturbations:

$$g_{\mu\nu}(t,\mathbf{x}) = \bar{g}_{\mu\nu}(t,\mathbf{x}) + h_{\mu\nu}(t,\mathbf{x})\,. \tag{5.98}$$

Naively, $h_{\mu\nu}$ contains ten independent components, but four are constrained by the Einstein equations, which are only first order in time derivatives for certain components. This follows from the contracted Bianchi identity,

$$\nabla^\mu G_{\mu\nu} = 0\,. \tag{5.99}$$

which implies a conservation law. Explicitly,

$$\partial_t G^{t\nu} = -\partial_m G^{m\nu} - \Gamma^\alpha_{\alpha\gamma} G^{\nu\gamma} + \Gamma^\nu_{\alpha\gamma} G^{\alpha\gamma}\,. \tag{5.100}$$

Since the right-hand side contains at most two derivatives of the metric, $G^{t\nu}$ has at most one time derivative. Consequently, four components of the Einstein equations serve as constraint equations, reducing the number of physical degrees of freedom.

A full analysis of the linearised Einstein equations requires more effort (see GRII), but for a spatially flat FLRW background, the result simplifies dramatically in a convenient gauge:

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)(\delta_{mn} + \gamma_{mn})\mathrm{d}x^m \mathrm{d}x^n,\,, \tag{5.101}$$

where the tensor perturbation γ_{mn} is symmetric, traceless ($\gamma^m_m = 0$), and transverse ($\partial^m \gamma_{mn} = 0$), leaving two independent degrees of freedom—the helicity-2 modes of the graviton.

A convenient gauge choice is given by

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(\delta_{mn} + \gamma_{mn})\mathrm{d}x^m \mathrm{d}x^n\,, \tag{5.102}$$

where γ is transverse, i.e. $\partial^m \gamma_{mn} = 0$ and traceless $\gamma^m_m = 0$. So we are left with two independent components. Expanding the Einstein Hilbert action to quadratic order in the fluctuations we find

$$S_2 = \frac{M_{\rm Pl}^2}{8}\int \mathrm{d}^3\mathbf{x}\,\mathrm{d}t\, a^2\left(\gamma'_{mn}\gamma'^{mn} - \partial_m\gamma_{nk}\partial^m\gamma^{nk}\right). \tag{5.103}$$

This action can be derived by rigorously linearising Einsteins equations, but alternatively it could easily have been guessed by writing the simplest action consistent with the symmetries of the problem. As we did for the scalar we can expand the graviton in plane waves by writing

$$\gamma_{mn}(x) = \int \mathrm{d}^3\mathbf{k}\sum_{s=\pm}\epsilon^s_{mn}(\mathbf{k})\gamma_s(\mathbf{k})\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}, \tag{5.104}$$

where $\epsilon^s_{mn}(\mathbf{k})$ are the polarisation tensors, which are generally complex and satisfy

$$\begin{aligned}
\epsilon^s_{mm}(\mathbf{k}) =& \mathbf{k}^m\epsilon^s_{mn}(\mathbf{k}) = 0\,, && \text{Transverse and traceless}\,,\\
\epsilon^s_{mn}(\mathbf{k}) =& \epsilon^s_{nm}(\mathbf{k})\,, && \text{Symmetric}\,,\\
\epsilon^s_{mn}(\mathbf{k})\epsilon^s_{nk}(\mathbf{k}) =& 0\,, && \text{Light-like}\,,\\
\epsilon^s_{mn}(\mathbf{k})\epsilon^{s'mn}(\mathbf{k}) =& \delta^{ss'}\,, && \text{Unit normalised}\,,\\
\epsilon^s_{mn}(\mathbf{k})^* =& \epsilon^s_{mn}(-\mathbf{k})\,, && \gamma_{ij}\ \text{ is real}\,.
\end{aligned} \tag{5.105}$$

From these properties we can derive explicit expressions for the polarisation vector and rewrite the action as

$$S_2 = \frac{M_{\rm Pl}^2}{4}\int \mathrm{d}^3\mathbf{k}\,\mathrm{d}t\, a^2\sum_{s=\pm}\left(\gamma'_s(\mathbf{k})\gamma'_s(-\mathbf{k}) - \frac{k^2}{a^2}\gamma_s(\mathbf{k})\gamma_s(-\mathbf{k})\right). \tag{5.106}$$

Now this action consists of two independent copies of the action for a (canonically normalised) massless scalar field, up to a normalisation factor $\frac{M_{\rm Pl}^2}{2}$. To quantise the gravitons we therefore can proceed exactly as above. We can promote $\gamma_s(\mathbf{k})$ to an operator and expand it in creation and annihilation operators,

$$\gamma_s(\mathbf{k}) = \frac{\sqrt{2}}{M_{\rm Pl}}\left(f_\mathbf{k} a^s_\mathbf{k} + f^*_\mathbf{k} a^{s\dagger}_\mathbf{k}\right), \tag{5.107}$$

where for both signs $s = \pm$ the creation and annihilation operators satisfy the canonical commutation relations. If we now assume a de Sitter background we can explicitly compute the graviton power spectrum in the same way as we did for the massless scalar field. We find

$$\begin{aligned}
\left\langle\gamma_{mn}(\mathbf{k})\gamma_{mn}(\mathbf{k}')\right\rangle =& \sum_{s,s'}\epsilon^s_{mn}(\mathbf{k})\epsilon^{s'}_{mn}(\mathbf{k}')\left\langle\gamma_s(\mathbf{k})\gamma_{s'}(\mathbf{k}')\right\rangle\\
=& \frac{2}{M_{\rm Pl}^2}\sum_{s,s'}\epsilon^s_{mn}(\mathbf{k})\epsilon^{s'}_{mn}(\mathbf{k}')(2\pi)^3\delta(\mathbf{k}+\mathbf{k}')|f_\mathbf{k}|^2\\
=& \frac{2}{M_{\rm Pl}^2}\frac{H^2}{2k^3}\sum_{s,s'}\delta_{ss'}(2\pi)^3\delta(\mathbf{k}+\mathbf{k}')|f_\mathbf{k}|^2\\
=& (2\pi)^3\delta(\mathbf{k}+\mathbf{k}')P_T\,,
\end{aligned} \tag{5.108}$$

with

$$P_T = \frac{4}{k^3}\frac{H^2}{M_{\rm Pl}^2}\,. \tag{5.109}$$

This power spectrum provides us with a clear prediction for the CMB spectrum after a period of inflation starting from the Bunch-Davies vacuum. This spectrum can then be compared to the observational data from cosmological experiments and gives an excellent match. However, the match is not exact. This is to be expected, since in this course we take the coarse approximation that all fields are free. It turns out that this is an excellent approximation to predict the CMB radiation but recent experiment have nonetheless found tiny non-Gaussianities hidden in the observed radiation. To properly account for such effects one has to introduce interactions. This goes beyond the scope of these lectures, but we refer the interested reader to [Paj20] for more details.

5.3 Thermal QFT

Another surprisingly simple application of quantum field theory in curved space are quantum field theories at finite temperature T. In statistical mechanics thermal ensembles are of great importance. In particular, the canonical ensemble describes a system in contact with a heat reservoir at a fixed temperature T.[4] Energy can be interchanged between the system and the reservoir, but the particle number and volume are fixed. In this section we will formalise how to formulate quantum field theory at non-zero temperature. To see what temperature has to do with curved backgrounds, let us briefly review some aspects of thermal states in quantum field theory.

Thermal states

Thermal states are a feature of statistical physics at a temperature T, the equilibrium state is a probability distribution of physical states. In quantum mechanics such a distribution is given in the form of a density matrix,

Definition 5.1. A density matrix is an element $\rho \in \mathcal{H}\otimes\mathcal{H}^*$ that is Hermitian, positive definite and has unit trace $\operatorname{tr}\rho = 1$.

Such matrices are always diagonalizable and an orthonormal basis of states $|n\rangle$ can be found such that they can be expressed in the form

$$\rho = \sum_n p_n \,|n\rangle\,\langle n| \tag{5.110}$$

where the coefficients are positive $p_n \geq 0$ and $\sum_n p_n = 1$. In this context, the coefficients p_n can be thought of as the probability of the ensemble to be in the state $|n\rangle$ such that the expectation value of an observable A can be computed as

$$\langle A\rangle_\rho = \operatorname{tr}\rho A = \sum_n \rho_n \langle n|A|n\rangle\,. \tag{5.111}$$

[4] One can consider various other ensembles such as the grand canonical ensemble, where also particle number and volume can be be exchanged with the reservoir.

A density matrix represents a pure state when $\rho = |\psi\rangle\langle\psi|$ for some normalized state $|\psi\rangle$, i.e., ρ has rank 1, otherwise states are said to be mixed. Mixed states arise naturally when part of a quantum system is hidden. Consider for example a pure state $|\psi\rangle \in \mathcal{H} = \mathcal{H}_L \otimes \mathcal{H}_R$ where the Hilbert space $\mathcal{H}_L$ is hidden for an observer. The state observed by the observer is then obtained by the partial trace over $\mathcal{H}_L$. More precisely, consider the state $|\psi\rangle$

$$|\psi\rangle = \sum_{r,s} \psi_{r,s} |r\rangle_L \otimes |s\rangle_R \,. \tag{5.112}$$

where $\{|r\rangle\}$ represents a basis of $\mathcal{H}_L$ and $\{|s\rangle\}$ of $\mathcal{H}_R$. The partially traced density matrix perceived by the observer is then given by

$$\rho_R = \mathrm{Tr}_L |\psi\rangle\langle\psi| = \sum_r \bar{\psi}_{r,s_1} \psi_{r,s_2} |s_2\rangle\langle s_1| \,. \tag{5.113}$$

The observed state is mixed if the density matrix ρ_R has rank greater than one. In this case the systems R and L are said to be entangled. Given a Hamiltonian H, we define a thermal state as follows,[5]

Definition 5.2. A thermal state of temperature T is a state of the form

$$\rho_\beta = \frac{1}{Z_\beta} \exp(-\beta H) = \frac{1}{Z_\beta} \sum_n \mathrm{e}^{-\beta E_n} |n\rangle\langle n| \,, \qquad \beta = \frac{1}{T} \,, \tag{5.114}$$

where $|n\rangle$ is a basis of energy eigenstates of energy E_n and

$$Z_\beta = \mathrm{Tr} \exp(-\beta H) = \sum_n \mathrm{e}^{-\beta E_n} \,, \tag{5.115}$$

is the partition function.

Note that we use units where the Boltzmann's constant $k_B = 1$. This is in exact analogy with the canonical ensemble in statistical mechanics.

Example 5.2 (Bose-Einstein distribution)**.** *In the context of a harmonic oscillator (i.e., a single mode of a quantum field), we can compute the thermal expectation of the number operator N using $E_n = (n + \frac{1}{2})\omega$ to obtain*

$$\mathrm{Tr}(\rho_\beta N) = \frac{\sum_n n \mathrm{e}^{-\beta n \omega}}{\sum_n \mathrm{e}^{-\beta n \omega}} = \frac{1}{\omega} \frac{d}{d\beta} \log(1 - \mathrm{e}^{-\beta\omega}) = \frac{1}{\mathrm{e}^{\beta\omega} - 1} \,. \tag{5.116}$$

A useful way to characterize thermal states in quantum mechanics and quantum field theory was described by Kubo, Martin and Schwinger [Kub57, MS59] and is called the KMS condition. Similarly a state satisfying the KMS condition is called a KMS state.

Definition 5.3 (KMS state)**.** A KMS state is a state for which the time evolution of operators $A \to A_t$ can be continued to complex time in such a way that for a time-independent operator, B, we have

$$\langle A_t B\rangle_{KMS} = \left\langle B A_{t+i\beta} \right\rangle_{KMS} \,, \tag{5.117}$$

[5] One can decorate this definition with chemical potentials when the particle number is allowed to change, i.e. in a grand canonical ensemble.

where $\langle A_z B\rangle_{KMS}$ and $\langle BA_z\rangle_{KMS}$ are analytic functions of z in the strip $0 < \mathrm{im}\, z < \beta$.

For finite systems the definition of the KMS condition is equivalent to the the definition of thermal states above. To see this recall that in the Heisenberg representation, an operator A has time dependence $A_t = \mathrm{e}^{iHt} A_0 \mathrm{e}^{-iHt}$. For our thermal density matrix above we can compute

$$\begin{aligned}\langle A_t B\rangle_\beta &= \frac{1}{Z_\beta}\,\mathrm{tr}(\mathrm{e}^{-\beta H} A_t B) = \frac{1}{Z_\beta}\,\mathrm{tr}(\mathrm{e}^{-\beta H + iHt} A_0 \mathrm{e}^{-iHt} B)\\ &= \frac{1}{Z_\beta}\,\mathrm{tr}(A_{t+i\beta}\mathrm{e}^{-\beta H} B) = \frac{1}{Z_\beta}\,\mathrm{tr}(\mathrm{e}^{-\beta H} B A_{t+i\beta})\\ &= \langle B A_{t+i\beta}\rangle_\beta\,,\end{aligned}$$

where we have used the cyclic property of the trace. This is often interpreted as the property that our system can be analytically continued to Euclidean signature with periodicity in imaginary time. In infinite dimensions these manipulations are a lot more subtle as we might encounter phase transitions, spontaneous symmetry breaking, operators that are not trace class and so on. However, the content of the KMS condition is precisely that a similar relation continues to hold in the thermodynamic limit.[6]

Matsubara formalism or QFT on $\mathbb{R}^d \times S^1$

The Matsubara (or imaginary time) formalism makes the above observations manifest and reformulates quantum field theory at finite temperature as Euclidean quantum field theory on $\mathbb{R}^d \times S^1$. In this setup it will be often more convenient to work in the path integral formulation of quantum field theory. We will not introduce this in detail but instead refer to one of the many standard text on quantum field theory.

We can now apply the same logic to Green's functions in a thermal state. The thermal Green's function can be obtained as the analytic continuation of the Wightman function or alternatively as the Green's function for the relevant operator, i.e., the Laplacian on $\mathbb{R}^3 \times S^1_\beta$ where now S^1 is a circle of length β.

Definition 5.4. The thermal Green's function is defined as

$$G_\beta(x,y) = \langle \phi(x)\phi(y)\rangle_\beta = \frac{\mathrm{Tr}(\mathrm{e}^{-\beta H}\phi(x)\phi(y))}{\mathrm{Tr}(\mathrm{e}^{-\beta H})}\,. \tag{5.118}$$

Furthermore, when our system enjoys time translation symmetry, the thermal Green's function is of the form $G_\beta(x,y) = G_\beta(t-t', \mathbf{x}, \mathbf{x}')$ and satisfies the KMS condition, i.e. it is periodic in imaginary time with period $i\beta$. This property follows directly as above from the KMS condition.

Example 5.3. *In flat space, the thermal propagator for a massless free scalar can be constructed by images in imaginary time of period β. Thus we can identify the thermal greens function on Minkowski space for the massless wave equation as*

$$G_\beta(x,x') = \sum_{n\in\mathbb{Z}} \frac{1}{4\pi^2((t-t'+in\beta+i\epsilon)^2 - \mathbf{x}\cdot\mathbf{x})}\,. \tag{5.119}$$

[6]If a phase transition takes place or if some symmetry is spontaneously broken, the KMS state might not be uniquely defined.

Defining the imaginary time coordinate $\tau = i\tau$, it immediately follows from the KMS condition it immediately follows that $\phi(0,\mathbf{x}) = \pm\phi(\beta,\mathbf{x})$ where the sign is determined by whether the fields commute or anti-commute, i.e. whether they are bosonic or fermionic. On $\mathbb{R}^d \times S^1$ it is convenient to Fourier decompose the field

$$\phi(\tau,\mathbf{x}) = \sum_n \phi(\omega_n,\mathbf{x})e^{i\omega_n\tau}\,. \tag{5.120}$$

In order to satisfy the KMS condition we only allow discrete frequencies

$$\begin{aligned} \omega_n &= \frac{2\pi n}{\beta} && \text{Bosonic fields} \\ \omega_n &= \frac{2\pi(n+1)}{\beta} && \text{Fermionic fields} \end{aligned} \tag{5.121}$$

These frequencies are called the Matsubara frequencies.

A fundamental object in thermal field theory is the partition function, $Z_\beta = \mathrm{Tr}\, e^{-\beta H}$, where H is the Hamiltonian of the system. In terms of the Euclidean path integral on $\mathbb{R}^d \times S^1$ this takes the form

$$Z_\beta = \int \mathcal{D}\phi\, \langle\phi|\, e^{-\beta H}\, |\phi\rangle = \int \mathcal{D}\phi e^{-S[\phi]}\,. \tag{5.122}$$

with the Euclidean action, $S = \int d^{d+1}x\mathcal{L}$, defined over an imaginary time interval $\tau \in [0,\beta]$ with appropriate boundary conditions (5.121) encoding thermal equilibrium.

In general it is very hard to compute the partition function, but for a free scalar we can explicitly carry out the computation. Substituting the Fourier expansion into the expression for the partition function we obtain

$$\begin{aligned} Z_\beta = \int \mathcal{D}\phi \exp\Bigg\{ -\beta \int_0^\beta d\tau \int d^d\mathbf{x} \sum_{l,n} \int \frac{d^d\mathbf{p}d^d\mathbf{q}}{(2\pi)^{2d}} \phi(\omega_l,\mathbf{p})\phi(\omega_n,\mathbf{q}) \times \\ \times\, e^{i(\omega_l\tau+\mathbf{p}\cdot\mathbf{x})}e^{i(\omega_n\tau+\mathbf{q}\cdot\mathbf{x})}\frac{-\omega_n^2-\mathbf{q}^2-m^2}{2}\Bigg\} \end{aligned} \tag{5.123}$$

After some manipulations the thermal partition function for the free scalar reduces to

$$Z_\beta = \exp\left\{\int \frac{d^d\mathbf{k}}{(2\pi)^d}\left(-\log\beta + \frac{1}{2}\sum_n \omega_n^2 - \frac{1}{2}\sum_n(\omega_n^2+\mathbf{k}^2+m^2)\right)\right\} \tag{5.124}$$

Exercise 5.9. *Derive the expression (5.124) for the thermal partition function of a free scalar. Hint: Remember the following representation of the δ-function* $\int_0^\beta d\tau e^{i(\omega_l+\omega_n)\tau} = \beta\delta(m+n)$.

With this expression for the partition function at hand we can compute a variety of thermodynamical properties such as the pressure and entropy, using the standard relations,

$$P = \frac{1}{\beta}\frac{\partial \ln Z}{\partial V}\,, \qquad S = -\beta\frac{\partial \ln Z}{\partial \beta}\,. \tag{5.125}$$

Exercise 5.10. *Compute the pressure and entropy for a free scalar. Hint: It might be useful to put the* *theory in finite volume and take the infinite volume limit at the end of the computation.*

Show that at large temperature the pressure is given by Stefan's law for the pressure of black body radiation,

$$P = \frac{\pi^2}{90} T^4 , \tag{5.126}$$

while at lower temperatures quantum effects modify this result.

Remark. In this section, we have focused exclusively on the Matsubara formalism for quantum field theory at finite temperature, which is based on Euclidean time compactification and is particularly well-suited for equilibrium systems. However, there exists an alternative approach known as the real-time formalism or Schwinger–Keldysh formalism, which is especially powerful when dealing with non-equilibrium phenomena or real-time correlation functions. Roughly speaking, this formalism involves doubling the degrees of freedom and defining the theory on a contour in complex time that runs forward and then backward along the real axis, possibly with a vertical leg into imaginary time to incorporate initial thermal conditions. This closed time path (CTP) formalism allows one to compute expectation values of time-ordered, anti-time-ordered, and Wightman functions in a unified way, making it ideal for studying dissipative processes and transport phenomena. For an introduction to the real-time formalism see for example [CSHY85, NvW87, Mac07, KL09].

Thermal mass and symmetry restoration

At finite temperature, the properties of quantum fields are modified by interactions with the thermal bath. One key effect is the generation of temperature-dependent corrections to the mass of fields, known as thermal masses. These corrections can significantly alter the vacuum structure of the theory. In particular, in theories with spontaneous symmetry breaking at zero temperature, thermal effects can restore the symmetry at high temperatures by driving the effective mass squared positive. This mechanism, known as symmetry restoration, plays a central role in understanding phase transitions in quantum field theory and early universe cosmology.

To illustrate this phenomenon, consider a real scalar field with a $\mathbb{Z}_2$ symmetry in 3+1 dimensions:

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m_0^2\phi^2 - \frac{\lambda}{4!}\phi^4 . \tag{5.127}$$

In an interacting field theory, the bare mass m_0 appearing in the Lagrangian gets renormalised through the self energy and the physical mass can be defined as

$$m = m_0^2 + \delta m^2 . \tag{5.128}$$

We are interested in computing how the mass term is modified at finite temperature.

At one-loop, the self-energy correction comes from a tadpole diagram:

$$\delta m^2(T) = \frac{\lambda}{2}\int \frac{\mathrm{d}^4 p_E}{(2\pi)^4}\frac{1}{p_E^2 + m^2} . \tag{5.129}$$

At finite temperature, we compactify Euclidean time and can replace the integral over p_0 by a sum over the Matsubara frequencies:

$$\delta m^2(T) = \frac{\lambda}{2} T \sum_{n=-\infty}^{\infty} \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3} \frac{1}{\omega_n^2 + \vec{p}^{\,2} + m^2}, \tag{5.130}$$

where ω_n are bosonic Matsubara frequencies. This expression contains both a zero-temperature part and a thermal correction:

$$\delta m^2(T) = \delta m^2(0) + \Delta m^2(T). \tag{5.131}$$

The thermal part is

$$\Delta m^2(T) = \frac{\lambda}{2} \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\omega_{\vec{p}}} \frac{1}{e^{\beta\omega_{\vec{p}}} - 1}, \quad \omega_{\vec{p}} = \sqrt{\vec{p}^{\,2} + m^2}, \tag{5.132}$$

which is a convergent integral. The divergent part is independent of temperature so the counterterms remain the same as at $T = 0$.

In the high-temperature limit $T \gg m$, we can approximate the thermal correction as

$$\Delta m^2(T) \approx \frac{\lambda}{2} \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{|\vec{p}|} \frac{1}{e^{\beta|\vec{p}|} - 1}. \tag{5.133}$$

This is a standard Bose integral which we can straightforwardly integrate, resulting in

$$\Delta m^2(T) \approx \frac{\lambda}{24} T^2. \tag{5.134}$$

We therefore found that the field acquires a temperature-dependent effective mass,

$$m^2_{\text{eff}}(T) \approx m^2 + \frac{\lambda}{24} T^2 + \mathcal{O}(\lambda^2). \tag{5.135}$$

This result has some very interesting consequences. Namely, if the theory exhibits spontaneous symmetry breaking at $T = 0$, e.g., $m^2 < 0$, we can write the potential as

$$V(\phi) = -\frac{1}{2}|m^2|\phi^2 + \frac{\lambda}{4!}\phi^4. \tag{5.136}$$

At zero temperature the vacuum expectation value of the scalar field is then given by

$$\langle\phi\rangle = \pm v, \quad v = \sqrt{\frac{6|m^2|}{\lambda}}. \tag{5.137}$$

At finite T, the thermal mass can drastically modify this behaviour. In particular, there is a critical temperature T_c where the effective mass vanishes,

$$T_c \approx \sqrt{\frac{24|m^2|}{\lambda}}. \tag{5.138}$$

For $T > T_c$, $m^2_{\text{eff}} > 0$, and the minimum of the potential is at $\langle\phi\rangle = 0$. In other words, the symmetry is restored at large temperature.[7]

Thermal fluctuations screen the scalar field and increase its effective mass. This mechanism explains how broken symmetries at zero temperature can be restored at high temperature – a key idea in early universe cosmology and the study of phase transitions in QFT.

5.4 Adiabatic expansions

In the previous sections, we explored a number of instructive examples where the high degree of symmetry allowed for an exact mode decomposition and explicit construction of Green's functions. These cases, while illuminating, are rather special. In a generic curved background – where the geometry lacks time translation invariance or spatial homogeneity – such analytic control is typically lost. Nevertheless, physical intuition tells us that if the spacetime curvature is small and evolves slowly, we may still extract useful information through a systematic expansion. This leads us to the adiabatic expansion, a perturbative framework that allows us to approximate quantum effects in slowly varying backgrounds by organising corrections in terms of derivatives of the metric. In what follows, we introduce the basic ideas and technical machinery behind this expansion, and illustrate how it can be used to compute Green's functions and understand particle creation in time-dependent spacetimes.

Let us once more consider a free scalar field in a generic background. To proceed with the quantisation, we expand this field in modes,

$$\phi(x) = \sum_n \left(a_n u_n(x) + a_n^\dagger u_n^*(x)\right), \tag{5.139}$$

appropriate for a given choice of vacuum. The Wightman function G^+ (and similarly G^-) can then be computed as

$$G^+(x,x') = \sum_n u_n(x) u_n^*(x'). \tag{5.140}$$

Furthermore, since the field ϕ satisfies the wave equation we necessarily have

$$(\Box_x + m^2 + \xi R) G^\pm(x,x') = 0. \tag{5.141}$$

Although conceptually clear, finding the appropriate basis of functions and computing the Wightman functions is often prohibitively hard and only in very special cases will we be able to obtain analytic expressions. Such cases are usually characterised by some sort of additional symmetry. The more symmetries are present in our setup the more constrained are the Green's functions. Another case where computations simplify dramatically is when we have a conformal vacuum, i.e. a conformal theory on a conformally flat manifold. In this case the Green's functions can simply be obtained through a conformal transformation from Minkowski space.

[7] Note that although conventional wisdom tells us ordered phases should only exist at low temperatures, there are examples where ordered phases exist for arbitrarily high temperatures or even phase transitions where disordered phases become ordered at higher temperature. The most famous example is the Pomeranchuk effect in ^{3}He. At $T < 10^{-7}$K (at a pressure of 30atm) we find a liquid, while for $10^{-7} < T < 1$K we find a solid [Pom50].

In the absence of symmetries, explicitly solving for mode functions or constructing Green's functions is generally hopeless. Fortunately, progress is still possible through perturbative methods in many relevant situations. In this section, we outline the key results while omitting several technical steps; readers are referred to [BD84, PT09] for a more thorough treatment.

When the space-time curvature is small and varies slowly, we do not expect the creation of particles with arbitrarily high energies. Intuitively, if the background metric evolves sufficiently slowly – i.e., adiabatically – the number of particles in a given mode should remain approximately constant. This intuition can be formalized by introducing a scaling parameter T in the metric,

$$g_{\mu\nu}(t,\mathbf{x}) \to g^T_{\mu\nu}(t,\mathbf{x}) = g_{\mu\nu}\left(\frac{t}{T},\frac{\mathbf{x}}{T}\right). \tag{5.142}$$

An expansion in inverse powers of T is called the adiabatic expansion, which loosely counts the number of derivatives acting on the metric. For instance, the scalar curvature R is of adiabatic order two. While we will not always keep the parameter T explicit throughout, it can be reintroduced at any point by counting derivatives. When we write terms like $\mathcal{O}(T^{-3})$, this refers to contributions of third adiabatic order and beyond.

To illustrate this idea, let us consider a free, neutral scalar field in four-dimensional space-time obeying the equation,

$$(\Box + m^2 + \xi R)\phi = 0\,. \tag{5.143}$$

Solving this equation exactly to obtain the mode functions and Green's function is rarely feasible. However, one can show that Feynman's Green's function admits a well-defined adiabatic expansion. As in flat spacetime, it satisfies

$$(\Box + m^2 + \xi R)G_F(x,x') = -\delta(x,x')\,, \qquad \delta(x,x') = |g(x)|^{-1/2}\delta(x-x')\,, \tag{5.144}$$

where the minus sign is conventional, and both $\delta(x,x')$ and $G_F(x,x')$ transform as bi-scalars.

To construct the adiabatic expansion, we write the propagator in the form

$$G_F(x,x') = -\mathrm{i}\int_0^\infty \mathrm{d}s\mathrm{e}^{-\mathrm{i}m^2 s}K(x,x';s)\,, \tag{5.145}$$

where m^2 is understood to have a small imaginary part $m^2 - \mathrm{i}\epsilon$ so that there is no divergence as $s\to\infty$. The kernel K satisfies the Schrödinger type equation

$$\mathrm{i}\partial_s K(x,x';s) = (\Box_x + \xi R)K(x,x';s)\,, \tag{5.146}$$

with the boundary condition that $K(x,x';s) \sim |g(x)|^{-1/2}\delta(x-x')$ as $s\to 0$.

Exercise 5.11. *Show that the kernel $K(x,x';s)$ satisfies the equation (5.146) with the stated boundary conditions.*

We now seek a short-time (i.e. small s) expansion of the kernel K. This is achieved by writing

$$K(x,x';s) = \mathrm{i}\frac{\Delta^{1/2}(x,x')}{(4\pi^2)(\mathrm{i}s)^2}\mathrm{e}^{\frac{\sigma(x,x')}{2\mathrm{i}s}}F(x,x';\mathrm{i}s)\,, \tag{5.147}$$

where $\Delta(x,x')$ is the Van Vleck-Morette determinant and $\sigma(x,x')$ is related to the proper distance along the geodesic from x to x',[8]

$$\Delta(x,x') = -|g(x)|^{-1/2}\det\left(-\partial_{x^\mu}\partial_{x'^\nu}\sigma(x,x')\right)|g(x')|^{-1/2}\,, \qquad \sigma(x,x') = \frac{1}{2}\tau(x,x')^2\,, \tag{5.148}$$

where τ is the proper distance along the geodesic. The adiabatic expansion of G_F can now be rephrased as the following expansion of the function F,

$$F(x,x';s) \sim a_0(x,x') + (\mathrm{i}s)a_1(x,x') + (\mathrm{i}s)^2 a_2(x,x') + \cdots\,. \tag{5.149}$$

where the first coefficients are given in the coincidence limit $x \to x'$ as

$$a_0(x) = 1\,, \qquad a_1(x) = \left(\frac{1}{6} - \xi\right)R\,, \tag{5.150}$$

$$a_2(x) = \frac{1}{180}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - \frac{1}{180}R_{\mu\nu}R^{\mu\nu} - \frac{1}{6}\left(\frac{1}{5} - \xi\right)\Box R + \frac{1}{2}\left(\frac{1}{6} - \xi\right)^2 R^2\,. \tag{5.151}$$

In this expansion we did not keep the parameter T explicit but by counting the derivatives on the metric, one can easily see that the term a_n is of adiabatic order $2n$. If the metric is smooth, one can continue this expression indefinitely and find a unique expansion. However, typically this expansion is asymptotic so the solutions will in general not be uniquely determined.

Exercise 5.12. *To make this more explicit, consider the FLRW metric with flat spatial slices,*

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2\mathrm{d}\mathbf{x}^2\,. \tag{5.152}$$

The wave equation for the scalar field it

$$\left(\Box + m^2 + \xi R\right)\phi = 0\,. \tag{5.153}$$

Hint: Note that the Ricci scalar for this metric is $R = 6\left(\frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a}\right)$

Show that the modes for the scalar field can be written as

$$f_{\mathbf{k}}(x) = \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}h_{\mathbf{k}}(t)\,, \tag{5.154}$$

where $h_{\mathbf{k}}(t)$ *satisfies*

$$\ddot{h}_{\mathbf{k}} + \left(\omega^2 + \sigma\right)h_{\mathbf{k}} = 0\,, \tag{5.155}$$

where $\omega = \sqrt{\frac{\mathbf{k}^2}{a^2} + m^2}$. *Find an expression for* σ *in terms of* a *and its derivatives.*

The adiabatic expansion for the field modes is based on the usual WKB ansatz,

$$h_{\mathbf{k}}(t) = \frac{1}{\sqrt{W_{\mathbf{k}}(t)}}\exp\left[-\mathrm{i}\int^t W_{\mathbf{k}}(t')\mathrm{d}t'\right]\,, \tag{5.156}$$

[8] Here we assume x' is in a normal neighbourhood of x such that only one geodesic goes from x to x'.

where $W_{\mathbf{k}}(t)$ can be expanded in an adiabatic expansion,

$$W_{\mathbf{k}}(t) = \omega^{(0)} + \omega^{(0)} + \omega^{(0)} + \mathcal{O}(T^{-2}), \tag{5.157}$$

where each $\omega^{(n)}$ is of nth adiabatic order, i.e. contains n derivatives with respect to t.

Show that $W_{\mathbf{k}}(t)$ satisfies the following equation,

$$W_{\mathbf{k}}^2 = \omega^2 + \sigma + \frac{3}{4}\frac{\dot{W}_{\mathbf{k}}^2}{W_{\mathbf{k}}^2} - \frac{1}{2}\frac{\ddot{W}_{\mathbf{k}}}{W_{\mathbf{k}}}, \tag{5.158}$$

and solve this equation perturbatively in a large T expansion. (Hint: explicitly reintroduce T and expand.)

Show that all odd $\omega^{(\mathrm{odd})}$ vanish and that

$$\begin{aligned}
\omega^{(0)} =& \omega\,, && (5.159)\\
\omega^{(2)} =& \frac{1}{2\omega^3}\left(\sigma\omega^2 + \frac{3}{4}\dot{\omega}^2 - \frac{1}{2}\omega\ddot{\omega}\right), && (5.160)\\
\omega^{(4)} =& \frac{1}{2\omega^3}\left(2\sigma\omega\omega^{(2)} - 5\omega^2\left(\omega^{(2)}\right)^2 + \frac{3}{2}\dot{\omega}\dot{\omega}^{(2)} - \frac{1}{2}(\omega\ddot{\omega}^{(2)} + \ddot{\omega}\omega^{(2)}\right). && (5.161)
\end{aligned}$$

Exercise 5.13. *Using the results from the previous exercise we can compute the Green's function in the adiabatic expansion. In the coincidence limit the Green's function takes the form*

$$G(x,x) \simeq \int_0^\infty \mathrm{d}k k^2 W_{\mathbf{k}}^{-1} \tag{5.162}$$

Expand this integral adiabatically and show that only the first to adiabatic orders contain divergences. More precisely, show that the Green's function can be written as,

$$G(x,x) = \frac{R}{288\pi^2} + \frac{1}{4\pi^2 a^3}\int_0^\infty \mathrm{d}k k^2\left[\frac{1}{\omega} - \left(\xi - \frac{1}{6}\right)\frac{R}{2\omega^3}\right]. \tag{5.163}$$

After removing the divergent terms, we find a finite result for the Green's function (at coincident points).

Note that the adiabatic expansion described above can be continued to arbitrary order and it is even possible to find analytic expressions for all of the higher order terms in terms of first two [dRNS15]. The perturbative expansion is therefore uniquely determined. However, this expansion is an asymptotic expansion and is only uniquely defined up to non-perturbative terms.

Chapter 6

The Unruh effect

A fundamental application of the formalism introduced in part I can already be seen in flat space. Indeed, the general principles of relativity state that it should be possible to express the laws of physics as being the same for all observers, even those undergoing acceleration. Indeed, the effects of constant acceleration are equivalent to those caused by the presence of a uniform gravitational field. Thus one can ask the question: in flat space, how does the Minkowski vacuum appear to an accelerating observer? The perhaps surprising answer to this question lies in the Unruh effect which states that this observer will perceive a thermal state!

Studying the physics of an accelerated observer will highlight many of the properties we discussed in the previous chapters. Due to its simplicity we will be able to exactly solve this problem and explicitly see the theory at work. Moreover, as we will see in the next chapter, many of the curious properties observed in this case immediately carry over to the study of an evaporating black hole.

6.1 Particle detectors

Since in general curved space-times the notion of a particle is observer-dependent it will prove useful to give a coordinate independent characterisation of the temperature. A useful way to achieve this is to consider an observer equipped with a so-called Unruh detector [Unr76, GH77b].

The detector will have some internal energy states and can interact with the scalar field by exchanging energy, i.e. by emitting or absorbing scalar particles. The detector could for example be constructed so that it emits a 'ping' whenever its internal energy state changes. All observers will agree on whether or not the detector has pinged, although they may disagree on whether the ping was caused by particle emission or absorption. Such a detector can be modelled by a coupling of the scalar field $\phi(x(\tau))$ along the world-line $x(\tau)$ of the observer to some operator $m(\tau)$ acting on the internal detector states

$$g\int_{-\infty}^{\infty} \mathrm{d}\tau\, m(\tau)\phi(x(\tau))\,, \tag{6.1}$$

where g is the strength of the coupling and τ is the proper time along the observer's world-line. Let H_0 denote the detector Hamiltonian, with energy eigenstates $|E_j\rangle$,

$$H_0 \left|E_j\right\rangle = E_j \left|E_j\right\rangle\,, \tag{6.2}$$

and let m_{ij} be the matrix element of the operator $m(\tau)$ at $\tau = 0$,

$$m_{ij} = \langle E_i|\, m(0) \left|E_j\right\rangle\,. \tag{6.3}$$

We will calculate the transition amplitude from a state $|0\rangle \otimes |E_i\rangle \in \mathcal{H}_\phi \otimes \mathcal{H}_{\text{det.}}$ in the tensor product of the scalar field and detector Hilbert spaces to the state $\langle E_j| \otimes \langle\psi|$, where $|\psi\rangle$ is any state of the scalar field. To first order in perturbation theory (for small g) the desired amplitude can be computed as

$$\mathcal{A} = g \int_{-\infty}^{\infty} \mathrm{d}\tau \left\langle E_j \right| \otimes \langle\psi| \, m(\tau)\phi(x(\tau)) \, |0\rangle \otimes |E_i\rangle \, . \tag{6.4}$$

Using (in the Heisenberg picture) that $m(\tau) = \mathrm{e}^{\mathrm{i}H\tau} m(0) \mathrm{e}^{-\mathrm{i}H\tau}$, this can be written as

$$\mathcal{A} = g m_{ji} \int_{-\infty}^{\infty} \mathrm{d}\tau \, \mathrm{e}^{\mathrm{i}(E_j - E_i)\tau} \, \langle\psi| \, \phi(x(\tau)) \, |0\rangle \, . \tag{6.5}$$

Since we are only interested in the probability for the detector to make the transition from E_i to E_j, we should square this amplitude and sum over the final state $|\psi\rangle$ of the scalar field, which will not be measured. Using the resolution of identity $\sum_\psi |\psi\rangle\langle\psi| = 1$ we find the probability

$$P(E_i \to E_j) = g^2 |m_{ij}|^2 \int_{-\infty}^{\infty} \mathrm{d}\tau \mathrm{d}\tau' \mathrm{e}^{-\mathrm{i}(E_j - E_i)(\tau' - \tau)} G_+\left(x(\tau'), x(\tau)\right) , \tag{6.6}$$

where G_+ is the Wightman function. Notice that the prefactor in (6.6) depends on the details of the detector, so it is useful to extract the piece which depends only on the scalar field and the world-line trajectory. For this reason we define the detector response function

$$\mathcal{F}(E) = \int_{-\infty}^{\infty} \mathrm{d}\tau \mathrm{d}\tau' \mathrm{e}^{-\mathrm{i}E(\tau' - \tau)} G_+\left(x(\tau'), x(\tau)\right) , \tag{6.7}$$

When the Wightman function only depends on $\Delta\tau = \tau' - \tau$ we can change variables to $\Delta\tau$ and $\bar{\tau} = \frac{\tau' + \tau}{2}$. The detector response function is then defined by removing the diverging volume factor coming from the integration over $\bar{\tau}$,

$$f(E) = \int_{-\infty}^{\infty} \mathrm{d}\Delta\tau \, \mathrm{e}^{-\mathrm{i}E\Delta\tau} G_+(\Delta\tau) . \tag{6.8}$$

Example 6.1. *First consider Minkowski space in the vacuum state. The Wightman function is then given by*

$$G_+(\Delta\tau) = \frac{1}{4\pi^2} \frac{1}{(\Delta\tau - \mathrm{i}\epsilon)^2} \, . \tag{6.9}$$

If we plug this in the formula (6.8), we can calculate the integral by residues. Since $E > 0$ we should close the contour for $\Delta\tau$ in the lower half-plane, since in this case the integral at the half-circle at infinity goes to zero due to the damping factor $\mathrm{e}^{-\mathrm{i}E\Delta\tau}$ (Jordan's lemma) and we conclude that

$$f(E) = 0 . \tag{6.10}$$

So, unsurprisingly, we find that there is no particle detection in the Minkowski vacuum.

Example 6.2. *We can also arrive at the Bose-Einstein distribution from the Green's function by computing the detector responds function for a detector at $\mathbf{x} = 0$. Inserting the thermal Green's function in (6.8) we* *find*

$$
\begin{aligned}
f(E) &= \int_{-\infty}^{\infty} \mathrm{d}\Delta\tau \, \mathrm{e}^{-\mathrm{i}E\,\Delta\tau}\, G_\beta(\Delta\tau) \\
&= \int_{-\infty}^{\infty} \mathrm{d}\Delta\tau \sum_{n\in\mathbb{Z}} \frac{-\mathrm{e}^{-\mathrm{i}E\,\Delta\tau}}{4\pi^2(\Delta\tau + \mathrm{i}n\beta + i\epsilon)^2}
\end{aligned}
\tag{6.11}
$$

The integral can be analytically continued so as to be evaluated by residues along the negative imaginary axis where it has double poles at $t = -n\beta i$. Thus we obtain the sum of residues

$$
f(E) = E \sum_{n\in\mathbb{Z}} \mathrm{e}^{-En\beta} = \frac{E}{\mathrm{e}^{\beta E} - 1}\,, \tag{6.12}
$$

in line with our expectation for a detector immersed in a thermal bath of temperature $T = 1/\beta$.

6.2 The Unruh effect

Having introduced these tools we are ready and well-equipped to study the Unruh effect. Here we will perform the calculation for massless modes in $1+1$ dimensions where we can use the conformal invariance of the wave equation allowing us to work out everything very explicitly. All the essential features will already be present in this setup. The analysis can be performed explicitly also for massive modes and be extended to higher dimensions but comes at the expense of having to deal with Bessel functions.[1]

Consider an observer $\mathcal{O}_a$ with constant acceleration a along the x-axis. The world-line for this observer can be parametrised as

$$
X(\tau) = (t(\tau), x(\tau)) = \frac{1}{a}(\sinh a\tau, \cosh a\tau)\,, \tag{6.13}
$$

where τ is the proper time of the observer. Note that these trajectories parametrise the hyperbolae, $x^2 - \tau^2 = a^2$. We will be asking the question as to how the accelerating observer sees the Minkowski vacuum. To do so we want to find the natural coordinates in Minkowski space-time adapted to this observer. I.e. we want the time coordinate to be its proper time, and the spatial coordinate to be characterized by the fact that the observed is at rest in it. Using the clock and radar method, the observer will set up coordinates (τ, ξ) related to inertial coordinates (t, x) by

$$
(t,x) = \frac{\mathrm{e}^{a\xi}}{a}(\sinh a\tau, \cosh a\tau)\,, \qquad \mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 = \mathrm{e}^{2a\xi}(-\mathrm{d}\tau^2 + \mathrm{d}\xi^2)\,. \tag{6.14}
$$

Indeed, in these coordinates, the path followed by the observer is given by $X(\tau) = (\tau, 0)$. The new coordinates have range $\{\tau, \xi\} \in (-\infty, \infty)$ but they only cover the part of Minkowski space with $\mathcal{R} : \{x > |t|\}$, called the Rindler wedge. This is the portion of space-time that the accelerating observer can measure and see. Minkowski space equipped with this metric will be denoted Rindler space. Note that Rindler space corresponds to the right wedge foliated by the world-lines of the accelerated observers, labelled by $\mathcal{R}$ in Figure 6.1.

[1] Going to higher dimensions is no worse than introducing a mass; in both cases we lose the conformal invariance that we exploit in this section.

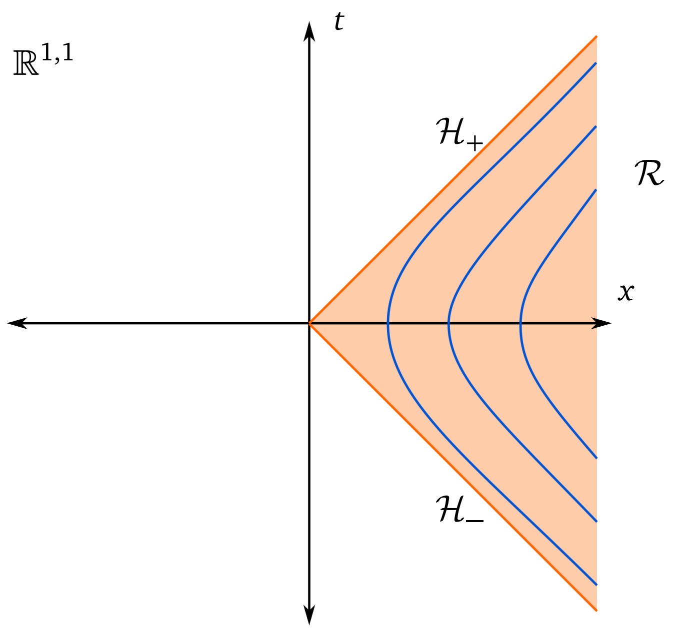

Figure 6.1: The Rindler wedge $\mathcal{R}$ in two-dimensional Minkowski space is denoted by the orange region. A uniformly accelerated observer $\mathcal{O}_a$ follows the blue hyperbolic trajectories in $\mathcal{R}$. $\mathcal{H}_\pm$ denote the Killing horizons which are the boundaries of $\mathbb{R}^{1,1}$ perceivable for this observer.

The inertial light-cone coordinates $(u, v) := (t - x, t + x)$ can be rewritten in terms of the light-cone coordinates adapted to the accelerating observer $(U_R, V_R) := (\tau - \xi, \tau + \xi)$,

$$(u, v) = \frac{1}{a}\left(-e^{-aU}, e^{aV}\right), \qquad ds^2 = du dv = e^{a(V_R - U_R)} dU_R dV_R . \tag{6.15}$$

The Rindler wedge is therefore the whole plane in the (U_R, V_R) coordinates but only the quadrant with $-u, v > 0$ in the (u, v) coordinates.

More generally, we see that the lines of constant ξ in the Rindler metric describe uniformly accelerated observers with acceleration

$$\alpha = a e^{-a\xi} , \tag{6.16}$$

and proper time τ. Therefore, Rindler space can be regarded as a foliation of Minkowski space by the trajectories of uniformly accelerated observers. Near the horizon where $\xi \to -\infty$, we have $\alpha \to \infty$ such that the observers feel an infinite proper acceleration.

Similarly, we can also cover the left wedge of Minkowski space, $x < |t|$, by defining the coordinates

$$t = -\frac{e^{a\xi}}{a} \sinh a\xi , \qquad x = -\frac{e^{a\xi}}{a} \cosh a\xi . \tag{6.17}$$

Notice that the events happening in the left Rindler wedge are causally disconnected from the world-lines of a Rindler observer in the right Rindler wedge, and the line $u = 0$ effectively behaves as an event horizon. This observation will be relevant in the context of our discussion of black holes in the next chapter.

Let us now consider the quantisation of a massless scalar field in Rindler space. Since the wave equation in this case is conformal, we can trivially solve it by applying a conformal transformation to the standard Minkowski space modes. In standard Minkowski space, we have respectively the left and right moving modes[2]

$$\phi_\omega(u) = \frac{1}{\sqrt{4\pi\omega}} \mathrm{e}^{-i\omega u}, \qquad \tilde{\phi}_\omega(v) = \frac{1}{\sqrt{4\pi\omega}} \mathrm{e}^{-i\omega v}, \tag{6.18}$$

which constitute the positive frequency modes for $\omega > 0$. In the adapted coordinates introduced above the metric is conformal to the standard Minkowski metric so the modes seen/measured by $\mathcal{O}_a$ of frequency ω will be respectively

$$\Phi^R_\omega(U_R) = \frac{\theta(-u)}{\sqrt{4\pi\omega}} \mathrm{e}^{-i\omega U_R}, \qquad \tilde{\Phi}^R_\omega(V_R) = \frac{\theta(v)}{\sqrt{4\pi\omega}} \mathrm{e}^{-i\omega V_R}. \tag{6.19}$$

where we added the Heaviside functions to denote that these modes are only non-zero in the right Rindler wedge.

Similarly, considering the left Rindler wedge L with lightcone coordinates U_L and V_L we find the modes for the quantum field there as,

$$\Phi^L_\omega(U_L) = \frac{\theta(u)}{\sqrt{4\pi\omega}} \mathrm{e}^{i\omega U_L}, \qquad \tilde{\Phi}^L_\omega(V_L) = \frac{\theta(-v)}{\sqrt{4\pi\omega}} \mathrm{e}^{i\omega V_L}. \tag{6.20}$$

Given that the left- and right-moving modes decouple, we can focus on the right-moving modes while the results for left-moving modes will follow identically.[3] In the standard Minkowski picture, we can define a general right-moving field operator as

$$\hat{\phi} = \int_0^\infty \frac{d\omega}{\sqrt{4\pi\omega}} [\mathrm{e}^{-i\omega u} a_\omega + \mathrm{e}^{i\omega u} a^\dagger_\omega], \tag{6.21}$$

while for an accelerating observer in the right Rindler wedge the right-moving field operators are defined as

$$\hat{\Phi}_R = \int_0^\infty \frac{\mathrm{d}\lambda}{\sqrt{4\pi\lambda}} [\mathrm{e}^{-i\lambda U_R} A^R_\lambda + \mathrm{e}^{i\lambda U_R} A^{R\dagger}_\lambda], \tag{6.22}$$

where a_ω and A^R_λ are the standard raising and lowering operators satisfying the commutation relations

$$[a_\omega, a^\dagger_{\omega'}] = \delta(\omega - \omega'), \qquad [A_\lambda, A^\dagger_{\lambda'}] = \delta(\lambda - \lambda'), \tag{6.23}$$

and analogous for the left Rindler wedge. Implicit in these definitions are the Minkowski vacuum $|0_M\rangle$, satisfying $a_\omega |0_M\rangle = 0$, and the Rindler vacuum satisfying $A_\lambda |0_R\rangle = 0$ for respectively all $\omega > 0$ and $\lambda > 0$.

As by now standard, we can build a Fock space $\mathcal{F}_{R/L}$ on the Rindler vacua $\left|0_{R/L}\right\rangle$ using the respective

[2]In two dimensions we have $|\mathbf{k}| = \omega$ where $\mathbf{k}$ only has one component. The right-moving waves have $k > 0$ while the left-moving ones have $k < 0$.

[3]The left and right movers can never mix under Bogoliubov transformations, since $\Phi^{R/L}_\omega(U_{R/L})$ only depends on $U_{R/L}$ while $\widetilde{\Phi}^{R/L}_\omega(V_{R/L})$ only depends on $V_{R/L}$.

creation and annihilation operators$A_\lambda^{R/L\dagger}$ and $A_\lambda^{R/L}$. These Fock spaces are based on the Rindler modes (6.19) or (6.20) measured by an observer $\mathcal{O}_a^{R/L}$ respectively in the right or left Rindler wedge. In particular the vacua $|0\rangle_{R/L}$ is the state in which $\mathcal{O}_a^{R/L}$ sees no particles. However, As it stands, there can be no identification between the Minkowski Fock space $\mathcal{F}_M$ and either of the Rindler Fock space $\mathcal{F}_{R/L}$ separately as they only determine the Minkowski fields respectively for $u<0$ or $u>0$ and are not defined in the opposite patch. To determine $\hat{\phi}$ and $\mathcal{F}_M$ from Rindler type data, we have to consider both the left and right Fock spaces $\mathcal{F}_R$ and $\mathcal{F}_L$ such that the Minkowski Fock space is now a tensor product

$$\mathcal{F}_M=\mathcal{F}_L\otimes\mathcal{F}_R\,. \tag{6.24}$$

Note however, that the Minkowski vacuum $|0\rangle_M$ might be an entangled state in this product, i.e. $|0\rangle_M\neq|0\rangle_L\otimes|0\rangle_R$.

Having discussed in detail the various vacua in the picture we now wish to compute the distribution of the number of particles of frequency λ detected by the observer $\mathcal{O}_a$ in the Minkowski vacuum. To do so we need to compute the Bogoliubov coefficients relating the Minkowski and Rindler modes in the right Rindler wedge,

$$\theta)(-u)\phi_\omega(u)=\int \mathrm{d}\lambda\left[\alpha^R_{\omega\lambda}\Phi^R_\lambda(U_R)+\beta^R_{\omega\lambda}\Phi^{R*}_\lambda(U_R)\right]. \tag{6.25}$$

Inserting the modes in the definition for the Bogoliubov coefficients, (4.33), [4] we find,

$$\begin{aligned}\alpha^R_{\omega\lambda}=&\mathrm{i}\int_{-\infty}^{\infty}\mathrm{d}U_R\,\Phi^{R*}_\lambda\overleftrightarrow{\partial_u}\,\xi_\omega\\=&\frac{1}{2\pi\sqrt{\omega\lambda}}\int_{-\infty}^{0}\mathrm{d}u\lambda\mathrm{e}^{-\mathrm{i}\omega u}(-au)^{-\frac{\mathrm{i}\lambda}{a}-1}\\=&\frac{1}{2\pi a}\sqrt{\frac{\lambda}{\omega}}\left(\frac{a}{\omega}\right)^{-\frac{\mathrm{i}\lambda}{a}}\Gamma\left(-\frac{\mathrm{i}\lambda}{a}\right)\mathrm{e}^{\frac{\pi\lambda}{2a}}\,.\end{aligned} \tag{6.26}$$

Similarly, for $\beta^R_{\omega\lambda}$ we find

$$\begin{aligned}\beta^R_{\omega\lambda}=&\mathrm{i}\int_{-\infty}^{\infty}\mathrm{d}U_R\,\Phi^{R}_\lambda\overleftrightarrow{\partial_u}\,\xi_\omega\\=&\frac{1}{2\pi a}\sqrt{\frac{\lambda}{\omega}}\left(\frac{a}{\omega}\right)^{\frac{\mathrm{i}\lambda}{a}}\Gamma\left(\frac{\mathrm{i}\lambda}{a}\right)\mathrm{e}^{-\frac{\pi\lambda}{2a}}\,.\end{aligned} \tag{6.27}$$

The verification of the intermediate steps are left as an exercise for the reader but mainly consist of rewriting the integral in terms of an integral representation of the Gamma function and using some Gamma function identities.

The main takeaway from this calculation is the relation

$$|\alpha_{\omega\lambda}|^2=\mathrm{e}^{2\pi\lambda/a}|\beta_{\omega\lambda}|^2\,, \tag{6.28}$$

as this allows us to compute the expectation value of the number operator N_λ of the modes with

[4] where the inner product is defined on the hypersurface $\xi=$ constant with normal vector $n^\mu=\mathrm{e}^{-a\xi}(1,0)$

frequency λ detected by $\mathcal{O}_a$ in the Minkowski vacuum,

$$\begin{aligned}\langle N_\lambda \rangle_M &:= \langle 0|_M A_\lambda^{R\dagger} A_\lambda^R |0_M\rangle \\ &= \int_0^\infty d\omega |\beta_{\omega\lambda}|^2 \end{aligned} \tag{6.29}$$

This can be simplified by using the normalization condition for Bogoliubov coefficients which in our context reads

$$\int_0^\infty d\omega (\alpha_{\omega\lambda}\bar{\alpha}_{\omega\lambda'} - \beta_{\omega\lambda}\bar{\beta}_{\omega\lambda'}) = \delta(\lambda - \lambda'). \tag{6.30}$$

Evaluating at $\lambda = \lambda'$, we reinterpret the right hand side $\delta(0) = V$ as the volume of space, regularized as usual by putting the system in a finite box. This allows us to deduce the particle number density as

$$\frac{\langle N_\lambda \rangle_M}{V} = \frac{1}{e^{2\pi\lambda/a} - 1}. \tag{6.31}$$

This is the main result of this section as we now recognize this as the Bose Einstein distribution with Unruh temperature

$$T_{\text{Unruh}} = \frac{a}{2\pi}. \tag{6.32}$$

We conclude that an observer moving with uniform acceleration through the Minkowski vacuum observes a thermal spectrum of particles. The Unruh temperature $T = \frac{a}{2\pi}$ is the temperature that would be measured by an observer moving along the path $\xi = 0$, which feels the acceleration $\alpha = a$. Any other path with $\xi =$ constant feels an acceleration

$$\alpha = a e^{-a\xi}, \tag{6.33}$$

and will thus measure thermal radiation at temperature $T = \frac{\alpha}{2\pi}$. As $\xi \to \infty$, the temperature approaches 0, in line with the fact that near ∞ the Rindler observer is nearly inertial. We conclude that not only does the choice of vacuum, and hence concept of particle become time-dependent, it is also observer-dependent, even in flat space-time.

Coming back to the relation between the Minkowski and Rindler vacua we have that the Unruh state ρ_U as measured by $\mathcal{O}_a$ is given by the density matrix

$$\rho_U = \operatorname{Tr}_{\mathcal{F}_L} |0\rangle_M \langle 0|_M . \tag{6.34}$$

A key point to analyse this state is that time translation ∂_τ for $\mathcal{O}_a$ in R is given by the boost Killing vector on $\mathcal{R}$ but in $\mathcal{L}$ it is given by minus the boost Killing vector

$$B := x\partial_t + t\partial_x = v\partial_v - u\partial_u = \frac{\partial}{\partial \tau_R} = -\frac{\partial}{\partial \tau_L} \tag{6.35}$$

However, since the Minkowski vacuum is Lorentz and hence boost invariant we must have an entangled product of the form

$$|0\rangle_M = \sum_n f_n |n\rangle_L \otimes |n\rangle_R, \tag{6.36}$$

for some f_n. Indeed, our calculations show that $f_n = \mathrm{e}^{-\beta E_n}$, where $\beta = 1/T_{\mathrm{Unruh}}$.

Alternatively we can recognize the thermal nature of the Rindler vacuum by looking at the KMS condition on the Wightman function in the Rindler vacuum. Since we are working in a conformal setup we can straightforwardly obtain the Rindler space Wightman function from the Minkowski Wightman function. Indeed, in two dimensions it is unchanged so we simply have to change coordinates to obtain the Wightman function in Rindler space

$$\begin{aligned}
G_+^M(x,x') = \langle 0|_M\, \hat{\phi}(x)\hat{\phi}(x')\,|0\rangle_M &= \frac{1}{(x^0 - x'^0 - i\epsilon)^2 - (\mathbf{x}-\mathbf{x}')^2} \\
&= \frac{-a^2}{\mathrm{e}^{2a\xi} + \mathrm{e}^{2a\xi'} + a^2\epsilon^2 - 2\mathrm{e}^{a(\xi+\xi')}\cosh a(\tau-\tau') + 2ai\epsilon\,(\mathrm{e}^{a\xi}\sinh a\tau + \mathrm{e}^{a\xi'}\sinh a\tau')} \\
&= \langle 0|_R\, \hat{\phi}(x)\hat{\phi}(x')\,|0\rangle_R \\
&= G_+^R(x,x')\,.
\end{aligned} \tag{6.37}$$

From this expression is is clear that in Rindler coordinates, G_+^R is periodic in complex τ with period $\beta = \frac{2\pi}{a}$. Thus, for $\mathcal{O}_a$, the Minkowski propagator is a thermal Green's function of temperature T_{Unruh}.

As before, we can make the previous remark more explicit by restricting to the accelerating world-line $x = x(\tau), x' = x(\tau = 0)$ and introducing a particle detector. Restricting to the world-line we obtain the Green's function

$$\begin{aligned}
G_+^R(x(\tau),x(0)) &= \frac{a^2}{(\sinh a\tau - ia\epsilon)^2 - (\cosh a\tau - 1)^2} \\
&= \frac{-a^2}{2(1-\cosh a\tau + 2ia\epsilon\sinh a\tau + a^2\epsilon^2/2)}\,.
\end{aligned} \tag{6.38}$$

With this expression at hand we can compute the detector response function $f(E)$ characterising the detection of field transitions at energy E as

$$\begin{aligned}
f(E) &= \int_{-\infty}^{\infty} \mathrm{d}\Delta\tau\, \mathrm{e}^{-iE\Delta\tau}\, G_+^R(\Delta\tau) \\
&= \int_{-\infty}^{\infty} \mathrm{d}\Delta\tau\, \frac{-a^2\mathrm{e}^{-\mathrm{i}E\Delta\tau}}{2(1-\cosh a\tau + ia\epsilon\sinh a\tau)}\,,
\end{aligned} \tag{6.39}$$

The integral can be analytically continued so as to be evaluated by residues along the negative imaginary axis where it has double poles at $a\tau = 2n\pi i$. We obtain the sum of residues

$$f(E) = E\sum_{n=0}^{\infty} \mathrm{e}^{-\frac{2\pi E n}{a}} = \frac{E}{\mathrm{e}^{\frac{2\pi E}{a}} - 1}\,, \tag{6.40}$$

as expected for a detector immersed in a thermal bath at the Unruh temperature T_{Unruh}.

Before moving on let us introduce an alternative way to detect the thermal nature of space-times. In terms of the Euclidean continuation of the space-time, it turns out thermal effects can be seen as the need to periodically identify the imaginary time coordinate [GH77a, GH94]. It is easy to see that the inverse is true. In a space-time which is periodic in imaginary time on can compute the Euclidean Green's function which consequentially will also be periodic. After analytically continuing to Lorentzian signature, the Euclidean Green's function becomes the Feynman Green's function which naturally inherits the complex periodicity of its Euclidean counterpart.

As an example, consider the the analytic continuation of the Rindler wedge metric,

$$\mathrm{d}s^2 = a^{-2}(\mathrm{d}\rho^2 + \rho^2 \mathrm{d}\theta^2)\,, \tag{6.41}$$

where we defined the coordinate $\rho = \mathrm{e}^{a\xi}$ and $\theta = \mathrm{i}a\tau$. This is usual metric on flat Euclidean space but generically, it has a conical singularity at $\rho = 0$. In order to avoid this singularity we need to periodically identify θ with period 2π. This is essential for regularity at the horizon and gives rise to the imaginary periodicity $\tau \sim \tau + i\beta$ for $\beta = 1/T_{\text{Unruh}}$. This is a theme that can be taken much further in curved space-times where similar consideration prove very useful in studying black hole backgrounds.

Remark. Let us finish this chapter with some apparent paradoxes, and their resolutions, in Rindler space. First, note that a Rindler observer with smaller constant ξ coordinate are accelerating faster to keep up. This may seem surprising because in Newtonian physics, observers who maintain constant relative distance must share the same acceleration. In relativistic physics, this is no longer true and we see that the trailing endpoint of a rod which is accelerated by some external force (parallel to its symmetry axis) must accelerate a bit faster than the leading endpoint, or else it must ultimately break. This is a manifestation of Lorentz contraction. As the rod accelerates, its velocity increases and its length decreases. Since it is getting shorter, the back end must accelerate harder than the front. Another way to look at it is: the back end must achieve the same change in velocity in a shorter period of time. This leads to a differential equation showing that, at some distance, the acceleration of the trailing end diverges, resulting in the Rindler horizon. This phenomenon is the basis of a well known "paradox", Bell's spaceship paradox. However, it is a simple consequence of relativistic kinematics. One way to see this is to observe that the magnitude of the acceleration vector is just the path curvature of the corresponding world line. But the world lines of our Rindler observers are the analogues of a family of concentric circles in the Euclidean plane, so we are simply dealing with the Lorentzian analogue of a fact familiar to speed skaters: in a family of concentric circles, inner circles must bend faster (per unit arc length) than the outer ones.

The main observation of this chapter was that an accelerated observer detects particles in the Minkowski vacuum state. An inertial observer would say that the same state is completely empty, the expectation value of the energy momentum tensor $\langle T_{\mu\nu}\rangle_M = 0$. If there is no energy momentum how can the Rindler observer detect particles? If the Rindler observer is to detect background particles, they must carry a detector. This must be coupled to the particle being detected. However, if a detector is being maintained at constant acceleration, energy is not conserved. From the point of view of the Minkowski observer the Rindler detector emits as well as absorbs particles, once the coupling is introduced the possibility of emission is unavoidable. When the detector registers a particle the inertial observer would say that it had emitted a particle and felt a radiation-reaction force in response. Ultimately the energy needed to excite the Rindler detector does not come from the background energy momentum tensor but from the energy we put into the detector to keep it accelerating.

Chapter 7

Hawking radiation

The creation of particles by black holes is necessary for maintaining the second law of thermodynamics in their presence. This process of radiation and evaporation of black holes is an important facet in the fundamental search for a microscopic explanation of the entropy of black holes; a search which appears to be leading to new and exciting physics connecting gravitation and quantum theory. In this chapter we will explore the effect of Hawking radiation and introduce some of the challenges this phenomenon generates.

7.1 Quantum fields in a black hole background

The goal of this section is to explore quantum fields in a black hole background. The simplest, and prototypical example of such a background is the Schwarzschild background, with metric,

$$\mathrm{d}s^2 = -\left(1-\frac{2M}{r}\right)\mathrm{d}t^2 + \left(1-\frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 . \tag{7.1}$$

Black holes, such as the Schwarzschild black hole and its rotating and/or charged cousins were discussed at length in the course general relativity II. In Appendices C and E we collect all the necessary background information for these notes to be self-contained.

One might be surprised that anything interesting can happen since the Schwarzschild black hole is a static space-time. Surely one can use the Schwarzschild time-like Killing vector (at least at large distances) to define positive and negative frequency and proceed with the quantisation just like in Minkowski space. The point of this chapter is to show that interesting things do happen! We will do so in steps and start with a simple toy model exhibiting many of the relevant phenomena.

$(1+1)$-dimensional toy model

We start our exploration with a massless scalar field in a two-dimensional "black hole" background with the same time-radial part of the metric as the Schwarzschild black hole

$$\begin{aligned}\mathrm{d}s^2 &= -\left(1-\frac{2M}{r}\right)\mathrm{d}t^2 + \left(1-\frac{2M}{r}\right)^{-1}\mathrm{d}r^2 \\ &= \left(1-\frac{2M}{r}\right)(-\mathrm{d}t^2 + \mathrm{d}r_*^2) \\ &= -\frac{2M}{r}\mathrm{e}^{-\frac{r}{2M}}\mathrm{d}U\mathrm{d}V .\end{aligned} \tag{7.2}$$

where we introduced the tortoise coordinate $r_* = r + 2M \log\left(\frac{r}{2M} - 1\right)$ and the Kruskal-Szekeres and Eddington-Finkelstein null coordinates U, V and u, v are defined as,

$$U = -4M\mathrm{e}^{-\frac{u}{4M}}\,, \qquad V = 4M\mathrm{e}^{-\frac{v}{4M}}\,, \qquad u = t + r^*\,, \qquad v = t - r^*,. \tag{7.3}$$

Note that this $(1+1)$-dimensional model merely serves to illustrate some properties of the $(3+1)$-dimensional black hole and should not be taken seriously on its own. In itself it is not even a solution to the (vacuum) Einstein equation.

A first hint that this picture is related with the Unruh effect can already be seen from the coordinate change (7.3), which is identical to the transformation between null coordinates in Rindler and Minkowski space upon substituting $a = (4M)^{-1}$. Indeed, the problem is very similar to the one in Rindler space. The two coordinate systems we consider, Eddington-Finkelstein and Kruskal-Szekeres are respectively very similar to the Rindler and Minkowski coordinates.

Consider for simplicity the massless minimally coupled scalar. We could introduce mass and a coupling to the Ricci scalar but note that in the four-dimensional background the Ricci scalar vanishes so we won't consider it. The addition of mass breaks conformality so for the sake of keeping things simple we will not include it here but comment on it later. The Eddington-Finkelstein coordinates are adapted to an observer sitting very far from the black hole, where the metric approaches Minkowski space $\mathrm{d}s^2 \to -\mathrm{d}u\mathrm{d}v$. In these coordinates it's straightforward to solve the wave equation and find a complete set of incoming and outgoing modes

$$\psi_\omega = \frac{1}{\sqrt{4\pi\omega}}\mathrm{e}^{-\mathrm{i}\omega u}\,, \qquad \tilde{\psi}_\omega = \frac{1}{\sqrt{4\pi\omega}}\mathrm{e}^{-\mathrm{i}\omega v}\,. \tag{7.4}$$

To these modes we can associate a vacuum called the Boulware vacuum which is defined by

$$b_\omega \left|0\right\rangle_B = 0\,. \tag{7.5}$$

The Boulware vacuum contains no particles from the point of view of a distant observer. However, since the Eddington-Finkelstein coordinates do not cover the whole of space-time, only the first quadrant of Penrose diagram. This is somewhat similar to the Rindler coordinates, this vacuum can be thought of as the analog of the Rindler vacuum of an accelerated observer.

Similarly, in Kruskal-Szekeres coordinates we can solve the wave equation finding the following set of positive frequency, incoming and outgoing, modes,

$$\xi_\omega(U) = \frac{1}{\sqrt{4\pi\omega}}\mathrm{e}^{-\mathrm{i}\omega U}\,, \qquad \tilde{\xi}_\omega(V) = \frac{1}{\sqrt{4\pi\omega}}\mathrm{e}^{-\mathrm{i}\omega V}\,. \tag{7.6}$$

The corresponding Kruskal vacuum is defined as,

$$a_\omega \left|0\right\rangle_K = 0\,. \tag{7.7}$$

In Kruskal-Szekeres coordinates, the metric near the black hole horizon approaches $\mathrm{d}s^2 \to \mathrm{d}U\mathrm{d}V$, so the Kruskal vacuum is the appropriate one for an observer sitting next to the black hole horizon. Since Kruskal–Szekeres coordinates cover the whole of space-time, they are the analogue of the Minkowski vacuum that we studied in the quantization of a scalar field in Rindler space.

We can now ask the following question: if a Kruskal observer is in the vacuum state, what does the Boulware observer see? Since the relation between both systems is the same as we found before in the case of the Unruh effect, the calculation of the Bogoliubov coefficients will be the identical as the one for the Unruh effect. The only difference lies in replacing the acceleration a by the surface gravity $a \to \kappa = \frac{1}{4M}$. We conclude that the Boulware observer sees a thermal spectrum with temperature

$$T_H = \frac{\kappa}{2\pi} = \frac{1}{8\pi M}\,. \tag{7.8}$$

Since the Kruskal-Szekeres coordinates U and V are well defined both in quadrant I and II of the Penrose diagram, the expansion in modes $\xi_\omega(U)$ is valid both outside and inside the horizon. On the other hand, the Eddington-Finkelstein coordinates, u and v only cover region I, so the expansion of b operators in terms of a is only valid there. This implies that while the expression b_ω in terms of a_λ and $a^\dagger_\lambda$ is complete, the inverse relation expressing a_ω also involves some other operators, $\tilde{b}$, whose modes have support only in region II inside the horizon. In particular, we can write

$$a_\omega = \int \mathrm{d}\lambda \left(\alpha^*_{\omega\lambda} b_\lambda + \beta^*_{\omega\lambda} b^\dagger_\lambda + \tilde{\alpha}^*_{\omega\lambda}\tilde{b}_\lambda + \tilde{\beta}^*_{\omega\lambda}\tilde{b}_\lambda\right), \tag{7.9}$$

In particular one can easily see that

$$\left[H, b^\dagger_\omega\right] = \omega b^\dagger_\omega\,, \qquad \left[H, \tilde{b}^\dagger_\omega\right] = -\omega \tilde{b}^\dagger_\omega\,. \tag{7.10}$$

Hence, while the creation operator $b^\dagger_\omega$ raises the energy by ω, the creation operator $\tilde{b}^\dagger_\omega$ lowers the energy by ω. The modes created by $\tilde{b}^\dagger_\omega$ are in fact necessary for energy conservation. Every time a particle with positive energy is created and propagates away from the black hole horizon, a particle with negative energy is also created, and falls into the horizon. These particles with opposite energy are entangled and there is a large entanglement between the radiation propagating outside the horizon and the inside. The resulting state for our quantum field in the black hole background is described by the repeated action of $\tilde{b}^\dagger_\omega b^\dagger_\omega$ on the vacuum $|0\rangle_{b,\tilde{b}}$ and takes the form of a squeezed state as derived in chapter 4,

$$|0\rangle_K \propto \exp\left\{\int \mathrm{d}\omega \mathrm{e}^{-\frac{\omega}{2T_H}} b^\dagger_\omega \tilde{b}^\dagger_\omega\right\} |0\rangle_{b,\tilde{b}}\,. \tag{7.11}$$

The Boulware vacuum above is the state obtained by tracing out the tilded part of the Hilbert space.

In conclusion, we can interpret the Hawking emission process as arising from particle pair creation close to the horizon, with a negative energy particle falling into the black hole and a positive energy particle escaping to infinity. One may be surprised by the appearance of propagating negative energy modes. However, one should recall that here the energy is the conserved charge associated with a Killing vector that generates time translations far away from the horizon, lets say t. This vector is timelike outside the horizon, but becomes space-like inside the horizon. The charge of a space-like Killing vector is momentum, and this can be either positive, or negative so there is no worry. We see that since Hawking radiation needs a time-like Killing vector becoming space-like. This is exactly what happens in the vicinity of a Killing horizon.

$(3+1)$-dimensional Schwarzschild

The two-dimensional toy model discussed above was extremely simple but included all the necessary ingredients to observe the Hawking temperature. However, we are really interested in the four-dimensional Schwarzschild black hole. In this case we lose conformality and we will not be able to exactly solve the problem. However, using some approximations we will still be able to come to a similar conclusion as in the toy model above.

Let us consider again a massless scalar field but now in the full Schwarzschild background (7.1). As in Schwarzschild, and similarly in the Kerr black hole we have $R = 0$ we can safely ignore the coupling to the Ricci scalar.[1] As the problem is entirely symmetric on the two-sphere it will prove useful to decompose our field in spherical harmonics,

$$\phi = f_{lm}(t,r)Y_{lm}(\theta,\phi)\,, \tag{7.12}$$

where $Y_{lm}(\theta,\phi)$ are the spherical harmonics. Substituting this expansion in the wave equation results in

$$\Box^{(4)}\phi = 0 \Rightarrow \left(\Box^{(2)} + V_l(r)\right) f_{lm}(r,t)\,, \tag{7.13}$$

where $\Box^{(d+1)}$ denotes the $(d+1)$-dimensional Laplacian on respectively the full Schwarzschild space-time or the 2d time-radial slice considered in the above. The potential $V_l(r)$ is given by

$$V_l(r) = \left(1 - \frac{2M}{r}\right)\left(\frac{2M}{r^3} + \frac{l(l+1)}{r^2}\right). \tag{7.14}$$

So we see that the massless scalar in $(3+1)$ dimensions decomposes in infinitely many massless scalars in $(1+1)$ dimensions in the presence of a potential. The only change from the story above is therefore that a wave escaping the black hole needs to propagate through the potential barrier caused by $V_l(r)$. Even though we cannot solve this problem analytically, note that the potential falls off exponentially in r^* as $r^* \to -\infty$, i.e. when one approaches the horizon, and falls off polynomially as $r \to \infty$. For this reason we can use the same asymptotic states as above. Hence, the only effect of the potential is that it decreases the intensity of the wave and changes the resulting spectrum of emitted particles by a greybody factor $0 < \Gamma_l(\omega) < 1$,

$$\langle n_\omega \rangle = \frac{\Gamma_l(\omega)}{\mathrm{e}^{\frac{\omega}{T_H}} - 1}\,. \tag{7.15}$$

The greybody factor is entirely due to the potential outside the black hole horizon. It is clear that this factor is not directly related to the quantum origin of the Hawking radiation and therefore the basic features of the derivation above survive without significant alterations. This result can be generalized for the case of a massive scalar field, and also for vector and spinor fields. The conclusion is that the black hole must emit all possible species of particles, each having the Hawking thermal spectrum corrected by the corresponding greybody factor.

[1] In the collapse picture below, in the collapsing phase the Ricci scalar might be non-vanishing. However, this will not change the late time spectrum and so we will keep ignoring this coupling.

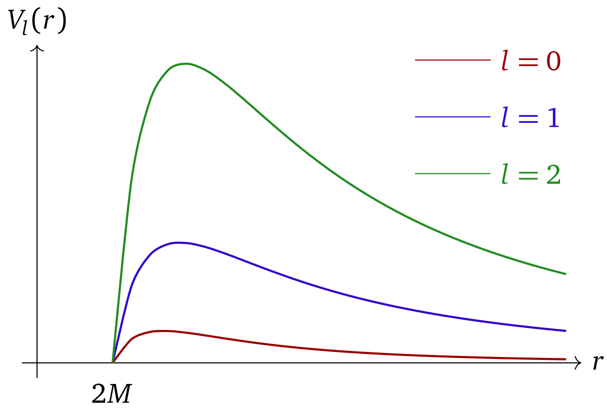

Figure 7.1: The effective potential $V_l(r)$ experienced by the spherically symmetric modes f_{lm} for the values of $l = 0, 1, 2$.

Black hole formed through collapse

The eternal black hole described above is rather unphysical and we don't expect to see the full Kruskal extension. A more physical picture would be to consider a ball of spherically symmetric dust collapsing to form a black hole. The Penrose diagram for this space-time is given in Figure 7.2.

We can now proceed as before and quantise the massless scalar field in this background. The past null hypersurface $\mathscr{I}_-$ is a Cauchy hypersurface, hence we can quantise the scalar field using this hypersurface and write

$$\phi = \int \mathrm{d}\omega \left(a_\omega f_\omega + a_\omega^\dagger f_\omega^* \right), \tag{7.16}$$

where the f_ω are a complete set of orthonormal solutions to the wave equation with associated annihilation and creation operators a_ω and $a_\omega^\dagger$. Far outside the collapsing body at early times, the definition of physical particles that would be detected by inertial observers, or equivalently of positive frequency solutions of the wave equation, is unambiguous. We choose the f_ω such that they form a complete set of incoming positive frequency solutions of energy ω. Their asymptotic form on past null infinity is

$$f_\omega \sim \frac{1}{\sqrt{16\pi^3\omega}} \mathrm{e}^{-\mathrm{i}\omega v} Y_{lm}(\theta, \phi), \qquad \langle f_\omega, f_{\omega'} \rangle = \delta(\omega - \omega'), \tag{7.17}$$

where we suppress the discrete quantum numbers l and m in labelling the functions f_ω.

At late times on the other hand, we know that $\mathscr{I}^+$ is not a Cauchy hypersurface. Instead we have to consider boundary data both at future null infinity and the event horizon $\mathcal{H}$. On $\mathscr{I}^+$, just like on $\mathscr{I}^-$, the definition of positive frequency modes is unambiguous and we can find a complete set $\{p_\omega, p_\omega^*\}$ of orthonormal solutions on $\mathscr{I}^+$. The asymptotic form of these functions on $\mathscr{I}^+$ is

$$p_\omega \sim \frac{1}{\sqrt{16\pi^3\omega}} \mathrm{e}^{-\mathrm{i}\omega u} Y_{lm}(\theta, \phi), \tag{7.18}$$

where u is the outgoing null coordinate at $\mathscr{I}^+$. A general solution, incoming from the past, will also have a part that is incoming at the event horizon. Therefore we must introduce a second complete basis of orthonormal functions q_ω on the horizon which have zero Cauchy data on $\mathscr{I}^+$. Since the

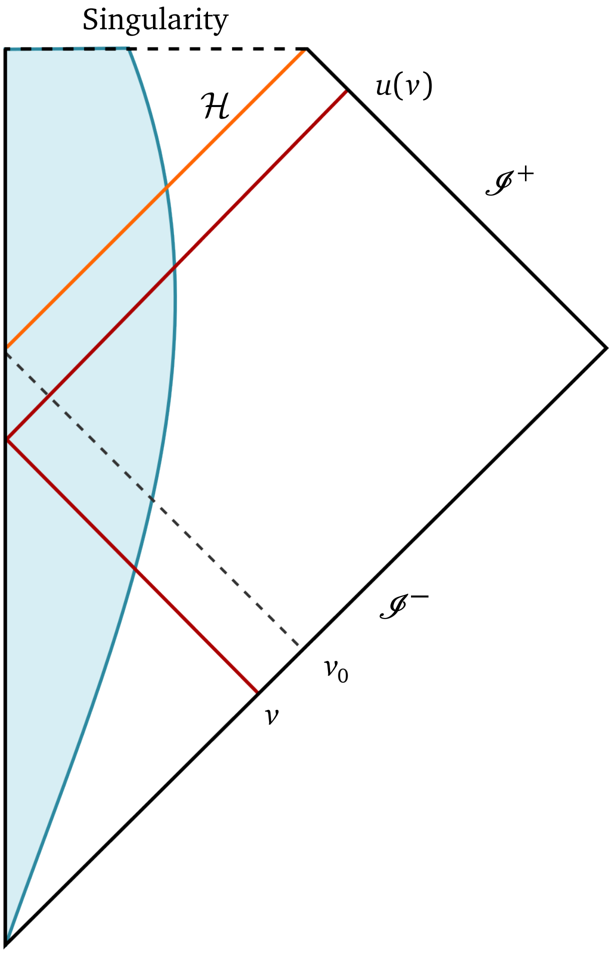

Figure 7.2: The Penrose diagram for collapse to the Schwarzschild black hole. The singularity is located on top and shielded by the horizon $\mathcal{H}$ in orange. The collapsing cloud of dust is pictured in blue. Once the cloud enters the horizon, a black hole is formed. The incoming ray with $v = v_0$ is the last one that reaches the centre of the collapsing body and makes it to $\mathscr{I}^+$. Rays with $v < v_0$ fall into the black hole.

functions p_ω and q_ω are supported in disjoint regions at late times, their (conserved) scalar product must vanish $\langle q_\omega, p_{\omega'}\rangle = 0$ and similarly for their complex conjugates. For this reason the precise form of the functions q_ω will not affect observations on $\mathscr{I}^+$. The details are therefore not important since we will trace over the modes at the horizon. We can thus expand the field ϕ in the entire space-time as

$$\phi = \mathrm{d}\omega \left(b_\omega p_\omega + c_\omega q_\omega + b^\dagger_\omega p^*_\omega + c^\dagger_\omega q^*_\omega \right) , \tag{7.19}$$

with b_ω and c_ω the annihilation operators for outgoing particles at late times. The vacuum at $\mathscr{I}_+$ defined by $b_\omega |0\rangle_B$ is the Boulware vacua as defined before, while the vacuum at past null infinity $|0\rangle_-$, defined by $a_\omega |0\rangle_-$ will take up the role of the Kruskal vacuum. The task we have to do is then clear, we want to compute the number density of particles observed by a Boulware observer in the "Kruskal" vacuum. Although conceptually clear the computation is rather involved and some details will be left to fill in by the reader.

To compute the density of emitted particles we have to compute the Bogoliubov coefficients

$$\alpha_{\omega\omega'} = \langle f_{\omega'}, p_\omega \rangle \,, \qquad \beta_{\omega\omega'} = -\left\langle f^*_{\omega'}, p_\omega \right\rangle . \tag{7.20}$$

To determine these coefficients we need to trace back in time the function p_ω along an outgoing geodesic at a large value of u, close to the horizon. Such a geodesic is illustrated in Figure 7.2 as the red line and passes through the center of the collapsing cloud just before the event horizon is formed and emerges as an incoming geodesic characterised by a value of v close to v_0. The value of u depending at $\mathscr{I}^+$ depending on v can be computed by analysing the null geodesics in this space-time, see Appendix C, and is given by

$$u(v) = -4M \log\left(\frac{v_0 - v}{K}\right) , \tag{7.21}$$

where K is some positive constant. Inserting this expression into expressions (7.18) for the functions p_ω we can compute the Bogoliubov coefficients as

$$\begin{aligned}
\alpha_{\omega\omega'} &= C \int_{-\infty}^{v_0} \mathrm{d}v \left(\frac{\omega'}{\omega}\right)^{\frac{1}{2}} \mathrm{e}^{\mathrm{i}\omega' v - \mathrm{i}\omega u(v)} \,, \\
\alpha_{\omega\omega'} &= C \int_{-\infty}^{v_0} \mathrm{d}v \left(\frac{\omega'}{\omega}\right)^{\frac{1}{2}} \mathrm{e}^{-\mathrm{i}\omega' v - \mathrm{i}\omega u(v)} \,,
\end{aligned} \tag{7.22}$$

where C is a constant. Substituting $s = v_0 - v = \mathrm{i}z$ we can compute

$$\begin{aligned}
\alpha_{\omega\omega'} &= -C \int_{\infty}^{0} \mathrm{d}s \left(\frac{\omega'}{\omega}\right)^{\frac{1}{2}} \mathrm{e}^{-\mathrm{i}\omega'(s - v_0)} \exp\left(4\mathrm{i}\omega M \log\frac{s}{K}\right) \\
&= -\mathrm{i}C\mathrm{e}^{\mathrm{i}\omega' v_0} \int_{-\infty}^{0} \mathrm{d}z \left(\frac{\omega'}{\omega}\right)^{\frac{1}{2}} \mathrm{e}^{\omega' z} \exp\left(4\mathrm{i}\omega M \log\frac{\mathrm{i}z}{K}\right) \\
&= -\mathrm{i}C\mathrm{e}^{\mathrm{i}\omega' v_0} \mathrm{e}^{2\pi\omega M} \left(\frac{\omega'}{\omega}\right)^{\frac{1}{2}} \int_{-\infty}^{0} \mathrm{d}z \, \mathrm{e}^{\omega' z} \exp\left(4\mathrm{i}\omega M \log\frac{|z|}{K}\right) ,
\end{aligned} \tag{7.23}$$

and similarly,

$$\beta_{\omega\omega'} = \mathrm{i}C\mathrm{e}^{-\mathrm{i}\omega' v_0} \mathrm{e}^{-2\pi\omega M} \left(\frac{\omega'}{\omega}\right)^{\frac{1}{2}} \int_{-\infty}^{0} \mathrm{d}z \, \mathrm{e}^{\omega' z} \exp\left(4\mathrm{i}\omega M \log\frac{|z|}{K}\right) . \tag{7.24}$$

We immediately find that

$$|\alpha_{\omega\omega'}|^2 = \mathrm{e}^{8\pi M\omega} |\beta_{\omega\omega'}|^2 \,, \tag{7.25}$$

for the part of the wave packet that was propagated back in time through the collapsing body just before it formed a black hole.

For the components p_ω of this part of the wave packet, we have the scalar product,

$$\langle p_\omega, p_{\omega'} \rangle = \Gamma(\omega)\delta(\omega - \omega') \,, \tag{7.26}$$

where $\Gamma(\omega)$ is the fraction of the wave packet that would propagate back in time through the collapsing body. Indeed, we can divide the functions p_ω in two parts,

$$p_\omega = p_\omega^{(1)} + p_\omega^{(2)} . \tag{7.27}$$

The part $p_\omega^{(1)}$ propagates backwards in time outside of the collapsing body and reaches $\mathscr{I}_-$ at some value $v > v_0$. This part of the wave will interact minimally with the collapsing matter and consequentially the frequency will not change significantly from $\mathscr{I}_-$ to $\mathscr{I}_+$. For this reason we can ignore this part of the wave when asking questions about particle production. Indeed, since $p_\omega^{(1)}$ and $p_\omega^{(2)}$ have disjoint support on $\mathscr{I}_-$ (resp. $v > v_0$ and $v < v_0$), they do not interact and we can safely ignore the parts $p_\omega^{(1)}$. Their only effect is the introduction of the function Γ. From the normalisation condition (7.26) we therefore find

$$\Gamma(\omega)\delta(\omega-\omega') = \int \mathrm{d}\lambda \left(\alpha^*_{\omega\lambda}\alpha_{\omega'\lambda} - \beta^*_{\omega\lambda}\beta_{\omega'\lambda}\right) \tag{7.28}$$

where now the Bogoliubov coefficients refer to the coefficients in the expansion of $p_\omega^{(2)}$ only.

As before, this allows us to compute the density of emitted particles as

$$\langle N_\omega\rangle_- = \langle 0|_K\, b^\dagger_\omega b_\omega\, |0\rangle_- \simeq \langle 0|_-\, b^{(2)\dagger}_\omega b^{(2)}_\omega\, |0\rangle_- = \int \mathrm{d}\omega' |\beta_{\omega\omega'}|^2 . \tag{7.29}$$

The resulting integral is again divergent but can be regularised by putting the system in a box and computing the density of emitted particles instead,

$$n_\omega = \frac{1}{V}\langle N_\omega\rangle_- = \frac{\Gamma(\omega)}{2\pi\left(\mathrm{e}^{8\pi M\omega}-1\right)} . \tag{7.30}$$

Hence after this long computation we come to exactly the same conclusion as before and find that the collapsing black hole emits and absorbs radiation exactly like a gray body of absorptivity $\Gamma_{lm}(\omega)$ and Hawking temperature $T_H = (8\pi M)^{-1}$!

For large black holes this temperature $T_H \sim 6\times 10^{-8}\frac{M_\odot}{M} K$ is extremely small for large black holes with $M \gg M_\odot$, where $M_\odot$ is the mass of the sun. For this reason our assumption that the background does not back-react against this radiation seems to be justified. For small black holes the back-reaction cannot be ignored and a more sophisticated treatment is needed.

Indeed, from energy conservation one can estimate the rate of loss of mass. Stefan's law for the evaporation of a black body states that

$$\frac{dE}{dt} \propto -AT^4 , \tag{7.31}$$

where A is the area. With $E \propto M$, $A \propto M^2$ and $T \propto M^{-1}$ this leads to the rate of mass loss to be proportional to

$$\frac{\mathrm{d}M}{\mathrm{d}t} \simeq -\frac{c}{M^2} , \tag{7.32}$$

where c is a positive constant that depends on the number and type of quantised matter fields that couple to gravity. From this expression we it becomes indeed apparent that for large black holes $M \gg \frac{\mathrm{d}M}{\mathrm{d}t}$ justifying our assumption of ignoring back-reaction. For reference, this leads to the black-hole evaporating in a finite time of the order of $10^{71}\left(\frac{M}{M_\odot}\right)^3$ seconds.

Finally, before moving on, note that a static observer $\mathcal{O}$ at finite radius r measures a blue-shifted temperature $T_{\mathcal{O}} = \frac{T_H}{|g_{00}|}$. As $r \to \infty$ this approaches the Hawking temperature, but it diverges at the horizon where $|g_{00}| \to 0$ due to the infinite acceleration of the static observer at the horizon. This is precisely the Unruh effect we observed in the previous section. A freely falling observer however sees no divergence as they cross the horizon.

7.2 The Hawking thermal state and friends

In analogy with the Rindler case, we can easily observe the thermal nature of the $|0\rangle_-$ vacuum for a Boulware observer. The observer in the Boulware vacuum $|0\rangle_B$ has access to a Fock space $\mathcal{F}_B$ built by acting on $|0\rangle_B$ with the creation operators $b^\dagger_\omega$. As in the Rindler case however, and from the fact that $\mathscr{I}^+$ is not a Cauchy surface, we know that this is not enough to construct the full Fock space as seen by an observer at past null infinity. Indeed, the full Fock space is obtained as the tensor product $\mathcal{F}_- = \mathcal{F}_B \otimes \mathcal{F}_H$.

As before for Rindler, the late time vacuum is a complicated state of the form

$$|0\rangle_- \propto \sum_n f_n \,|n\rangle_H \,|n\rangle_B \ , \tag{7.33}$$

Considering the associated density matrices, we construct the Boulware vacuum by tracing over the horizon modes

$$\rho_B = \mathrm{Tr}_{\mathcal{F}_H} \rho_- = \mathrm{Tr}_{\mathcal{F}_H} |0\rangle_- \langle 0|_- \propto \sum_n \mathrm{e}^{-n\pi\Omega/\kappa} |n\rangle_{BB}\langle n| \tag{7.34}$$

which gives the desired thermal state.

The Hawking state was what arose from an essentially Minkowskian vacuum at $\mathscr{I}^-$ in the collapsing scenario, but other states are natural for the eternal Schwarzschild black hole where we start from the Kruskal vacuum defined near the horizon. In this case the Cauchy surface in the far past consists of two components, $\mathscr{I}_- \cup \mathcal{H}_-$, as can be seen from the Penrose diagram in Figure E.1. Having said so, it becomes clear that there are various 'natural' choices for the vacuum in the past depending on what we define as positive frequency states. The options are summarised in Table 7.1 below for positive frequencies $\omega > 0$.

Vacuum	Positive modes on $\mathcal{H}_-$	Positive modes on $\mathscr{I}_-$
Boulware vacuum $\|B\rangle$	$\mathrm{e}^{-\mathrm{i}\omega u}$	$\mathrm{e}^{-\mathrm{i}\omega v}$
Unruh vacuum $\|U\rangle$	$\mathrm{e}^{-\mathrm{i}\omega U}$	$\mathrm{e}^{-\mathrm{i}\omega v}$
Hartle-Hawking vacuum $\left\|H^2\right\rangle$	$\mathrm{e}^{-\mathrm{i}\omega U}$	$\mathrm{e}^{-\mathrm{i}\omega V}$

Table 7.1: The three natural vacua in the eternal Schwarzschild black hole. The options differ by the choice of positive frequency modes in the two component of the far past Cauchy surface $\mathcal{H} \cup \mathscr{I}_-$.

The three options defined in this table each have a distinct physical interpretation.

- The Boulware vacuum corresponds to our familiar concept of an empty state defined far away from the black hole and is defined with respect to a static observer. It is pathologic in the sense that the expectation value of the stress tensor diverges at the horizon. This is similar to the Rindler vacua becoming singular at the Killing horizon seen in the previous chapter.
- The Unruh vacuum is regular on the future horizon, but not on the past horizon. At infinity, this vacuum corresponds to an outgoing flux of blackbody radiation at the black hole temperature. The black hole collapse studied in the previous section brings about the Unruh state.
- The Hartle-Hawking vacuum does not correspond to our usual notion of a vacuum. It is well-behaved both on the future and past horizon but the price we have to pay for this is that the state is not empty at infinity, but instead corresponds to a thermal distribution of quanta at the Hawking temperature. That is, the Hartle-Hawking vacuum corresponds to a black hole in (unstable) equilibrium with an infinite bath of blackbody radiation.

All these vacua are interesting in their own right and have been studied for a variety of reasons. However, from the point of view of the 'physical' collapse picture described above, it seems that the Unruh vacuum best approximates the state obtained following the gravitational collapse of a massive cloud of dust.

There are various ways to investigate the thermal nature of the various vacua above. As mentioned above, the Hartle-Hawking state is a thermal state both at $\mathscr{I}_{\pm}$. Moreover, it is an example of a Thermal Green's function

$$G^{H^2}(x,x') := \left\langle H^2 \right| \hat{\phi}(x)\hat{\phi}(x') \left| H^2 \right\rangle = G_\beta(x,x'). \tag{7.35}$$

We will not attempt to explicitly compute the Green's function (see for example [CJ86] for the result) but a key statement is that it analytically extends to complex time and is periodic in imaginary time with period,

$$\beta = 1/T_H = 8\pi M, \tag{7.36}$$

An alternative, and easier way to recognize the thermal nature of black holes is to study the Euclideanised background. Wick rotating $t \to \mathrm{i}\tau$, we find the positive definite background with metric[2]

$$\mathrm{d}s^2 = \left(1 - \frac{2M}{r}\right)\mathrm{d}\tau^2 + \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2. \tag{7.37}$$

Substituting $r = 2M + \epsilon$ and expanding in small ϵ we find

$$\mathrm{d}s^2 \approx \frac{2M}{\epsilon}\mathrm{d}\epsilon^2 + \frac{\epsilon}{2M}\mathrm{d}\tau^2 + 4M^2\mathrm{d}\Omega^2. \tag{7.38}$$

At $\epsilon = 0$ we see that the angular part nicely factorises out. Changing coordinates to $\rho = \sqrt{8M\epsilon}$ the metric becomes

$$\mathrm{d}s^2 \approx \mathrm{d}\rho^2 + \frac{\rho^2}{16M^2}\mathrm{d}\tau^2 + 4M^2\mathrm{d}\Omega^2. \tag{7.39}$$

Hence we clearly see that in order to avoid a conical singularity at the origin we need to impose the

[2]It's even easier to see this in Kruskal coordinates, where the metric is given by $\mathrm{d}s^2 = \frac{2M}{r}\mathrm{e}^{-\frac{r}{2M}}\mathrm{d}U\mathrm{d}V$. Remembering that $U = -4M\mathrm{e}^{\frac{r^*}{4M}}\mathrm{e}^{-\frac{\mathrm{i}\tau}{4M}}$ it is equally clear that unless τ has period $8\pi M$ this metric has a conical singularity at the origin.

periodicity $\tau \sim \tau + 8\pi M$. This gives alternative evidence for the Hawking temperature, analogous to the Euclideanisation argument for the Unruh temperature in the previous section. In Figure 7.3 we sketch the topology of the various Euclideanisations discussed so far. This property is an important clue for the presence of thermal states and remains valid much more general in various dimensions with various types of matter content. The Euclideanisation of a black hole has the characteristic topology of a cigar.

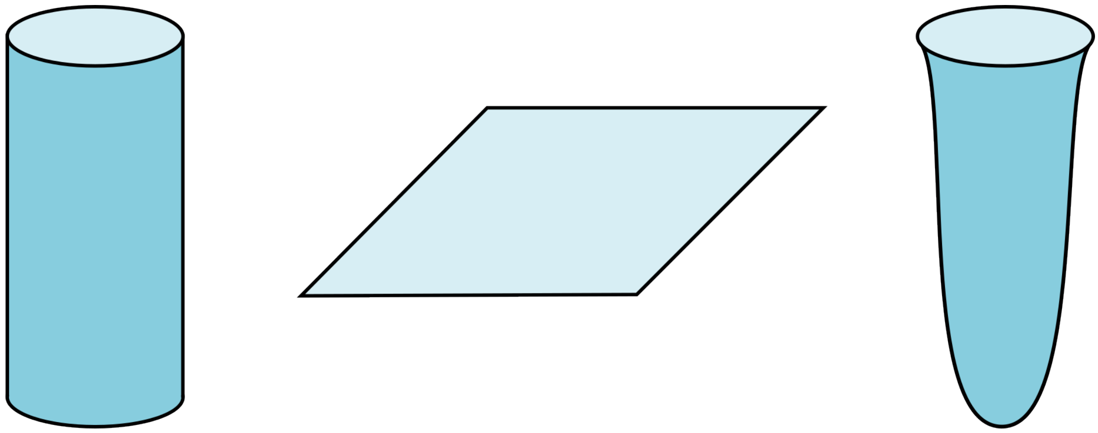

Figure 7.3: From left to right, the Euclideanised Minkowski space $\mathbb{R} \times S^1$, Rindler space $\mathbb{R}^2$ and (r, t) plane of the Schwarzschild black hole. The cigar topology background is characteristic for black hole backgrounds.

7.3 Black hole thermodynamics

Prior to the discovery of Hawking radiation of black holes Bekenstein already conjectured that black holes must have a non-vanishing intrinsic entropy [Bek73]. He came to this conclusion through the following thought experiment. Consider a black hole that absorbs matter with non-zero entropy. If the black hole entropy were vanishing then the total entropy in the system would decrease, violating the second law of thermodynamics. Based on this reasoning Bekenstein concluded that the second law can only be preserved if a black hole has an intrinsic entropy S_{BH} proportional to its surface area. However, the proportionality constant could not be fixed until the discovery of Hawking radiation.

Differentiating the expression for the surface area $A = 16\pi M^2$, we find

$$dM = \frac{1}{8\pi M} d\frac{A}{4} . \tag{7.40}$$

Recognising the coefficient on the left hand side as the Hawking temperature this looks precisely like the first law of thermodynamics

$$dE = T dS , \tag{7.41}$$

Following this analogy we conclude that the black hole (or Bekenstein-Hawking) entropy must be equal to

$$S_{BH} = \frac{A}{4} = 4\pi M^2 . \tag{7.42}$$

In line with its thermodynamic counterpart, the first law of black hole thermodynamics can be generalised to closed systems with rotation and charge as follows,

$$\mathrm{d}E = T\mathrm{d}S + \Omega \mathrm{d}J + \Phi \mathrm{d}Q\,, \tag{7.43}$$

where we interpret Φ as the electric potential at the horizon and Q the total charge. Similarly, J is the angular momentum and Ω the angular velocity.

Exercise 7.1. *Consider the Reissner-Nordstrom solution*

$$\mathrm{d}s^2 = \frac{\Delta(r)}{r^2}\mathrm{d}t^2 - \frac{r^2}{\Delta(r)}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2\,, \qquad \Delta(r) = r^2 - 2Mr + Q^2\,, \tag{7.44}$$

This is a solution to the Einstein-Maxwell equations with electromagnetic potential

$$A = \frac{Q}{r}\mathrm{d}t\,. \tag{7.45}$$

Assuming $Q < M$, state the second law of thermodynamics by differentiating the area as a function of mass and show that the coefficient is indeed equal to the Hawking temperature. (Hint: the Hawking temperature has the same expression as for Schwarzschild when expressed in terms of the surface gravity.)

Exercise 7.2. *If you are feeling courageous, repeat the previous exercise for the Kerr black hole.*

The entropy of astrophysical black holes is extremely large, for a solar mass black hole for example one finds $S_{BH}^{\odot} \sim 10^{76}$. Interpreting this as a statistical entropy implies that a quantum mechanical black hole has an enormous number of microstates corresponding to the unique classical black hole. Finding a microscopic derivation of this entropy is an active area of modern research. In asymptotically flat space-time such a derivation has been given through string theory [SV96] but in asymptotically AdS or dS space-times this remains an open question.

Taking into account the entropy of a black hole, we can state the generalised second law of thermodynamics as follows.

$$\delta S_{\text{total}} = \delta S_{\text{matter}} + \delta S_{BH} \geq 0\,. \tag{7.46}$$

I.e. the total entropy of all black holes and matter combined can never decrease. In classical general relativity, one can prove that the combined area of all black hole horizons cannot decrease. This applies not only to adiabatic processes but also to strongly out of equilibrium processes such as collisions and mergers of black holes.

Ordinary thermodynamic systems can be in a stable equilibrium with an infinity heat reservoir. However, this is not true for black holes because they have a negative heat capacity! In other words, black holes get colder when they absorb energy. Indeed, with $E(T) = M = (8\pi T)^{-1}$, we find

$$C_{BH} = \frac{\partial E}{\partial T} = -\frac{1}{8\pi T^2} < 0\,. \tag{7.47}$$

This means that a black hole surrounded by an infinite thermal bath at temperature $T < T_H$ will emit radiation and become even hotter. The process of evaporation is not halted in an infinite thermal reservoir with constant temperature. Similarly, putting a black hole in a bath with $T > T_H$ will make the black hole colder! In either case no stable equilibrium is possible. Stable equilibrium is only possible in a finite reservoir. In this case the radiation of the black hole changes the temperature of the bath until both reach the same temperature.

7.4 The information paradox

In this chapter we have discussed quantum fields in a black hole background and discovered that black holes have a temperature. But where precisely does this radiation come from? The answer, discovered by Hawking, is that we must consider quantum processes, more precisely quantum fluctuations of the vacuum. In the vacuum pairs of particles and antiparticles are continuously being created and annihilated. Consider such fluctuations for electron-positron pairs. Suppose we apply a strong electric field in a region which is pure vacuum. When an electron–positron pair is created, the electron gets pulled one way by the field and the positron gets pulled the other way. Thus instead of annihilation of the pair, we can get creation of real (instead of virtual) electrons and positrons which can be collected on opposite ends of the vacuum region. Thus we get a current flowing through the space even though there is no material medium filling the region where the electric field is applied. This is called the 'Schwinger effect'.

A similar effect happens with the black hole, with the effect of the electric field now replaced by the gravitational field. We do not have particles that are charged in opposite ways under gravity. But the attraction of the black hole falls off with radius, so if one member of a particle–antiparticle pair is just outside the horizon it can flow off to infinity, while if the other member of the pair is just inside the horizon then it can get sucked into the hole. The particles flowing off to infinity represent the 'Hawking radiation' coming out of the black hole. Doing a detailed computation, one finds that the rate of this radiation is given by (7.31). Thus we seem to have a very nice thermodynamic physics of the black hole. The hole has entropy, energy, and temperature and radiates as a thermal body should.

So far, so good, but there is a deep problem arising out of the way in which this radiation is created by the black hole [Haw76]. The radiation which emerges from the hole is not in a 'pure quantum state'. Instead, the emitted quanta are in a 'mixed state' with excitations which stay inside the hole. There is nothing wrong with this in this by itself, but the problem comes at the next step. The black hole loses mass because of the radiation and eventually disappears. Then the quanta in the radiation outside the hole are left in a state that is 'mixed', but we cannot see anything that they are mixed with! Thus the state of the system has become a 'mixed' state in a fundamental way. This does not happen in quantum mechanics. If we start with a pure state $|\psi\rangle$ and evolve it by some Hamiltonian H to $|\psi'\rangle = \mathrm{e}^{-\mathrm{i}Ht}|\psi\rangle$ we obtain another pure state at the end. Mixed states arise in usual physics when we coarse-grain over some variables and thereby discard some information about a system. This coarse-graining is done for convenience, so that we can extract the gross behaviour of a system without keeping all its fine details, and is a standard procedure in statistical mechanics. But there is always a 'fine-grained' description available with all information about the state, so that underlying the full system there is always a pure state. With black holes we seem to be getting a loss of information in a fundamental way. We are not throwing away information for convenience; rather we cannot get a pure state even if we wanted.

To make this discussion a bit more quantitative, let us introduce the von Neumann entropy, which is an extension of the Gibbs entropy from statistical mechanics to quantum statistical mechanics. For a quantum system described by a density matrix ρ, the von Neumann entropy is defined as,

$$S = -\mathrm{Tr}\,\rho\log\rho\,, \tag{7.48}$$

In a finite dimensional system we can always write the density matrix in a basis of eigenvectors $|n\rangle$ as

$$\rho = \sum_n p_n\,|n\rangle\langle n|\,, \tag{7.49}$$

which makes it clear that for pure state the von Neumann entropy vanishes, while its maximal value $S = \log\dim\mathcal{H}$ is reached for the maximally mixed state $\rho = \frac{1}{\dim\mathcal{H}}\sum|n\rangle\langle n|$. Now, let us consider the evaporation of a black hole à la Hawking. The black hole starts in a pure state, hence initially we have $S = 0$. After some time part of the black hole has evaporated where the radiation is in a mixed state. Hence, during the evaporation, the von Neumann entropy gradually increases until it reaches its maximum when the black hole is fully evaporated into thermal radiation. See Figure 7.4 for a graphical representation of the entropy as a function of time.

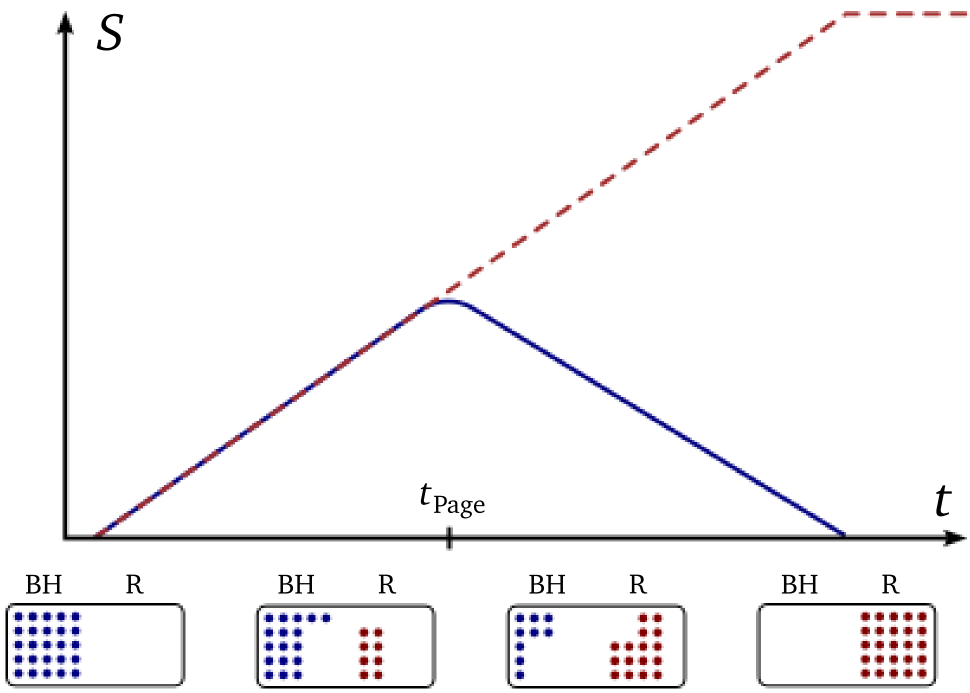

Figure 7.4: The red line represents the entropy of the radiation following Hawking's calculation, while the blue line is the page curve. The turning point at the page time occurs at the point in time where the entropy of Hawking radiation is equal to the Bekenstein-Hawking entropy of the black hole. The dots give a cartoon picture of the qubits of information transferred from the black hole to the radiation.

This paradox has been a guiding post for progress on quantum gravity since its discovery by Hawking in 1975. Hawking initially advocated that in the presence of gravity we should change our ideas about quantum mechanics and loosen the our demand of having purely unitary evolution. However, this is a very unsettling proposal which opens a Pandora's box of unwanted consequences and most physicists are not willing to abandon ordinary quantum mechanics when it works so well in all other contexts. Luckily, in the 90s and 2000s string theory provided various hints that information is not lost! But how can it be that we need string theory for this? The gravitational interactions at the event horizon for a large black hole are so incredibly small that we would expect that our semiclassical intuition should be valid here.

Around 2020 a new perspective emerged in the papers [Pen20, AEMM19] and many papers after that. In this paper an alternative semiclassical computation was performed that instead of Hawkings entropy curve produces the Page curve. As can be seen from Figure 7.4 this curve descends back to zero entropy at the end of the evaporation, therefore restoring unitarity! The fundamental idea behind these computations is to introduce a new tool called the 'quantum extremal surface' which takes into account the microscopic structure of the black hole as well as the coupling with the external fields. Performing the semiclassical computation using this surface, instead of the event horizon as in Hawkings computation results in a different prediction for the entropy where at the Page time a transition takes place after which the entropy starts to shrink, reproducing the Page curve. A full discussion of their formalism would lead us beyond the scope of this course so we refer the reader to the original literature.

This approach immediately brings us a whole range of new questions. Why are the equations for various quantities modified by quantum gravity when a black hole is involved? And should they then also be modified when studying the sun or Mercury? The key in answering this question turns out to be complexity. Black holes are incredibly complex objects, they are maximally chaotic and pack information in the densest possible way. This characteristic sets them apart from the other astrophysical object where we find a similar curvature as at the event horizon of a black hole. The computation of the quantum extremal surface turn out to crucially depend on complexity. It turns out that usual semi-classical gravity is valid at low curvature and low complexity. However, at large complexity our semi-classical intuition has to be modified in order to predict the correct physical behaviour. Research in this direction continues until today and is an exciting area of new developments in quantum gravity.

Chapter 8

Quantum fields in AdS

de Sitter space plays a crucial role in cosmology, capturing the behaviour of our universe during both its early inflationary phase and its current accelerated expansion. Anti-de Sitter (AdS) space, on the other hand, is valuable for entirely different reasons. It provides the natural setting for exploring the holographic principle, particularly through the AdS/CFT correspondence. The rich symmetry structure of AdS, characterized by the $SO(2,d)$ isometry group, heavily constrains the dynamics of quantum field theories defined on this background. Even with mass deformations, these symmetries guide us toward solvable structures. Moreover, the intrinsic length scale of AdS acts as a built-in infrared regulator, taming the long-distance behaviour of interacting field theories.

In many respects, placing a quantum theory in AdS improves its behaviour — much like confining a system within a box. However, this simplification comes with its own set of subtleties. Notably, AdS is not globally hyperbolic, which means that to fully specify the dynamics, one must carefully choose boundary conditions at the conformal boundary. These boundary effects are not just technical details but often encode essential physics in the holographic framework.

8.1 A CFT primer

This section introduces the minimal set of CFT concepts required to understand the AdS/CFT correspondence, focusing on the Euclidean signature for simplicity. Our presentation emphasizes the embedding formalism, as it will be essential for the remainder of this chapter. For a more comprehensive treatment, we refer the reader to the course on conformal field theory or standard textbooks and lecture notes such as [DFMS97, Ryc16].

Conformal symmetry

A conformal field theory (CFT) is a quantum field theory invariant under the conformal group. We begin by introducing this group and its action. As discussed earlier, a conformal transformation is a coordinate transformation that preserves the metric up to a local scale factor:

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \frac{d\tilde{x}^\alpha}{dx^\mu}\frac{d\tilde{x}^\beta}{dx^\nu} g_{\alpha\beta} = \Omega(x)^2 g_{\mu\nu}\,, \qquad \Omega(x) \neq 0\,. \tag{8.1}$$

Exercise 8.1. *Show that, for $d > 2$, the most general infinitesimal conformal transformation takes the form $\tilde{x}^\mu = x^\mu + \epsilon^\mu(x)$ with*

$$\epsilon^\mu(x) = a^\mu + \lambda x^\mu + m^{\mu\nu}x_\nu + x^2 b^\mu - 2x^\nu b_\nu x^\mu\,. \tag{8.2}$$

In $d > 2$ dimensions, conformal transformations of $\mathbb{R}^d$ form the group $\mathrm{SO}(1, d+1)$. Its generators are:

$$P^\mu\,, \qquad M_{\mu\nu}\,, \qquad K_\mu\,, \qquad D\,, \tag{8.3}$$

where P_μ and $M_{\mu\nu}$ generate translations and rotations (as in any Poincaré-invariant QFT), while D and K_μ generate dilatations and special conformal transformations, respectively. Special conformal transformations can be understood as the composition of an inversion, followed by a translation, followed by another inversion. The inversion acts as: [1]

$$x^\mu \to \frac{x^\mu}{x^2}\,. \tag{8.4}$$

Exercise 8.2. *Verify that inversion is a conformal transformation.*

The action of conformal generators on fields can be derived from their infinitesimal transformations:

$$\phi\left(x^\mu + \epsilon^\mu(x)\right) = \left[1 + \mathrm{i}\, a^\mu P_\mu - \lambda D + \frac{\mathrm{i}}{2} m^{\mu\nu} M_{\mu\nu} + \mathrm{i}\, b^\mu K_\mu\right] \phi\left(x^\mu\right)\,. \tag{8.5}$$

from which we can find a representation of the generators acting on operators. Acting on an operator with conformal dimension Δ and $\mathrm{SO}(d)$ representation R we have, [2]

$$\begin{aligned}
\left[P_\mu, \mathcal{O}(x)\right] &= -\mathrm{i}\partial_\mu \mathcal{O}(x)\,,\\
[D, \mathcal{O}(x)] &= -(\Delta + x^\mu \partial_\mu)\mathcal{O}(x)\,,\\
\left[M_{\mu\nu}, \mathcal{O}(x)\right] &= -\mathrm{i}\left(x_\mu \partial_\nu - x_\nu \partial_\mu\right)\mathcal{O}(x) + \mathrm{i}\, S^R_{\mu\nu} \cdot \mathcal{O}(x)\,,\\
\left[K_\mu, \mathcal{O}(x)\right] &= 2\mathrm{i}x_\mu \Delta \mathcal{O}(x) + 2\mathrm{i}x_\mu x^\nu \partial_\nu \mathcal{O}(x) - \mathrm{i}\, x^2 \partial_\mu \mathcal{O}(x) + 2\mathrm{i}x^\rho S^R_{\rho\mu} \cdot \mathcal{O}(x)\,,
\end{aligned} \tag{8.6}$$

where $S^R_{\mu\nu}$ is the spin generator in the representation R.

Exercise 8.3. *Show that the generators obey the following commutation relations*

$$\begin{aligned}
\left[D, P_\mu\right] &= P_\mu\,, & \left[M_{\mu\nu}, P_\alpha\right] &= \mathrm{i}\left(\delta_{\mu\alpha} P_\nu - \delta_{\nu\alpha} P_\mu\right)\,,\\
\left[D, K_\mu\right] &= -K_\mu\,, & \left[M_{\mu\nu}, K_\alpha\right] &= \mathrm{i}\left(\delta_{\mu\alpha} K_\nu - \delta_{\nu\alpha} K_\mu\right)\,,\\
\left[K_\mu, P_\nu\right] &= 2\delta_{\mu\nu} D - 2\mathrm{i}\, M_{\mu\nu}\,, & \left[M_{\alpha\beta}, M_{\mu\nu}\right] &= \mathrm{i}\left(\delta_{\alpha\mu} M_{\beta\nu} + \delta_{\beta\nu} M_{\alpha\mu} - \delta_{\beta\mu} M_{\alpha\nu} - \delta_{\alpha\nu} M_{\beta\mu}\right)\,.
\end{aligned} \tag{8.7}$$

Local operators

Local operators in CFTs are classified as primaries or descendants. Descendants are (linear combinations of) derivatives of other local operators, while primaries are not expressible as such. A primary operator $\mathcal{O}(0)$ inserted at the origin is annihilated by the generators of special conformal transformations, is an eigenvector of the dilatation operator, and transforms in an irreducible representation of the rotation group $\mathrm{SO}(d)$,

$$\left[K_\mu, \mathcal{O}(0)\right] = 0\,, \qquad [D, \mathcal{O}(0)] = \Delta\, \mathcal{O}(0)\,, \qquad \left[M_{\mu\nu}, \mathcal{O}_A(0)\right] = \left(S^R_{\mu\nu}\right)_A{}^B \mathcal{O}_B(0)\,. \tag{8.8}$$

[1] Note that inversions lie outside the component of the conformal group connected to the identity. Thus, it is possible to have CFTs that are not invariant under inversion. In fact, CFTs that break parity also break inversion.

[2] We define the dilatation generator D in a non-standard fashion so that it has real eigenvalues in unitary CFTs.

Under conformal transformations, correlation functions of scalar primary operators transform covariantly:

$$\langle \mathcal{O}_1(\tilde{x}_1)\dots\mathcal{O}_n(\tilde{x}_n)\rangle = \left|\frac{\partial \tilde{x}}{\partial x}\right|_{x_1}^{-\frac{\Delta_1}{d}} \dots \left|\frac{\partial \tilde{x}}{\partial x}\right|_{x_n}^{-\frac{\Delta_n}{d}} \langle \mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle \tag{8.9}$$

where Δ_i is the scaling dimension of $\mathcal{O}_i$. This transformation law, together with Poincaré invariance, dilatation invariance, and invariance under inversion highly constrains the correlation functions. In particular, all vacuum one-point functions $\langle \mathcal{O}(x)\rangle$ necessarily vanish unless $\mathcal{O} = 1$, the identity operator, for which $\Delta = 0$. Two- and three-point functions are constrained to take the form,

$$\begin{aligned} \langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle &= \frac{\delta_{\Delta_1,\Delta_2}}{|x_{12}|^{2\Delta_1}} \,, \\ \langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle &= \frac{C_{123}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}\,|x_{13}|^{\Delta_1+\Delta_3-\Delta_2}\,|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}} \,, \end{aligned} \tag{8.10}$$

where $|x_{ij}| = |x_i - x_j|$ and we have normalised all scalar primaries to have unit two-point function.[3]

Exercise 8.4. *Deduce the form of the two- and three-point functions above by imposing conformal invariance.*

Four-point functions are not completely fixed by symmetry, since two independent conformally invariant cross-ratios can be constructed,

$$u = z\bar{z} = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \,, \qquad v = (1-z)(1-\bar{z}) = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} \,. \tag{8.11}$$

The general form of the scalar four-point function is then

$$\Big\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\Big\rangle = \frac{f(u,v)}{\prod_{i<j}|x_{ij}^2|^{\delta_{ij}}} \,, \tag{8.12}$$

where $\sum_{j\neq i}\delta_{ij} = \Delta_i$ and f could in principle be any function of the cross-ratios.

State-operator map and the OPE

In conformal field theories, there exists a one-to-one correspondence between local operators and states, known as the state-operator map. This becomes manifest upon mapping flat Euclidean space $\mathbb{R}^d$ to the cylinder $\mathbb{R}\times S^{d-1}$ via a Weyl transformation. Under this map a local operator inserted at the origin of $\mathbb{R}^d$ prepares a state on the cylinder at $\tau = -\infty$. Conversely, a state on a time slice of the cylinder can be evolved backwards to define a boundary condition on a small sphere around the origin, thus defining a local operator. Time translations on the cylinder correspond to dilatations on $\mathbb{R}^d$, so the spectrum of the dilatation operator coincides with the energy spectrum of the theory on $\mathbb{R}\times S^{d-1}$.[4]

[3]This normalization is only consistent for primaries not subject to Ward identities. For example, the coefficients C_T and C_J appearing in the stress tensor and current two-point functions encode physical information.

[4]More precisely, there can be a constant shift equal to the Casimir energy of the vacuum on S^{d-1}, which is related with the Weyl anomaly. In $d = 2$, this gives the usual energy spectrum $\left(\Delta - \frac{c}{12}\right)\frac{1}{L}$ where c is the central charge and L is the radius of S^1.

The operator product expansion (OPE) expresses the product of two local operators as a sum over local operators:

$$\mathcal{O}_i(x)\mathcal{O}_j(0) = \sum_k \frac{C_{ijk}}{|x|^{\Delta_i+\Delta_j-\Delta_k}} \left[\mathcal{O}_k(0) + \beta \underbrace{x^\mu \partial_\mu \mathcal{O}_k(0) + \dots}_{\text{descendants}} \right] \tag{8.13}$$

where the coefficients of descendants (e.g. β) are fixed by conformal symmetry. We have shown only the contribution of a scalar primary; in general, the OPE includes operators of arbitrary spin.

Exercise 8.5. *Compute β by using this OPE inside a three-point function.*

The OPE converges inside correlation functions within a finite radius, a fact that follows from the state-operator map and radial quantization. By iteratively applying the OPE, any n-point function reduces to a sum over one-point functions, which vanish in the vacuum except for the identity operator. Thus, all correlation functions are determined by the CFT data, $\{\Delta_i, C_{ijk}\}$, where Δ_i are the scaling dimensions and C_{ijk} the OPE coefficients.[5]

The CFT data is highly constrained. In particular, associativity of the OPE requires that different OPE channels in correlation functions yield the same result, leading to the conformal bootstrap equations. There must exist a conserved stress tensor $T_{\mu\nu}$, a spin-2 primary with dimension $\Delta = d$, whose correlators satisfy conformal Ward identities. The theory is assumed to be unitary, which in Euclidean signature implies reflection positivity. This enforces bounds on operator dimensions and allows for a real basis in which all OPE coefficients are real.

Embedding Space Formalism

The conformal group $\mathrm{SO}(1, d+1)$ acts naturally on the space of light rays through the origin of $\mathbb{R}^{1,d+1}$,

$$-\left(P^0\right)^2 + \left(P^1\right)^2 + \dots + \left(P^{d+1}\right)^2 = 0\,. \tag{8.14}$$

A section of this light-cone is a d-dimensional manifold where the CFT lives. For example, it is easy to see that the Poincaré section $P^0 + P^{d+1} = 1$ is just $\mathbb{R}^d$. To see this we can parameterise this section using

$$P^0(x) = \frac{1+x^2}{2}\,, \qquad P^\mu(x) = x^\mu\,, \qquad P^{d+1}(x) = \frac{1-x^2}{2}\,, \tag{8.15}$$

with $\mu = 1, \dots, d$ and $x^\mu \in \mathbb{R}^d$ and compute the induced metric. In fact, any conformally flat manifold can be obtained as a section of the light-cone in the embedding space $\mathbb{R}^{d+1,1}$. Using the parameterisation $P^A = \Omega(x) P^A(x)$ with $x^\mu \in \mathbb{R}^d$, one can easily show that the induced metric is simply given by $\mathrm{d}s^2 = \Omega^2(x)\delta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$.

Exercise 8.6. *Consider the parameterisation $P^A = \left(P^0, P^\mu, P^{d+1}\right) = (\cosh\tau, \Omega^\mu, -\sinh\tau)$ of the global section $\left(P^0\right)^2 - \left(P^{d+1}\right)^2 = 1$, where Ω^μ ($\mu = 1, \dots, d$) parameterises a unit $(d-1)$—dimensional sphere,*

[5] For primary operators $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$ transforming in non-trivial irreps of $\mathrm{SO}(d)$ there are several OPE coefficients C_{123}. The number of OPE coefficients C_{123} is given by the number of symmetric traceless tensor representations that appear in the tensor product of the 3 irreps of $\mathrm{SO}(d)$ associated to $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$.

$\Omega \cdot \Omega = 1$. *Show that this section has the geometry of a cylinder exactly like the one used for the state-operator map.*

With this in mind, it is natural to extend a primary operator from the physical section to the full light-cone with the following homogeneity property,

$$\mathcal{O}(\lambda P) = \lambda^{-\Delta} \mathcal{O}(P) \,, \qquad \lambda \in \mathbb{R} \,. \tag{8.16}$$

In other words, this implements the Weyl transformation. One can then compute correlation functions directly in the embedding space, where the constraints of conformal symmetry are just homogeneity and SO$(1, d+1)$ Lorentz invariance. Physical correlators are simply obtained by restricting to the section of the light-cone associated with the physical space of interest. This idea goes back to Dirac [Dir36] and has been further develop by many authors [MS69, BBP70, FGG73a, FGG73b, CCP10, Wei10, CPPR11].

Exercise 8.7. *Using the embedding space formalism, show that the form of two- and three-point functions of scalar primary operators in $\mathbb{R}^d$ take the form*

$$\langle \mathcal{O}_1(P_1)\mathcal{O}_2(P_2)\rangle = \frac{\delta(\Delta_1 - \Delta_2)}{P_{12}^{\Delta_1}} \,, \tag{8.17}$$

$$\langle \mathcal{O}_1(P_1)\mathcal{O}_2(P_2)\mathcal{O}_3(P_3)\rangle = \frac{C_{123}}{P_{12}^{\frac{\Delta_1+\Delta_2-\Delta_3}{2}} P_{23}^{\frac{\Delta_2+\Delta_3-\Delta_1}{2}} P_{31}^{\frac{\Delta_3+\Delta_1-\Delta_2}{2}}} \,, \tag{8.18}$$

where we defined $P_{ij} = -2P_i \cdot P_j$.

It is easy to see that these expressions are the only ones consistent with SO$(1, d+1)$ invariance and the degree of homogeneity of each $\mathcal{O}_i(P_i)$. Using the identity,

$$-2P_{x_i} \cdot P_{x_j} = x_{ij}^2 \,, \qquad x_{ij} = |x_i - x_j| \,, \tag{8.19}$$

which holds in each physical section, we can straightforwardly reduce the three-point function to the familiar,

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \frac{C_{123}}{|x_{12}^2|^{\frac{\Delta_1+\Delta_2-\Delta_3}{2}} |x_{23}^2|^{\frac{\Delta_2+\Delta_3-\Delta_1}{2}} |x_{31}^2|^{\frac{\Delta_3+\Delta_1-\Delta_2}{2}}} \,, \tag{8.20}$$

Exercise 8.8. *Similarly, vector primary operators can also be extended to the embedding space. In this case, we impose*

$$P^A \mathcal{O}_A(P) = 0 \,, \qquad \mathcal{O}_A(\lambda P) = \lambda^{-\Delta} \mathcal{O}_A(P) \,, \qquad \lambda \in \mathbb{R} \,, \tag{8.21}$$

where the physical operator is obtained by projecting the indices to the section,

$$\mathcal{O}_\mu(x) = \frac{\partial P^A}{\partial x^\mu} \mathcal{O}_A(P) \bigg|_{P^A = P^A(x)} \,. \tag{8.22}$$

Notice that this implies a redundancy, $\mathcal{O}_A(P) \to \mathcal{O}_A(P) + P_A \Lambda(P)$ gives rise to the same physical operator $\mathcal{O}(x)$, for any scalar function $\Lambda(P)$ such that $\Lambda(\lambda P) = \lambda^{-\Delta-1}\Lambda(P)$. This redundancy, together with the constraint $P^A \mathcal{O}_A(P) = 0$, removes 2 degrees of freedom of the $(d+2)$-dimensional vector $\mathcal{O}_A(P)$.

Show that the two-point function of vector primary operators is given by

$$\left\langle \mathcal{O}^A(P_1)\mathcal{O}^B(P_2)\right\rangle = const\,\frac{\eta^{AB}\,(P_1\cdot P_2) - P_2^A P_1^B}{(-2P_1\cdot P_2)^{\Delta+1}}\ , \tag{8.23}$$

up to redundant terms.

Conformal correlation functions extended to the light-cone of $\mathbb{R}^{1,d+1}$ are annihilated by the generators of $SO(1,d+1)$

$$\sum_{i=1}^{n} J_{AB}^{(i)}\,\langle \mathcal{O}_1(P_1)\ldots\mathcal{O}_n(P_n)\rangle = 0\ , \tag{8.24}$$

where $J_{AB}^{(i)}$ is the generator

$$J_{AB} = -i\left(P_A\frac{\partial}{\partial P^B} - P_B\frac{\partial}{\partial P^A}\right)\ , \tag{8.25}$$

acting on the point P_i. For a given choice of light cone section, some generators will preserve the section and some will not. The first are Killing vectors (isometry generators) and the second are conformal Killing vectors. The commutation relations give the usual Lorentz algebra

$$[J_{AB},J_{CD}] = i\,(\eta_{AC}J_{BD} + \eta_{BD}J_{AC} - \eta_{BC}J_{AD} - \eta_{AD}J_{BC})\ . \tag{8.26}$$

Exercise 8.9. *Check that the conformal algebra (8.7) follows from (8.26) and*

$$\begin{aligned} D &= -i\,J_{0,d+1}\ , & P_\mu &= J_{\mu 0} - J_{\mu,d+1}\ ,\\ M_{\mu\nu} &= J_{\mu\nu}\ , & K_\mu &= J_{\mu 0} + J_{\mu,d+1}. \end{aligned} \tag{8.27}$$

Exercise 8.10. *Show that equation (8.24) for $J_{AB} = J_{0,d+1}$ implies time translation invariance on the cylinder*

$$\sum_{i=1}^{n}\frac{\partial}{\partial\tau_i}\,\langle \mathcal{O}_1(\tau_1,\Omega_1)\ldots\mathcal{O}_n(\tau_n,\Omega_n)\rangle = 0\ , \tag{8.28}$$

and dilatation invariance on $\mathbb{R}^d$

$$\sum_{i=1}^{n}\left(\Delta_i + x_i^\mu\frac{\partial}{\partial x_i^\mu}\right)\langle \mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n)\rangle = 0\ . \tag{8.29}$$

In this case, you will need to use the differential form of the homogeneity property $P^A\frac{\partial}{\partial P^A}\mathcal{O}_i(P) = -\Delta_i\mathcal{O}_i(P)$. It is instructive to do this exercise for the other generators as well.

8.2 Geometry of AdS

Anti-de Sitter (AdS) space is the maximally symmetric spacetime with constant negative curvature. A convenient way to construct it is by considering the isometric embedding of a $d+1$-dimensional hyperboloid into a flat ambient space $\mathbb{R}^{2,d}$. More precisely, AdS space is defined as the universal cover of the manifold

$$-\left(X^0\right)^2 + \left(X^1\right)^2 + \cdots + \left(X^d\right)^2 - \left(X^{d+1}\right)^2 = -L^2\ , \tag{8.30}$$

embedded in $\mathbb{R}^{d,2}$. We refer the reader to Appendix E for an overview of the various coordinate systems. For concreteness, we work here in global coordinates, for which the embedding is parametrised as

$$\begin{aligned} X^0 &= L\cos t\cosh\rho \\ X^\mu &= L\,\omega^\mu\sinh\rho \\ X^{d+1} &= L\sin t\cosh\rho \end{aligned} \tag{8.31}$$

where ω^μ ($\mu=1,\dots,d$) parametrises a unit $(d-1)$—dimensional sphere. This parametrisation yields the global AdS metric,

$$\mathrm{d}s^2 = L_{\mathrm{AdS}}^2\left(-\cosh^2\rho\,\mathrm{d}t^2+\mathrm{d}\rho^2+\sinh^2\rho\,\mathrm{d}\Omega_{d-1}^2\right). \tag{8.32}$$

By taking $\rho\geq 0$ and $0\leq t<2\pi$, we cover the entire hyperboloid once. However, since the time coordinate t is periodic, the spacetime contains closed timelike curves. To restore causality, we pass to the universal cover by unwrapping the time direction: $t\in\mathbb{R}$.

This coordinate system reveals several notable features of AdS geometry. For instance, any light ray can reach spatial infinity in finite coordinate time. To see this, consider a null trajectory at fixed angular position. The radial null condition gives:

$$\Delta t = L_{\mathrm{AdS}}\int \mathrm{d}t = L_{\mathrm{AdS}}\int_0^\infty \frac{\mathrm{d}\rho}{\cosh\rho} = \frac{\pi L_{\mathrm{AdS}}}{2}. \tag{8.33}$$

This implies that observers anywhere in AdS can communicate with each other within a finite (proper) time.

Causality and the conformal boundary

As $\rho\to\infty$, the metric (8.32) diverges. The surface at $\rho=\infty$ does not belong to the AdS manifold proper. However, as discussed in Chapter 2, one can perform a conformal compactification to formally include the hypersurface $\mathscr{I}_{\mathrm{AdS}}$, also known as the conformal boundary. More precisely, the conformal boundary is defined as the conformal equivalence class of metrics $\mathrm{d}\tilde{s}^2=\mathrm{e}^{-2\rho}\mathrm{d}s^2$ with boundary $\mathbb{R}^{1,d-1}$ at $\rho=\infty$.

Particularly interesting for our purposes is the relation between the conformal compactification of AdS and flat space. It is well-known that Euclidean flat space can be compactified to the d-sphere, S^d by adding a point at infinity. On the other hand, Euclidean AdS$_{d+1}$, which is simply the hyperbolic space, can be conformally mapped to the $(d+1)$-dimensional disc. Therefore the boundary of the compactified Euclidean AdS space is the compactified Euclidean plane.

Similarly, in Lorentzian signature, by changing to hyperspherical coordinates, i.e. performing the coordinate change $\tanh\rho=\sin r$ from global coordinates, we obtain

$$\mathrm{d}s^2=\frac{L_{\mathrm{AdS}}^2}{\cos^2\theta}\left(-\mathrm{d}t^2+\mathrm{d}r^2+\sin^2 r\,\mathrm{d}\Omega_{d-1}^2\right), \tag{8.34}$$

which after a conformal rescaling becomes the metric of the Einstein static universe. However, it is only half of the Einstein static universe since θ is restricted to the range $[0, \pi/2)$ rather then $[0, \pi)$. The boundary of this space is at $r = \pi/2$ and is given by $\mathbb{R} \times S^{d-1}$. This is identical to the conformal compactification of d-dimensional Minkowski space. This identification will play an essential role in the AdS/CFT correspondence.

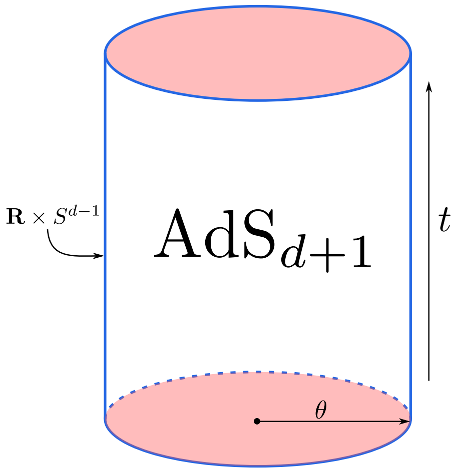

Figure 8.1: AdS_{d+1} can be conformally mapped into one half the Einstein static universe. This space has boundary $\mathbf{R} \times S^{d-1}$ which is exactly the conformal compactification of Minkowski space.

For future reference it will also be useful to introduce the conformal compactification from the point of view of the embedding coordinates. The (conformal) boundary of AdS is not part of the hyperboloid itself but can be seen by rescaling the metric. To do so, let us introduce null rays in embedding space,

$$P^A \sim \lambda P^A\,, \qquad P^2 = 0\,, \qquad P^A \in \mathbb{R}^{2,d}\,. \tag{8.35}$$

Such null rays define coordinates on the projective null cone $\mathcal{N}$ embedded in $\mathbb{R}^{2,d}$. Consider now a point X^A on the hyperboloid. As we approach infinity, this point comes arbitrarily close to the null cone, but never quite reaches it due to the defining constraint (8.30). The null lines do not lie on the hyperboloid, but they represent its conformal boundary. In other words,

$$\partial\,\mathrm{AdS} \simeq \left\{P^A \in \mathbb{R}^{2,d} | P^2 = 0\right\} / \sim\,, \qquad \text{where} \quad P^A \sim \lambda P^A\,. \tag{8.36}$$

The quotient identifies points along the light ray and is the space on which the conformal structure lives. This space inherits the group of linear transformations preserving $P^A = 0$, i.e. $\mathrm{SO}(2, d)$, which is the conformal group acting on the boundary spacetime.

Before moving on to considering quantum fields in AdS there are a few causal issues that need our attention. A peculiarity of AdS spaces is that an initially radially outward trajectory from $\rho = 0$ will begin to re-converge to its starting point following a period of $\frac{\pi}{2}$. Remembering the discussion in Chapter 2, a Lorentzian spacetime is globally hyperbolic if and only if it contains a Cauchy hypersurface, i.e. a hypersurface whose domain of dependence covers all of the spacetime. Famously, AdS is not globally hyperbolic. This becomes clear by looking at Figure 8.2, where the domain of dependence for the spatial hypersurface Σ is indicated by the blue diamonds.

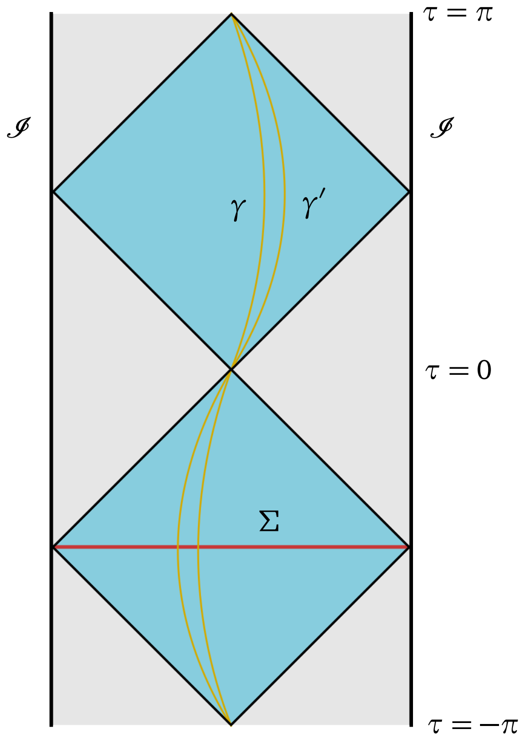

Figure 8.2: The Penrose diagram of AdS. The blue domain blue diamonds denote the domain of dependence of the spatial hypersurface Σ. Two geodesics, γ and γ' are denoted in yellow. In the universal covering the diagram continues indefinitely in the vertical direction.

Therefore, there are regions for which a knowledge of events on Σ does not result in any predictive power. The underlying cause of these issues is the fact that conformal infinity in AdS is a time-like hypersurface. Indeed, as a massless particle can reach spatial infinity in finite time, it can then propagate along $\mathscr{I}$ and move outside of $D(\Sigma)$. In this way information is 'lost' from Σ to spatial infinity. Similarly AdS space allows for information to be introduced from spatial infinity.

To resolve these issues, and restore the predictability in the entire space, we have to provide boundary conditions at spatial infinity. In these lectures we will insist on having a closed system, hence reflective boundary conditions are the natural choice [AIS78]. In the next section, when discussing the solutions to the wave equation we will discuss these boundary conditions at length. Note that such boundary conditions amount to requiring that there is no net flux across spatial infinity.

Asymptotically locally AdS spaces

Through the AdS/CFT correspondence, empty AdS corresponds to the vacuum state of the dual CFT. If we want to access more general states we can consider non-trivial fillings with the same conformal boundary. In other words we will no longer have an exact AdS space but only an asymptotically locally AdS (AlAdS) space. However, a lot of the machinery we will develop for pure AdS spaces will still be valid. By adding matter to the theory the bulk of AdS will change but, as we discussed in Chapter 2, the conformal boundary is a rather robust characteristic of spaces with a negative cosmological constant. It takes infinite energy to change the asymptotics of such spaces.

In particular we can expand the metric of such spaces near the conformal boundary as

$$\mathrm{d}s^2 = \frac{L_{\mathrm{AdS}}^2}{z^2}\left(-\mathrm{d}z^2 + g_{mn}(z,x)\mathrm{d}x^m\mathrm{d}x^n\right), \tag{8.37}$$

where we use Poincaré coordinates and $g_{mn}(z,x)$ is smooth and finite as $z \to 0$ and can be expanded in powers of the radial coordinate as

$$g_{ab}(z,x) = \sum_{n=0}^{\infty} z^n g_{ab}^{(n)}(x). \tag{8.38}$$

where $g^{(0)}$ reproduces the pure AdS metric, while the $g^{(n)}$ with $n > 0$ parametrise the deformations away from it. Similarly, the matter fields coupled to gravity can be expanded near the conformal boundary. This expansion is called the Fefferman-Graham expansion. Since the Einstein equations are second order PDEs, plugging in the Fefferman-Graham expansion leads to a second order recursion relation for the $g^{(n)}$. After specifying $g^{(0)}$ one finds that all $g^{(n)}$ with $n < d$ are uniquely determined. However, new data enters in $g^{(d)}$, which can be derived from the Hamiltonian and momentum constraints along surfaces with z constant. Once $g^{(d)}$ is determined all higher terms are again determined in terms of $g^{(0)}$ and $g^{(d)}$. For even d the situation is slightly more complicated, since at order d we must allow for logarithmic terms in the expansion.

8.3 Quantum fields in AdS

We have seen that AdS_{d+1} acts like a box for classical massive particles. Quantum mechanically, this confining potential gives rise to a discrete energy spectrum. Consider the Klein-Gordon equation

$$\left(\Box - m_\xi^2\right)\phi = 0, \tag{8.39}$$

in global coordinates. Since the Ricci scalar is a constant in AdS we define the effective mass m_ξ as

$$m_\xi^2 = m^2 + \xi\frac{d(d+1)}{L^2}. \tag{8.40}$$

To avoid excessive notation we will mostly suppress the subscript ξ. The phase space in this case is again defined as the solution space of this equation, but unlike for globally hyperbolic spacetimes we now have impose additional boundary conditions at conformal infinity. There are various options for these boundary conditions, but since we are interested in closed quantum systems, a natural set of choices is given by reflective boundary conditions. These can be divided in three cases, Dirichlet, Neumann or mixed/Robin boundary conditions

$$\begin{aligned} \phi\big|_{z=0} &= 0 \qquad \text{(Dirichlet)} \\ \nabla_{\mathbf{n}}\phi\big|_{z=0} &= 0 \qquad \text{(Neumann)} \\ \left(K(z,x)\phi + \nabla_{\mathbf{n}}\phi\right)\big|_{z=0} &= 0 \qquad \text{(Robin)} \end{aligned} \tag{8.41}$$

Here $\mathbf{n}$ denotes the unit normal to the conformal boundary at $z = 0$ and K is a function. The usual plane wave incoming and outgoing modes do not satisfy such boundary conditions as they parameterise solutions with ingoing or outgoing energy flux at conformal infinity. Instead, the basic solutions will be of the form of 'standing wave' solutions, $\phi \simeq \exp(-\mathrm{i}\omega u) \pm \exp(-\mathrm{i}\omega v)$.

To find the mode functions we will use an indirect route which has the advantage that it makes the correspondence with holography more explicit. Consider the action of the quadratic Casimir of the AdS isometry group on a scalar field[6]

$$\frac{1}{2} J_{AB} J^{BA} \phi = \left[-X^2 \partial_X^2 + X \cdot \partial_X \left(d + X \cdot \partial_X \right) \right] \phi \,. \tag{8.42}$$

By foliating the embedding space $\mathbb{R}^{2,d}$ with AdS surfaces of different AdS radii L, we can obtain the Laplacian in the embedding space as

$$\partial_X^2 = -\frac{1}{L^{d+1}} \frac{\partial}{\partial L} L^{d+1} \frac{\partial}{\partial L} + \Box_{\text{AdS}} \,. \tag{8.43}$$

Substituting this in (8.42) and noticing that $X \cdot \partial_X = L \partial_L$ we conclude that

$$\frac{1}{2} J_{AB} J^{BA} \phi = L^2 \Box_{\text{AdS}} \phi \,. \tag{8.44}$$

Therefore, we should identify $m_\xi^2 L^2$ with the quadratic Casimir of the conformal group. We can use this fact to construct the phase space and Hilbert space mimicking our CFT discussion.

For this is it useful to introduce the Lorentzian version of the conformal generators,

$$D = -J_{0,d+1} \,, \qquad P_\mu = J_{\mu 0} + \mathrm{i} J_{\mu,d+1} \,, \tag{8.45}$$

$$M_{\mu\nu} = J_{\mu\nu} \,, \qquad K_\mu = J_{\mu 0} - \mathrm{i} J_{\mu,d+1} \,. \tag{8.46}$$

Exercise 8.11. *Show that, in global coordinates, the conformal generators take the form*

$$\begin{aligned} D &= \mathrm{i} \frac{\partial}{\partial t} \,, \\ M_{\mu\nu} &= -\mathrm{i} \left(\omega_\mu \frac{\partial}{\partial \omega^\nu} - \omega_\nu \frac{\partial}{\partial \omega^\mu} \right) \,, \\ P_\mu &= -\mathrm{i}\, e^{-\mathrm{i}t} \left[\omega_\mu \left(\partial_\rho - \mathrm{i} \tanh\rho \, \partial_t \right) + \frac{1}{\tanh\rho} \nabla_\mu \right] \,, \\ K_\mu &= -\mathrm{i}\, e^{\mathrm{i}t} \left[\omega_\mu \left(\partial_\rho + \mathrm{i} \tanh\rho \, \partial_t \right) + \frac{1}{\tanh\rho} \nabla_\mu \right] \,, \end{aligned}$$

where $\nabla_\mu = \frac{\partial}{\partial \omega^\mu} - \omega_\mu \omega^\nu \frac{\partial}{\partial \omega^\nu}$ is the covariant derivative on the unit sphere S^{d-1}.

In analogy with the construction of the Hilbert space in a CFT, we can look for primary states, which are annihilated by K_μ and are eigenstates of the Hamiltonian, $D\phi = \Delta\phi$. The condition $K_\mu \phi = 0$ splits in one term proportional to ω_μ and one term orthogonal to ω_μ. The second term implies that ϕ is independent of the angular variables ω^μ while the first term reduces to $\left(\partial_\rho + \Delta \tanh\rho \right) \phi = 0$.

[6] Formally, we are extending the function ϕ from AdS, defined by the hypersurface $X^2 = -L^2$, to the embedding space. However, the action of the quadratic Casimir is independent of this extension because the generators J_{AB} are interior to AdS, i.e. $\left[J_{AB}, X^2 + L^2 \right] = 0$.

This implies that

$$\phi \propto \left(\frac{e^{-it}}{\cosh\rho}\right)^{\Delta} = \left(\frac{L}{X^0 - X^{d+1}}\right)^{\Delta} . \tag{8.47}$$

This is the lowest "energy" state with eigenvalue Δ for the dilatation operator. Starting from this state one can construct excited states by acting on it with the generator P_μ. Notice that all such states have the same value for the quadratic Casimir

$$\frac{1}{2}J_{AB}J^{BA}\phi = \Delta(\Delta - d)\phi . \tag{8.48}$$

Hence in this way we can generate all normalisable solutions of the Klein-Gordon equation with $m^2L^2 = \Delta(\Delta - d)$. This shows that the one-particle "energy" spectrum is given by $\omega = \Delta + l + 2n$ where $l = 0, 1, 2, \dots$ is the spin, generated by acting with the traceless generators, $P_{\mu_1}\dots P_{\mu_l}$ − traces, and similarly, the quantum number $n = 0, 1, 2, \dots$ is generated by acting with $\left(P^2\right)^n$.

Exercise 8.12. *Given the symmetry of the metric (see (E.46)) we can look for solutions of the form*

$$\phi = \mathrm{e}^{\mathrm{i}\omega t} Y_l(\Omega) F(r) , \tag{8.49}$$

where $Y_l(\Omega)$ is a spherical harmonic with eigenvalue $-l(l+d-2)$ of the Laplacian on the unit sphere S^{d-1}. In addition, we changed coordinates with respect to the above such that $\tanh\rho = \sin r$ such that $r \in [0, \pi/2)$.

Derive a differential equation for $F(r)$ and show that it is solved by

$$F(r) = (\cos r)^{\Delta} (\sin r)^l \ {}_2F_1\left(\frac{l+\Delta-\omega}{2}, \frac{l+\Delta+\omega}{2}, l + \frac{d}{2}, \sin r\right) , \tag{8.50}$$

with $2\Delta = d + \sqrt{d^2 + 4(mL)^2}$. We chose this solution because it is smooth at $r = 0$. Now we also need to impose another boundary condition at the boundary of AdS, i.e. $r = \frac{\pi}{2}$. Imposing that there is no energy flux through the boundary leads to the quantization of the energies $|\omega| = \Delta + l + 2n$ with $n = 0, 1, 2, \dots$ (see for example [AGM+00]).

If there are no interactions between the particles in AdS, then the Hilbert space has a Fock representation and the energy of a multi-particle state is just the sum of the energies of particles. Turning on small interactions leads to small energy shifts of the multi-particle states.

Note that, for a given (effective) mass we can solve the equation $m^2L^2 = \Delta(\Delta - d)$ as

$$\Delta = \Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2L^2} . \tag{8.51}$$

Demanding that $\Delta_\pm \in \mathbb{R}$ requires having $m^2L^2 \geq -\frac{d^2}{4}$. Hence, we see that a range of tachyonic masses is allowed. In Minkowski space this would lead to an instability of the perturbative vacuum. In AdS space, whenever the mass-squared lies above this bound the free energy of the field is bounded from below and no instabilities arise. This bound is called the Breitenlohner-Freedman bound after [BF82].

When building the phase space above we restricted to $\Delta = \Delta_+$. This is because generically this is the only normalisable solution, i.e. with finite norm

$$||\phi||^2 = \int \mathrm{d}^{d+1}x\sqrt{|g|}|\phi|^2 < \infty\,. \tag{8.52}$$

Note however that in the window $-\frac{d^2}{4} \leq m^2L^2 \leq -\frac{d^2}{4}+1$, both choices are allowed. This in particular implies that above this window, only Dirichlet conditions can be considered, while in the window Dirichlet, Neumann as well as mixed (Robin) boundary conditions are allowed. This phenomenon goes under the name "alternate quantisation". In these notes we will usually avoid the subtleties that come with this and restrict ourselves to $m^2L^2 \geq -\frac{d^2}{4}+1$.

Green's functions

To find the Green's function, let us return to Euclidean signature and afterwards analytically continue the result to obtain the Lorentzian Feynman Green's function. For simplicity, we consider a free scalar field with action

$$S_E = \frac{1}{2}\int_{AdS} \mathrm{d}^{d+1}x\left(|\mathrm{d}\phi|^2 + m^2\phi^2\right)\,. \tag{8.53}$$

Similar as for the de Sitter space, we can obtain the two-point function $\langle\phi(X)\phi(Y)\rangle$ by exploiting the conformal symmetry of AdS. The Euclidean Green's function denoted by the propagator $\Pi(X,Y)$, has to obey,

$$\left(\Box_X - m^2\right)\Pi(X,Y) = -\delta(X,Y)\,. \tag{8.54}$$

From the symmetry of the problem it is clear that the propagator can only depend on the invariant $Z = -X\cdot Y$. From now on we will set $L = 1$ and all lengths will be expressed in units of the AdS radius.

Exercise 8.13. *Use (8.42) and (8.44) to show that*

$$\Pi(X,Y) = c_\Delta\,{}_2F_1\left(\Delta_+,\Delta_-,\frac{d+1}{2};\frac{1+Z}{2}\right), \tag{8.55}$$

where $\Delta_\pm = \frac{d\pm\sqrt{d^2+4m^2}}{2}$ and c_Δ a normalisation constant which can be fixed by demanding that the high energy behaviour is the same as in flat space.

For a free field, higher point functions are simply given by Wick contractions. For example,

$$\begin{aligned}\left\langle\phi(X_1)\phi(X_2)\phi(X_3)\phi(X_4)\right\rangle =&\Pi(X_1,X_2)\Pi(X_3,X_4) + \Pi(X_1,X_3)\Pi(X_2,X_4)\\ &+\Pi(X_1,X_4)\Pi(X_2,X_3)\,.\end{aligned} \tag{8.56}$$

Weak interactions of ϕ can be treated perturbatively. Suppose the action includes a cubic term,

$$S = \int_{\mathrm{AdS}} dX\left[\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{1}{3!}g\phi^3\right]\,. \tag{8.57}$$

Then, there is a non-vanishing three-point function

$$\langle\phi(X_1)\phi(X_2)\phi(X_3)\rangle = -g\int_{\text{AdS}} dY\,\Pi(X_1,Y)\Pi(X_2,Y)\Pi(X_3,Y) + O(g^3)\,,$$

and a connected part of the four-point function of order g^2. This is completely analogous to perturbative QFT in flat space.

8.4 Towards a conformal theory on the boundary

Given a correlation function in AdS we can consider the limit where we send all points to infinity. More precisely, we introduce the extrapolate dictionary to identify an operator on the conformal boundary with a field in the bulk.

$$\mathcal{O}(P) = \frac{1}{\sqrt{C_\Delta}}\lim_{\lambda\to\infty}\lambda^\Delta\,\phi\,(X=\lambda P+\dots)\,, \tag{8.58}$$

where P is a future directed null vector in $\mathbb{R}^{1,d+1}$ and the $\dots$ denote terms that do not grow with λ whose only purpose is to enforce the AdS condition $X^2=-1$. In other words, the operator $\mathcal{O}(P)$ is the limit of the field $\phi(X)$ when X approaches the boundary point P of AdS. Notice that, by definition, the operator $\mathcal{O}(P)$ obeys the homogeneity condition (8.16). Correlation functions of $\mathcal{O}$ are naturally defined by the limit of correlation functions of ϕ in AdS.

Let us consider this in detail for the two point function. As a first step we have to understand the asymptotic behaviour of the two-point function $\Pi(X,Y)$ as X and Y approach the boundary,

$$X=\lambda_1 P_1+\delta X\,,\qquad Y=\lambda_2 P_2+\delta Y\,,\qquad \lambda_i\to\infty\,. \tag{8.59}$$

In this limit $Z=-X\cdot Y\sim\lambda_1\lambda_2 P_1\cdot P_2\to\infty$. The asymptotic behaviour of the hypergeometric function for large Z is given by

$${}_2F_1\left(\Delta,d-\Delta,\frac{d+1}{2};\frac{1+Z}{2}\right)\sim AZ^{-\Delta}+BZ^{\Delta-d}+\cdots\,. \tag{8.60}$$

These two terms come precisely from the normalisable and non-normalisable mode. In standard quantisation, the leading behaviour comes from the A term, while the B term is subleading. Plugging this in the Green's function of the scalar we find

$$\Pi(X,Y)\sim\frac{c_\Delta A_\Delta}{(-2X\cdot Y)^\Delta}=\frac{c_\Delta A_\Delta}{(-2\lambda_1\lambda_2 P_1\cdot P_2)^\Delta}\,, \tag{8.61}$$

as λ_1, $\lambda_2\to\infty$. Finally, we can find the boundary correlator as

$$\begin{aligned}\langle\mathcal{O}(P_1)\mathcal{O}(P_2)\rangle =&\frac{1}{c_\Delta}\lim_{\lambda_1,\lambda_2\to\infty}\lambda_1^\Delta\lambda_2^\Delta\Pi(\lambda_1P_1+\dots,\lambda_2P_2+\dots)\\ =&\frac{1}{(-2P_1\cdot P_2)^\Delta}+O(g^2)\,.\end{aligned} \tag{8.62}$$

where the $O(g^2)$ terms come in at one-loop level in perturbation theory. This is exactly the CFT two-point function of a primary operator of dimension Δ.

Similarly, we can compute the three-point function of three identical scalars. At tree-level in perturbation theory this is given by

$$\langle \mathcal{O}(P_1)\mathcal{O}(P_2)\mathcal{O}(P_3)\rangle = -g\, C_\Delta^{-\frac{3}{2}} \int_{\text{AdS}} dX\, \Pi(X,P_1)\Pi(X,P_2)\Pi(X,P_3) + O(g^3)\,, \tag{8.63}$$

where we defined the bulk to boundary propagator

$$\Pi(X,P) = \lim_{\lambda\to\infty} \lambda^\Delta\, \Pi(X, Y = \lambda P + \dots) = \frac{C_\Delta}{(-2P\cdot X)^\Delta} \tag{8.64}$$

is the bulk to boundary propagator whose form can be derived through similar arguments as above.

Exercise 8.14. *Write the bulk to boundary propagator in Poincaré coordinates.*

Exercise 8.15. *Compute the following generalization of the integral in* (8.63),

$$\int_{AdS} dX \prod_{i=1}^{3} \frac{1}{(-2P_i\cdot X)^{\Delta_i}}\,, \tag{8.65}$$

and show that it reproduces the expected formula for the CFT three-point function $\langle \mathcal{O}_1(P_1)\mathcal{O}_2(P_2)\mathcal{O}_3(P_3)\rangle$*. It is helpful to use the integral representation*

$$\frac{1}{(-2P\cdot X)^\Delta} = \frac{1}{\Gamma(\Delta)} \int_0^\infty \frac{ds}{s} s^\Delta e^{2s\,P\cdot X} \tag{8.66}$$

to bring the AdS integral to the form

$$\int_{\text{AdS}} dX e^{2X\cdot Q} \tag{8.67}$$

with Q *a future directed timelike vector. Choosing the* X^0 *direction along* Q *and using the Poincaré coordinates* (E.42) *it is easy to show that*

$$\int_{\text{AdS}} dX e^{2X\cdot Q} = \pi^{\frac{d}{2}} \int_0^\infty \frac{dz}{z} z^{-\frac{d}{2}} e^{-z+Q^2/z}\,. \tag{8.68}$$

To factorize the remaining integrals over s_1, s_2, s_3 *and* z *it is helpful to change to the variables* t_1, t_2, t_3 *and* z *using*

$$s_i = \frac{\sqrt{z\, t_1 t_2 t_3}}{t_i}\,. \tag{8.69}$$

State-operator map

So far, we have established that the correlation functions of the boundary operator defined in (8.58) exhibit the expected homogeneity and manifest invariance under the conformal group $SO(1, d+1)$. These are precisely the properties characteristic of correlation functions of identical primary scalar operators with scaling dimension Δ in a conformal field theory. We now turn to the question of whether these operators also satisfy an associative operator product expansion (OPE). The argument closely parallels that of standard CFTs: we interpret the correlation functions as vacuum expectation values of time-ordered products, and proceed accordingly.

$$\langle \phi(X_1)\phi(X_2)\phi(X_3)\dots\rangle = \langle 0|\dots\hat{\phi}(\tau_3,\rho_3,\Omega_3)\hat{\phi}(\tau_2,\rho_2,\Omega_2)\hat{\phi}(\tau_1,\rho_1,\Omega_1)\,|0\rangle\ , \tag{8.70}$$

where we assumed the time ordering $\tau_1 < \tau_2 < 0 < \tau_3 < \cdots$. We now insert a complete basis of states at global time $\tau = 0$,

$$\begin{aligned}&\langle \phi(X_1)\phi(X_2)\phi(X_3)\dots\rangle \\ &= \sum_\psi \langle 0|\dots\hat{\phi}(\tau_3,\rho_3,\Omega_3)\,|\psi\rangle\,\langle\psi|\,\hat{\phi}(\tau_2,\rho_2,\Omega_2)\hat{\phi}(\tau_1,\rho_1,\Omega_1)\,|0\rangle\ .\end{aligned} \tag{8.71}$$

Using the relation $\hat{\phi}(\tau,\rho,\Omega) = \mathrm{e}^{\tau D}\hat{\phi}(0,\rho,\Omega)\mathrm{e}^{-\tau D}$, and working in a basis where the Hamiltonian $D = -\frac{\partial}{\partial\tau}$ is diagonal, it follows that the sum over states converges for the chosen time ordering.

The next step is to establish a one-to-one correspondence between states $|\psi\rangle$ and boundary operators. Every boundary operator defined via (8.58) determines a bulk state. For example, inserting the boundary operator at the point $P^A = \left(P^0, P^\mu, P^{d+1}\right) = \left(\frac{1}{2}, 0, \frac{1}{2}\right)$, which corresponds to the boundary point $\tau \to -\infty$ in global coordinates, we can write:

$$\langle \dots\phi(X_3)\mathcal{O}(P)\rangle = \langle 0|\dots\hat{\phi}(\tau_3,\rho_3,\Omega_3)\,|\mathcal{O}\rangle\ \ , \tag{8.72}$$

where

$$\begin{aligned}|\mathcal{O}\rangle &= \lim_{\tau\to-\infty}\left(e^{-\tau}\cosh\rho\right)^\Delta\,\hat{\phi}(\tau,\rho,\Omega)\,|0\rangle \\ &= \sum_\psi |\psi\rangle\,(\cosh\rho)^\Delta\lim_{\tau\to-\infty}\langle\psi|\,e^{\tau(D-\Delta)}\hat{\phi}(0,\rho,\Omega)\,|0\rangle\ .\end{aligned} \tag{8.73}$$

The limit $\tau\to-\infty$ projects onto the lowest-weight (i.e., primary) state, whose wavefunction matches that in (8.47).

To complete the state-operator correspondence, we use global time translation invariance. For any state $|\psi\rangle$, we define the corresponding boundary operator $\mathcal{O}_\psi(P)$ via:

$$\begin{aligned}&\langle 0|\dots\hat{\phi}(\tau_3,\rho_3,\Omega_3)\,|\psi(0)\rangle \\ &= \lim_{\tau\to-\infty}\langle 0|\dots\hat{\phi}(\tau_3,\rho_3,\Omega_3)e^{\tau D}\,|\psi(\tau)\rangle \equiv \left\langle\dots\phi(X_3)\mathcal{O}_\psi(P)\right\rangle\end{aligned} \tag{8.74}$$

where $|\psi(\tau)\rangle = e^{-\tau D}|\psi\rangle$ and $P^A = \left(\frac{1}{2}, 0, \frac{1}{2}\right)$ is again the boundary point defined by $\tau\to-\infty$ in global coordinates. The idea is that $|\psi(\tau)\rangle$ prepares a boundary condition for the path integral on a surface of constant τ and this surface converges to a small cap around the boundary point $P^A = \left(\frac{1}{2}, 0, \frac{1}{2}\right)$ when $\tau\to-\infty$. This is depicted in figure 8.3.

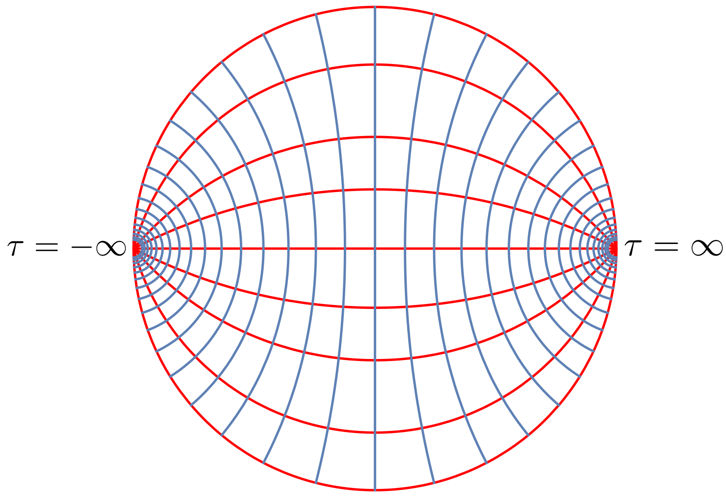

Figure 8.3: Curves of constant τ (in blue) and constant ρ (in red) for AdS_2 stereographically projected to the unit disk (Poincaré disk). This shows how surfaces of constant τ converge to a boundary bound when $\tau \to -\infty$. The Cartesian coordinates in the plane of the figure are given by $\vec{w} = \frac{1}{1+\cosh\rho\cosh\tau}(\cosh\rho\sinh\tau, \sinh\rho)$ which puts the AdS_2 metric in the form $\mathrm{d}s^2 = \frac{4\mathrm{d}\vec{w}^2}{1-\vec{w}^2}$.

Generating function

So far we have taken the approach of computing boundary correlators by starting from the bulk and extrapolating to the boundary. This approach captures the extrapolate dictionary.

Another common dictionary to define CFT correlation functions from a bulk QFT in AdS is via a generating functional. We introduce the functional

$$W[\phi_b] = \left\langle \mathrm{e}^{\int_{\partial\,\mathrm{AdS}} \mathrm{d}P \phi_b(P)\mathcal{O}(P)} \right\rangle , \tag{8.75}$$

where the integral is taken over a chosen section of the null cone in $\mathbb{R}^{d+1,1}$, equipped with its induced measure. The source $\phi_b(P)$ is required to transform homogeneously under rescalings as

$$\phi_b(\lambda P) = \lambda^{\Delta-d}\phi_b(P), \tag{8.76}$$

ensuring that the integral is invariant under changes of section – in other words, that it is conformally invariant. For instance, on the Poincaré section, the integral becomes the standard expression $\int \mathrm{d}^d x\, \phi_b(x)\mathcal{O}(x)$.

CFT correlation functions are then recovered as functional derivatives:

$$\langle \mathcal{O}(P_1)\ldots\mathcal{O}(P_n)\rangle = \frac{\delta}{\delta\phi_b(P_1)}\cdots\frac{\delta}{\delta\phi_b(P_n)} W[\phi_b]\bigg|_{\phi_b=0} . \tag{8.77}$$

To relate this construction to AdS QFT, we define the generating functional $W[\phi_b]$ as the ratio of AdS path integrals with specified boundary behaviour:

$$W[\phi_b] = \frac{\int_{\phi\to\phi_b}[d\phi]\,e^{-S[\phi]}}{\int_{\phi\to 0}[d\phi]\,e^{-S[\phi]}}\,, \tag{8.78}$$

where the path integral in the numerator is taken over bulk fields that asymptotically approach the source $\phi_b(P)$ at the boundary. Specifically, we impose the boundary condition

$$\lim_{\lambda\to\infty}\lambda^{d-\Delta}\phi(X=\lambda P+\dots) = \frac{1}{2\Delta-d}\frac{1}{\sqrt{C_\Delta}}\phi_b(P)\,, \tag{8.79}$$

With this prescription, the resulting boundary correlators computed via functional differentiation of $W[\phi_b]$ reproduce the correlators of $\mathcal{O}$ defined earlier as limits of bulk fields.

For a free scalar field, the path integral becomes Gaussian and can be evaluated using the classical solution ϕ_{cl} satisfying the AdS equations of motion and the boundary condition (8.79). A natural ansatz for this solution is

$$\phi(X) = \sqrt{C_\Delta}\int_{\partial AdS} dP\,\frac{\phi_b(P)}{(-2P\cdot X)^\Delta}\,. \tag{8.80}$$

which indeed solves the AdS Klein–Gordon equation $\nabla^2\phi = m^2\phi$. This follows because $\phi(X)$ is a homogeneous function of degree $-\Delta$ and satisfies $\partial_A\partial^A\phi = 0$ in embedding space (see equations (8.42) and (8.44)). To verify that the boundary condition (8.79) is also satisfied, it is convenient to work in the Poincaré patch:

Exercise 8.16. *In the Poincaré section (8.15) and using Poincaré coordinates (E.42), formula (8.80) reads*

$$\phi(z,x) = \sqrt{C_\Delta}\int d^d y\,\frac{z^\Delta\phi_b(y)}{(z^2+(x-y)^2)^\Delta} \tag{8.81}$$

and (8.79) reads

$$\lim_{z\to 0} z^{\Delta-d}\phi(z,x) = \frac{1}{2\Delta-d}\frac{1}{\sqrt{C_\Delta}}\phi_b(x)\,. \tag{8.82}$$

Show that (8.82) follows from (8.81). You can assume $2\Delta > d$.

Now consider the addition of interactions, such as a cubic term $\frac{1}{3!}g\phi^3$ in the bulk action. This modifies the classical solution $\phi = \phi_0 + \mathcal{O}(g)$ and introduces order-g corrections to $W[\phi_b]$. To compute the classical action, we must address a subtlety: the bulk action must include a boundary term to ensure a well-posed variational problem.

Exercise 8.17. *The coefficient β should be chosen such that the quadratic action*

$$S_2 = \int_{AdS}\text{vol}\left[\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\right] + \beta\int_{AdS}\text{vol}\,\nabla_a(\phi\nabla^a\phi) \tag{8.83}$$

is stationary around a classical solution obeying (8.82) for any variation $\delta\phi$ that preserves the boundary condition, i.e.

$$\delta\phi(z,x) = z^\Delta\left[f(x) + O(z)\right]\,. \tag{8.84}$$

Show that $\beta = \frac{\Delta - d}{d}$ and that the on-shell action is given by a boundary term

$$S_2 = \frac{2\Delta - d}{2d} \int_{AdS} \mathrm{vol}\, \nabla_\alpha \left(\phi \nabla^\alpha \phi\right) \, . \tag{8.85}$$

Finally, show that for the classical solution (8.81) *this action is given by* [7]

$$S_2 = -\frac{1}{2} \int d^d y_1 d^d y_2 \phi_b(y_1) \phi_b(y_2) K(y_1, y_2) \, , \tag{8.86}$$

where

$$\begin{aligned} K(y_1, y_2) &= C_\Delta \frac{2\Delta - d}{d} \lim_{z \to 0} \int \frac{d^d x}{z^{d-1}} \frac{z^\Delta}{(z^2 + (x - y_1)^2)^\Delta} \partial_z \frac{z^\Delta}{(z^2 + (x - y_2)^2)^\Delta} \\ &= \frac{1}{(y_1 - y_2)^{2\Delta}} \end{aligned} \tag{8.87}$$

is the CFT two point function (8.62).

Exercise 8.18. *Using $\phi = \phi_0 + O(g)$ with ϕ_0 given by* (8.80)*, show that the complete on-shell action is given by*

$$S = -\frac{1}{2} \int d^d y_1 d^d y_2 \phi_b(y_1) \phi_b(y_2) K(y_1, y_2) + \frac{1}{3!} g \int_{AdS} dX \left[\phi_0(X)\right]^3 + O(g^2) \, , \tag{8.88}$$

and that this leads to the three-point function (8.63).

Extra: Compute the terms of $\mathcal{O}(g^2)$ in the on-shell action.

8.5 The necessity of dynamical gravity

In the previous sections, we have seen that a quantum field theory (QFT) defined on an AdS background naturally gives rise to conformal correlation functions on the boundary of AdS. More precisely, we established that boundary states organize into representations of the d-dimensional conformal algebra. Via the state-operator correspondence, this implies that local operators on the boundary transform covariantly under conformal transformations, and that their operator product expansion (OPE) defines a consistent and convergent product structure on the space of boundary operators.

These features collectively characterize what we will refer to as a conformal theory (CT). As we have shown, this minimal set of properties already leads to significant constraints: correlation functions are highly structured, and all operators must have scaling dimensions above the unitarity bounds.

However, conformal theories in this sense are missing a crucial ingredient that would elevate them to full conformal field theories (CFTs): they lack a stress-energy tensor. That is, their spectrum does not include a conserved, symmetric spin-2 operator $T_{\mu\nu}$ of dimension $\Delta = d$. In the remainder of this section, we will explore why the presence of the stress tensor is so significant and how its universal

[7] This integral is divergent if the source ϕ_b is a smooth function and $\Delta > \frac{d}{2}$. The divergence comes from the short distance limit $y_1 \to y_2$ and does not affect the value of correlation functions at separate points. Notice that a small value of $z > 0$ provides a UV regulator.

properties are intimately connected to the emergence of gravity. Based on our previous discussions, it should already be clear that the stress tensor plays a key role in implementing conformal symmetries in a local and spacetime-resolved manner—local in the sense that the symmetry transformations are generated by insertions of $T_{\mu\nu}(x)$ at spacetime point x.

We begin by reviewing some general aspects of conserved currents. When a quantum field theory possesses a global symmetry, it is associated with a conserved current J_μ. In such cases, the theory can be coupled to a non-dynamical background gauge field A_μ that sources this current. For our boundary theory, the coupling takes the form

$$S = S + \int_{\partial\,\mathrm{AdS}} J^\mu A_\mu\,, \tag{8.89}$$

Here, A_μ denotes a fixed background gauge field defined on the boundary of AdS. In the context of the bulk-boundary system, coupling the boundary current to this background field is equivalent to introducing a dynamical gauge field in the bulk subject to Dirichlet boundary conditions:

$$A_\mu^{\mathrm{bulk}}\big|_{\partial\,\mathrm{AdS}} = A_\mu\,. \tag{8.90}$$

From the bulk perspective, gauging the global symmetry on the boundary corresponds to promoting A_μ to a dynamical field and relaxing the boundary conditions from Dirichlet-type to Neumann-type. This transition reflects the shift from treating A_μ as a fixed source to treating it as a fluctuating dynamical degree of freedom. At first glance, it may seem puzzling that a global symmetry on the boundary is dual to a gauge redundancy in the bulk. The resolution lies in the notion of a long-range gauge symmetry. In the bulk, there exist Wilson lines that can terminate on charged operators at the boundary, giving rise to non-trivial, physical observables. Gauging the boundary symmetry amounts to identifying these charged operators under a local redundancy, thereby rendering the corresponding Wilson lines trivial. In this way, the dual of a global symmetry is not a mere gauge redundancy, but a gauge symmetry with physical, long-range implications.

Extending the reasoning above to include spacetime symmetries, we consider the role of the stress-energy tensor $T^{\mu\nu}$. Just as a global symmetry current can be coupled to a background gauge field, the stress tensor naturally couples to a background metric $g_{\mu\nu}$. On the boundary, this coupling takes the form

$$S = S + \int_{\partial\,\mathrm{AdS}} T^{\mu\nu} g_{\mu\nu}\,. \tag{8.91}$$

where $g_{\mu\nu}$ is interpreted as the boundary value of a bulk dynamical field – namely, the spacetime metric. This is the gravitational analogue of coupling a conserved current to a background gauge field. But crucially, to account for dynamical responses of the theory (e.g. to compute correlation functions involving $T^{\mu\nu}$), we must allow $g_{\mu\nu}$ to fluctuate. That is, it is a dynamical field in the bulk.

This leads to a profound conclusion: if the boundary theory possesses a conserved stress tensor – that is, if we are dealing with a full-fledged conformal field theory – then the bulk dual must include a dynamical metric. In other words, the bulk theory must contain gravity. More precisely, since the dual theory lives in a spacetime that asymptotically approaches AdS, this must be a theory of quantum gravity in asymptotically AdS space.

This connection makes gravity in the bulk unavoidable: the very existence of a conserved stress tensor in the boundary theory forces us to include a dynamical spin-2 field, i.e. the bulk graviton. Thus, a conformal field theory on the boundary theory implies dynamic gravitaty in the bulk.

The next exercise also shows that a free QFT in AdS_{d+1} can not be dual to a local CFT_d.

Exercise 8.19. *Consider a gas of free scalar particles in global AdS. Since the particles are bosonic and non-interacting, multiparticle states can be formed by populating each single-particle state with any number of quanta. The full partition function is thus a product over single-particle states, and is completely determined by the single-particle partition function.*

Let $q = \mathrm{e}^{-\frac{1}{RT}}$, where T is the temperature and R is the AdS radius. The energy eigenvalues E_ψ of the single-particle states are determined by the AdS global Hamiltonian $D = -\partial_\tau$. Show that:

$$F = -T\log Z = -T\log\prod_{\psi_{sp}}\left(\sum_{k=0}^{\infty} q^{kE_{\psi_{sp}}}\right) = -T\sum_{n=1}^{\infty}\frac{1}{n}Z_1(q^n)\,, \tag{8.92}$$

$$Z_1(q) = \sum_{\psi_{sp}} q^{E_{\psi_{sp}}} = \frac{q^\Delta}{(1-q)^d}\,, \tag{8.93}$$

where $Z_1(q)$ is the single-particle partition function and Δ is the conformal dimension of the scalar field.

Show that in the high-temperature limit ($T \gg \frac{1}{R}$), the free energy behaves as

$$F \approx -\zeta(d+1)R^d T^{d+1} \tag{8.94}$$

where $\zeta(s)$ is the Riemann zeta function.

Compute the entropy from the free energy using the thermodynamic identity:

$$S = -\frac{\partial F}{\partial T}\,. \tag{8.95}$$

Compare this with the expected scaling of the entropy of a conformal field theory on a spatial sphere S^{d-1} of radius R, which behaves as

$$S \sim (RT)^{d-1}\,, \tag{8.96}$$

Discuss the physical meaning of this discrepancy in scaling, particularly in light of the difference between a theory in AdS and a boundary CFT. For further insights and a more detailed discussion, see section 4.3 of [ESP12].

8.6 The AdS/CFT correspondence

Finally, we are ready to state the AdS/CFT correspondence in its full form. The central statement of the duality relates the partition function of string theory in an asymptotically AdS spacetime to the generating functional of correlation functions in a conformal field theory living on the boundary of that spacetime. This is succinctly expressed as:

$$Z_{\text{bulk}}[\phi(x,z)|_{z=0} = \phi_0(x)] = \left\langle \mathrm{e}^{\int \mathrm{d}^d x \phi_0(x)\mathcal{O}(x)}\right\rangle_{\text{CFT}}\,. \tag{8.97}$$

On the left-hand side, we have the partition function of a quantum gravitational theory (often string theory or a low-energy effective supergravity theory) in a bulk spacetime that is asymptotically Anti-de Sitter (AdS). This partition function is computed with the boundary condition that the bulk field $\phi(z,x)$ approaches a fixed value $\phi_0(x)$ at the conformal boundary $z=0$. On the right-hand side, $\phi_0(x)$ is interpreted as a source for a local operator $\mathcal{O}(x)$ in the boundary conformal field theory. The exponential generates all correlation functions of $\mathcal{O}(x)$ via functional differentiation with respect to $\phi_0(x)$ as introduced above.

This expression encapsulates the core principle of the AdS/CFT correspondence: every bulk field $\phi(z,x)$ is dual to a boundary operator $\mathcal{O}(x)$, and their dynamics are intricately linked via the equality of partition functions.

In practice, computing the full quantum gravitational path integral is an incredibly difficult – if not impossible – task, especially in strongly curved or stringy regimes. However, substantial progress can be made in the semi-classical limit, where the bulk theory is weakly coupled. This limit is typically controlled by a large parameter, such as the string tension $1/\alpha'$ or the rank N of the gauge group in the boundary theory. In this regime, the bulk path integral can be approximated via a saddle-point expansion around classical solutions to the equations of motion:

$$Z_{\text{bulk}} \simeq \sum_{\text{saddles}} \mathrm{e}^{-S_{\text{cl}}(\phi_{\text{saddle}})+\mathcal{O}(\alpha',1/N)} + \mathcal{O}(\text{loops})\,, \tag{8.98}$$

Here S_{cl} denotes the classical action evaluated on a saddle-point ϕ_{saddle}. α' corrections correspond to higher curvature corrections, while $\frac{1}{N}$ corrections correspond to quantum corrections.

This approximation is valid when the gravitational theory is weakly coupled, and quantum and stringy corrections are suppressed. The validity of this approximation depends on two independent but related criteria:

- Small String Length vs. Curvature Scale:

  The first condition is that higher derivative corrections, controlled by the parameter α', must be small in units of the AdS length, $\frac{\alpha'}{L^2} \ll 1$. This condition ensures that higher-curvature corrections to the supergravity action, which scale as powers of α'/L^2, are negligible. When this inequality holds, the gravitational dynamics can be described by classical gravity rather than full quantum gravity.

- Weak Bulk Coupling (Large Effective Planck Mass):

  The second condition is that quantum corrections from bulk loops are suppressed. These are governed by the bulk Newton constant G_N, or equivalently, by the Planck length ℓ_P. The effective loop expansion parameter is $\frac{G_N}{L^{d-1}} \ll 1$. This ensures that one-loop and higher-loop contributions to the path integral are subleading, allowing us to keep only the tree-level (classical) contribution.

Up to this point, we have approached AdS/CFT from a "bottom-up" perspective, treating the bulk theory as an effective field theory of gravity coupled to matter fields in an AdS background. This approach is flexible and allows for general insights into the structure of holography, but it leaves open the question of UV completeness: what is the full theory that consistently includes quantum gravity?

To make the correspondence precise and UV-complete, we must turn to string theory. String theory is currently the only known consistent framework for quantum gravity, and it provides a "top-down" construction of the AdS/CFT correspondence. In this setting, AdS/CFT emerges as a specific limit of a duality between string theory on certain ten-dimensional spacetimes and (often supersymmetric) gauge theories in lower dimensions.

The canonical example

The best-understood and most precisely formulated example of the AdS/CFT correspondence is the duality between:

- Type IIB string theory on $\mathrm{AdS}_5 \times S^5$ with N units of five-form flux,
- $4d$ $\mathcal{N}=4$ $\mathrm{SU}(N)$ super Yang-Mills (SYM) theory.

Type IIB string theory on $\mathrm{AdS}_5 \times S^5$ (with equal radii L) and $\int_{S^5} F_5 \sim N$ is a maximally supersymmetric background with isometry group $\mathrm{SO}(2,4) \times \mathrm{SO}(6)$. This background can be obtained as the gravitational backreaction of a stack of N D3-branes in flat space. $\mathcal{N}=4$ $\mathrm{SU}(N)$ SYM on the other hand is a maximally supersymmetric conformal field theory who's field content consists of an $\mathrm{SU}(N)$ gauge field, four Majorana fermions (gaugini) and six scalars, all transforming in the adjoint representation of $\mathrm{SU}(N)$. The global symmetry group of this theory is $\mathrm{SO}(2,4) \times \mathrm{SO}(6)$, precisely matching the IIB isometry group, where the first factor is the four-dimensional conformal group and the second factor corresponds to the R-symmetry (rotating the various supercharges). This CFT can be obtained as the world-volume theory living on the same stack of N D3-branes, where the SYM fields arise as the open string modes ending on the branes. In string theory the duality can be understood as open/closed string duality relating.

The AdS/CFT correspondence asserts that the partition function of type IIB string theory on $\mathrm{AdS}_5 \times S^5$ with appropriate boundary conditions for the bulk fields is equal to the generating functional of connected correlators in $\mathcal{N}=4$ SYM with sources

$$Z_{\mathrm{CFT}} = Z_{\mathrm{IIB}} \tag{8.99}$$

This means that every local gauge-invariant operator in the CFT corresponds to a field (or excitation) in the bulk, and the dynamics of the two theories are fully equivalent.

The duality includes a precise map between the parameters of the two theories relating the string theory parameters with CFT parameters. We list the most important entries in Table 8.1.

In order for the semi-classical approximation to be valid, we need small string correction, which corresponds to large 't Hooft coupling,

$$\frac{L^2}{\alpha'} \sim \sqrt{\lambda} \gg 1\,. \tag{8.100}$$

This ensures that the curvature scale of the background is small compared to the string scale, suppressing higher-derivative corrections. In addition, in quantum corrections are suppressed if N is

Quantity	String theory	CFT
Gauge coupling	g_{YM}^2	$4\pi g_s$
't Hooft coupling	$\lambda = g_{YM}^2 N$	$4\pi g_s N$
AdS radius	$\sqrt{\lambda}$	L^2/ℓ_s^2
5d Newton constant	$G_N^{(5)}$	$\sim N^{-2}$

Table 8.1: Holographic dictionary between parameters. g_s is the string coupling and $\ell_s = \sqrt{\alpha'}$ the string length.

large,

$$g_s \sim \frac{\lambda}{N} \ll 1\,, \tag{8.101}$$

Putting these together, classical supergravity is valid when $N, \lambda \gg 1$. This corresponds to the strongly coupled, large-N limit of the gauge theory. In this regime, we can compute CFT correlators and observables by solving classical equations of motion in type IIB supergravity on $\mathrm{AdS}_5 \times S^5$. Conversely, at weak coupling ($\lambda \ll 1$), the gauge theory is accessible via perturbation theory, but the bulk dual becomes strongly curved and stringy and is no longer described by (semi-classical) supergravity.

In other words, the holographic duality is a strong-weak duality, meaning that when one side is weakly coupled, the other is strongly coupled. It provides a non-perturbative definition of string theory in certain backgrounds and gives a powerful tool for studying strongly coupled gauge theories.

The holographic correspondence is a rich and active research field, reaching far beyond what is discussed in these notes. A more complete discussion would go beyond the scope of this course, but let us offer some final remarks. Over the past two decades, AdS/CFT has grown from a bold conjecture into a versatile and powerful framework that connects quantum field theory, quantum gravity, and string theory in profound ways. Current research continues to expand the dictionary between bulk gravitational theories and boundary conformal (and non-conformal) field theories, exploring less symmetric setups, finite-coupling effects, and quantum corrections. There is also a strong drive to apply holographic ideas to more realistic systems, ranging from condensed matter physics to strongly coupled nuclear matter and beyond. Simultaneously, efforts to understand the emergence of spacetime geometry, the role of entanglement, and the microscopic origin of black hole entropy place holography at the heart of the quest to reconcile gravity and quantum mechanics. While much remains to be understood, the general trend is clear: holography provides a uniquely powerful lens to study the deep interplay between geometry, quantum information, and field theory, making it one of the most vibrant and promising areas in modern theoretical physics.

8.7 **Hawking radiation in** AdS

In this last section, which is left entirely as an exercise, we derive the Hawking effect for black holes in AdS space.

Exercise 8.20. *Hawking radiation in AdS*

1. *Consider the following metric for a black hole in* AdS_{d+1} *space,*

$$\mathrm{d}s^2 = f(r)\mathrm{d}t^2 - \frac{\mathrm{d}r^2}{f(r)} - r^2\mathrm{d}\Omega_{d-1}^2\,, \tag{8.102}$$

where $f(r) = r^2 + 1 - \frac{\mu}{r^{d-2}}$. *The mass is related to the parameter* μ *as*

$$(d-1)\mu = 8\pi^{\frac{2-d}{2}}\Gamma\left(\frac{d}{2}\right)M\,. \tag{8.103}$$

Argue why this metric represents a black hole,show that asymptotically it reduces to a metric on standard AdS_{d+1} *and find the horizon radius* r_H *for* $d=3$ *and* $d=4$.

2. *From now on, consider the same type of 2d toy model we considered in the Chapter 7. Discard the transverse sphere but keep the same function* $f(r)$ *as for* $d=4$. *As in flat space this suffices to predict the correct temperature. What changes when we reintroduce the sphere?*

   Introduce the tortoise coordinate analogous defined by $\mathrm{d}r^* = \frac{\mathrm{d}r}{f(r)}$ *which tends to* $-\infty$ *near the horizon and approaches a constant value near the asymptotic boundary.*

3. *Similar as in flat space, define the null coordinates* $u = t - r^*$ *and* $v = t + r^*$ *appropriate to describe incoming and outgoing waves for an asymptotic observer. Similar as in flat space, define the Kruskal coordinates*

$$U = -\mathrm{e}^{-\frac{f'(r_H)}{2}u}\,, \qquad V = \mathrm{e}^{\frac{f'(r_H)}{2}v}\,, \tag{8.104}$$

   appropriate to describe an observer outside the future or past horizon.

   Define the (plane wave) modes for both observers analogously as we did in the lectures for the Unruh effect or Hawking radiation in asymptotically flat space. Show that imposing reflective boundary conditions at $r=\infty$ *implies that the solutions have to be standing waves. (Consider Dirichlet and/or Neumann only)*

4. *Compute the Bogoliubov coefficients relating the two types of modes and use this to find the temperature of the Hawking radiation. (Hint: You can do this only for the outgoing modes depending on* u *resp.* U. *The computation for ingoing modes is analogous and essentially fixed by the boundary conditions.)*

5. *Physically speaking, what is the effect of the boundary conditions?*

Exercise 8.21. *Euclidean time periodicity*

An alternative way to detect the thermal nature of black hole backgrounds is to study the periodicity in Euclidean time. Show that for a generic black hole background with metric (8.102)[8], the absence of conical singularities predicts a Hawking temperature $T = \frac{1}{\beta}$ *with*

$$\beta = \frac{4\pi}{f'(r_h)}\,. \tag{8.105}$$

(Hint: expand the metric around the horizon and rewrite in the standard metric on $\mathbb{R}^{1,1} \times S^{d-1}$*.)*

[8]You do not need to assume anything about the asymptotic behaviour at $r \to \infty$.

Use this to verify your computation of the Hawking temperature of the AdS *Schwarzschild black hole in the previous exercise.*

Part III

APPENDICES

Appendix A

Conventions

In these notes we use natural (or Planck) units in which $\hbar = c = G_N = k_B = 1$. In these units the natural scales are given by

Quantity	Expression	Metric value
Length	$\ell_P = \sqrt{\frac{\hbar G_N}{c^3}}$	$1.616 \cdot 10^{-35}$ m
Mass	$m_P = \sqrt{\frac{\hbar c}{G_N}}$	$2.176 \cdot 10^{-8}$ kg
Time	$t_P = \sqrt{\frac{\hbar G_N}{c^5}}$	$5.391 \cdot 10^{-44}$ s
Temperature	$T_P = \sqrt{\frac{\hbar c^5}{G_N k_B^2}}$	$1.417 \cdot 10^{32}$ K

Using these normalised units, the cosmological constant of our observable universe is $\Lambda \sim 2.888 \cdot 10^{-122} \ell_P^{-2}$.

These notes we employ a plethora of indices, each with its own meaning. The various uses of indices are summarised in Table A.1 below.

Index	Range	Meaning
$\mu, \nu, \ldots$	$0, \ldots, d$	Curved spacetime indices
$m, n, \ldots$	$1, \ldots, d$	Space-like curved spacetime indices
$M, N, \ldots$	$0, \ldots, d+1$	Embedding space curved indices
$a, b, \ldots$	$1, \ldots, d$	Tangent space indices
$\alpha, \beta, \ldots$	$1, 2$	$\mathrm{SU}(2)_L$ indices
$\dot{\alpha}, \dot{\beta}, \ldots$	$1, 2$	$\mathrm{SU}(2)_R$ indices

Table A.1: The various indices used in these lecture notes. d is the dimension of spacetime. When considering four-dimensional spacetime we sometimes employ the exceptional isomorphism $\mathfrak{so}(4) = \mathfrak{su}(2)_L \times \mathfrak{su}(2)_R$.

A.1 Signs, signatures and curvature

Let $(\mathcal{M}, g_{\mu\nu})$ be our space-time. For the most part, we will take to be a four-dimensional manifold with metric $g_{\mu\nu}$ and here we restrict to this situation only. As usual curved indices are raised and lowered with respectively the metric and its inverse $g^{\mu\nu}$ while flat tangent space indices are raised and lowered with the Minkowski metric

$$\eta_{ab} = \mathrm{diag}(-1, 1, \cdots, 1), \tag{A.1}$$

and its inverse, where in these notes we choose mostly plus conventions.

In the literature many conventions are used, often causing confusion when comparing different sources. In order to easily compare with the literature we keep all the signs explicit in this appendix while in the main text we fix all signs to be one,

$$s_1 = s_2 = s_3 = s_4 = s_5 = 1\,. \tag{A.2}$$

The first choice of sign comes from the signature of the metric, which can be either mostly plus or mostly minus,

$$\eta_{ab} = s_1\,\mathrm{diag}\,(-1\,,1\,,\cdots\,,1\,,1)\,. \tag{A.3}$$

A second sign appears in the definition of the Riemann tensor,

$$R_{\mu\nu}{}^{\rho}{}_{\sigma} = -s_2\left(\partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\tau}\Gamma^\tau_{\nu\sigma} - \Gamma^\rho_{\nu\tau}\Gamma^\tau_{\mu\sigma}\right)\,. \tag{A.4}$$

A third sign appears in the definition of the Ricci tensor

$$s_2 s_3 R_{\mu\nu} = R^{\rho}{}_{\nu\rho\mu}\,, \qquad R = g^{\mu\nu}R_{\mu\nu}\,. \tag{A.5}$$

This sign in turn gives rise to a sign in the Einstein equation

$$s_3\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) = \kappa^2 T_{\mu\nu}\,, \tag{A.6}$$

where by definition, T_{00} is always positive and $\kappa^2 = 8\pi G_N$. The signs s_1 and s_3 determine the signs of the kinetic terms of scalars and gravitons

$$\mathcal{L} = \frac{1}{2\kappa^2}\sqrt{|g|}\left\{s_1 s_3 R - s_1\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \cdots\right\}\,. \tag{A.7}$$

Hence, looking at the Lagrangian one can easily recognize the values of these signs.

When working with frame fields the curvature can also be obtained from the spin connection $\omega_\mu{}^{ab}$. The usual convention is that it is related to the curvature defined above in terms of the Christoffel symbols as

$$R_{\rho\sigma}{}^{\mu}{}_{\nu}(\Gamma) = R_{\rho\sigma}{}^{ab}(\omega)e^\mu_a e_{\nu b}\,. \tag{A.8}$$

However, there is an independent sign in

$$R_{\mu\nu}{}^{ab} = -s_4\left(\partial_\mu\omega^{ab}_\nu - \partial_\nu\omega^{ab}_\mu + \omega^{ac}_\mu\omega^{\ b}_{\nu c} - \omega^{ac}_\nu\omega^{\ b}_{\mu c}\right)\,. \tag{A.9}$$

This sign is relevant when considering the covariant derivatives of vectors and spinors, which are given by

$$\nabla_\mu\psi = \left(\partial_\mu + s_2 s_4\frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\right)\psi\,, \qquad \nabla_\mu V^a = \partial_\mu V^a + s_2 s_4\omega_\mu{}^{ab}V_b\,. \tag{A.10}$$

Furthermore we always (anti-)symmetrize with weight one, i.e.

$$A_{[ab]} = \frac{1}{2}(A_{ab} - A_{ba})\,, \qquad A_{(ab)} = \frac{1}{2}(A_{ab} + A_{ba})\,. \tag{A.11}$$

In some references the factor of 2 is omitted. Finally, the Levi-Civita tensor is defined as $\epsilon_{0123} = s_5$

and $\epsilon^{0123} = -s_5$.

To illustrate all these conventions above here are some useful formulae which depend on the choices of sign

$$\nabla_\mu e^a_\nu = \partial_\mu e^a_\nu + s_2 s_4 \omega_\mu{}^{ab} e_{\nu b} - \Gamma^\rho_{\mu\nu} e^a_\rho = 0\,, \tag{A.12}$$

$$\omega_\mu{}^{ab} = s_2 s_4 \left(2 e^{\nu[a} \partial_{[\mu} e^{b]}_{\nu]} - e^{\nu[a} e^{b]\sigma} e_{\mu c} \partial_\nu e^c_\sigma \right)\,, \tag{A.13}$$

$$\left[\nabla_\mu, \nabla_\nu\right] V_\rho = s_2 R^\sigma{}_{\rho\mu\nu} V_\sigma \tag{A.14}$$

Appendix B

Differential forms

Differential forms often simplify formulae both computationally and conceptually. In this appendix we briefly review the essentials of this framework, for a more comprehensive treatment, see for example [Nak90, BT13, Nab10].

On any manifold we can define the formal objects, $\mathrm{d}x^\mu$, called differentials. The composition of such differential forms is done through the exterior product and denoted by a wedge, $\wedge$. This product is associative and anti-symmetric,

$$\mathrm{d}x^{\mu_1}\wedge\ldots\wedge\mathrm{d}x^{\mu_p}=\mathrm{d}x^{[\mu_1}\wedge\ldots\wedge\mathrm{d}x^{\mu_p]}\equiv\frac{1}{p!}\sum_{\sigma\in\mathcal{S}_p}(-1)^{|\sigma|}\mathrm{d}x^{\mu_{\sigma(1)}}\wedge\ldots\wedge\mathrm{d}x^{\mu_{\sigma(p)}} \tag{B.1}$$

where $|\sigma|$ denotes the signature of the permutation σ. We define a p-form as an element of the linear vector space $\wedge^p(\mathcal{M})$ spanned by the the external composition of p differentials. Any p-form can thus be represented as a homogeneous polynomial of degree p in the exterior product of differentials,

$$\alpha=\alpha_{\mu_1\mu_2\ldots\mu_p}\mathrm{d}x^{\mu_1}\wedge\ldots\mathrm{d}x^{\mu_p}=\alpha_{[\mu_1\mu_2\ldots\mu_p]}\mathrm{d}x^{\mu_1}\wedge\ldots\mathrm{d}x^{\mu_p}\in\wedge^p(\mathcal{M})\,. \tag{B.2}$$

In a d-dimensional manifold, the direct sum of vector spaces $\wedge(\mathcal{M})=\bigoplus_{p=0}^{d}\wedge^p(\mathcal{M})$ is called the exterior algebra. In the exterior algebra, the exterior product is a map $\wedge(\mathcal{M})\times\wedge(\mathcal{M})\to\wedge(\mathcal{M})$ defined as

$$\alpha\wedge\beta\equiv\alpha_{[\mu_1\ldots\mu_p}\beta_{\mu_{p+1}\ldots\mu_{p+q}]}\mathrm{d}x^{\mu_1}\wedge\ldots\wedge\mathrm{d}x^{\mu_{p+q}}\in\wedge^{p+q}(\mathcal{M})\,, \tag{B.3}$$

where α is a p-form and β a q-form. This product is graded commutative

$$\alpha\wedge\beta=(-1)^{pq}\beta\wedge\alpha\,. \tag{B.4}$$

We also have the exterior derivative defined by

$$\mathrm{d}\alpha\equiv\partial_{[\mu_1}\alpha_{\mu_2\ldots\mu_{p+1}]}\mathrm{d}x^{\mu_1}\wedge\ldots\wedge\mathrm{d}x^{\mu_{p+1}}\;\in\wedge^{p+1}(\mathcal{M})\,. \tag{B.5}$$

Key features of the exterior derivative are:

Lemma B.1. *The exterior derivative does not depend on the choice of torsion-free covariant derivative. We have $\mathrm{d}^2\alpha=0$ for all α as a consequence of the commutation of partial derivatives (or symmetry of a torsion-free connection).*

Thus it is metric independent and can be defined just using the coordinate derivative in any coordinate system. The fact that $d^2 = 0$ allows us to define cohomology groups

$$H^p(M) = \{\alpha \in \Omega^p | d\alpha = 0\}/\{\alpha = d\beta\}\,, \tag{B.6}$$

since the exact forms, i.e. those that can be expressed as $d\beta$, are a subset of the closed forms, those that satisfy $d\alpha = 0$. Such cohomology groups encode important information about the topology of M because $d\alpha = 0$ implies that locally there exists a β with $\alpha = d\beta$ (Poincaré lemma).

Example B.1. *As an example, consider the circle S^1. Since the circle is connected, every two points are connected by a segment and are cohomologically equivalent. Indeed, this implies that $H^0(S^1) = \mathbb{R}$ which remains to be true for any connected manifold. Next, let us compute $H^1(S^1)$. Consider a generic one-form $\omega = f(\theta)\mathrm{d}\theta \in \Omega^1(S^1)$. This form is clearly closed so we are left to investigate whether it is exact, i.e. if we can find a globally well-defined function F such that $\omega = \mathrm{d}F$. Locally it is easy to see that we can find such a function,*

$$F(\theta) = \int_0^\theta f(\theta')\mathrm{d}\theta'\,. \tag{B.7}$$

In order for F to be globally well-defined we need to impose that $F(2\pi) = 0$. Defining the function

$$\lambda : \Omega^1(S^1) \to \mathbb{R} : \omega = f(\theta)\mathrm{d}\theta \mapsto \int_0^{2\pi} f(\theta')\mathrm{d}\theta'\,, \tag{B.8}$$

it is easy to see that the first cohomology group is given by

$$H^1(S^1) = \Omega^1(S^1)/\ker\lambda = \mathrm{im}\,\lambda = \mathbb{R}\,. \tag{B.9}$$

The exterior derivative satisfies the graded Leibniz rule

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^p \alpha \wedge d\beta\,. \tag{B.10}$$

Furthermore, we can also define the interior product with a vector V^a that takes a p-form α to the $p-1$-form[1]

$$(V \lrcorner \alpha)_{a_2 a_3 \dots a_p} = p V^{a_1} \alpha_{a_1 \dots a_p}\,. \tag{B.11}$$

This also satisfies a graded Leibniz property,

$$V \lrcorner (\alpha \wedge \beta) = (V \lrcorner \alpha) \wedge \beta + (-1)^p \alpha \wedge (V \lrcorner \beta)\,. \tag{B.12}$$

It plays a role in the Cartan formula for the Lie derivative of a form

$$\mathcal{L}_V \alpha = V \lrcorner d\alpha + d(V \lrcorner \alpha)\,. \tag{B.13}$$

When we have a metric, we can define Hodge duality: in d dimensions a p-form α is dualized to a $d-p$ form ${}^*\alpha$ by

$$({}^*\alpha)_{a_{p+1} \dots a_d} := \frac{1}{p!} \varepsilon_{a_1 \dots a_d} \alpha^{a_1 \dots a_p} \tag{B.14}$$

[1] Another common notation for the inner product is given by $\iota_V \alpha = V \lrcorner \alpha$.

where $\varepsilon_{a_1 \dots a_d} = \varepsilon_{[a_1 \dots a_d]}$ and $\varepsilon_{0\,1 \dots d-1} = \sqrt{-g}$ is the metric volume form.

A key application is to integration. Being a covariant tensor, a p-form naturally 'pulls back' under a map, and restricts to provide a p-form on a submanifold. On a p-dimensional submanifold, it can naturally be integrated subject to the choice of an orientation on the surface.

Definition B.1. A p-surface Σ^p is said to be orientable if it is possible to choose a non-vanishing p-form. Such a choice provides an *orientation* on Σ^p.

The key point is that under a change of coordinates on the p-surface Σ^p, a p-form transforms with the determinant of the Jacobian of the coordinate transformation, whereas the change of variables formula for integration requires the modulus of the determinant which can introduce additional signs, and so we must restrict the coordinate transformations to those that preserve the sign of the chosen form making sure that the sign in question is positive.[2] The standard example of a non-orientable manifold is $\mathbb{RP}^{2n} = S^{2n}/\mathbb{Z}_2$ where the $\mathbb{Z}_2$ acts by the antipodal map which reverses the sign of the volume form.

The main theorem concerning integration on manifolds is Stoke's theorem:

Theorem B.2 (Stokes)**.** *Let Σ be a p-surface with boundary S with compatible orientations (i.e., the orientation on S is obtained from that on Σ by use of an outward pointing normal vector), and let α be a $p-1$-form on Σ, then*

$$\int_\Sigma d\alpha = \int_S \alpha \,. \tag{B.15}$$

Another application is the Cartan formulation of connections and curvature.

B.1 Connections and curvature

Instead of working with the metric, it is often useful to define a orthonormal frame of one-forms, or vielbeine, $e^a = e^a_\mu \mathrm{d}x^\mu$ satisfying

$$g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu \,, \tag{B.16}$$

where $\eta_{ab} = \mathrm{diag}(1, -1, \cdots, -1)$ is the flat Lorentz metric. The vielbeine e^a_μ and their inverses e^μ_a can be used to freely convert curved spacetime indices to flat tangent space indices. Note that the global structure of spacetime manifolds does not always allow the vielbeine to be chosen globally. In other words, generic spacetimes do not admit a global framing. In general this description is only valid locally. However, for globally hyperbolic spacetimes with orientable spatial slices, it is valid globally.

The connection acting on this frame can be obtained from the Cartan structural equation

$$\mathrm{d}e^a + \omega^a{}_b \wedge e^b = 0 \,, \tag{B.17}$$

[2] The issue is seen in one dimension: under the transformation $y = -x$,

$$\int_a^b f(x)dx = \int_{-a}^{-b} -f(-y)dy = \int_{-b}^{-a} f(-y)dy \,,$$

so that there is no sign change if we are to integrate from the lower limit to the upper in each case.

where

$$\omega^{ab} = \omega^{[ab]} = \mathrm{d}x^\mu \omega_\mu{}^{ab}\,, \tag{B.18}$$

is the 1-form spin connection.[3] In terms of the spin connection, we can define the curvature 2-form,

$$R_a{}^b = \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu R_{\mu\nu a}{}^b = \mathrm{d}\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b\,. \tag{B.19}$$

Consistency then requires that this form satisfies the Bianchi identities

$$R^a{}_b \wedge e^b = 0\,, \qquad \mathrm{d}R^a{}_b + \omega^a{}_c \wedge R^c{}_b - R^a{}_c \wedge \omega^c{}_b = 0\,. \tag{B.20}$$

For a general 1-form we can then write the covariant derivative as

$$\nabla_a A_b = (\partial_a A_b - \omega_a{}^c{}_b A_c)\,, \tag{B.21}$$

and similar for higher forms. In addition, this formulation allows us to consider spinors in general spacetimes.

[3]Note that here, as always in this course, we assume the connection to be torsionless. In the presence of torsion, (B.17) has to be modified to $\mathrm{d}e^a + \omega^a{}_b \wedge e^b = \Theta$, where Θ is the torsion 2-form. Similarly, in the presence of torsion the Bianchi identities have to be modified.

Appendix C

Hypersurfaces

This appendix reviews some facts on hypersurfaces which will be useful in various computations in this course. More details and examples can be found in the book [Poi04].

Let $\mathcal{M}$ be a $(d+1)$-dimensional Lorentzian manifold. A hypersurface Σ can be defined by parametric equations of the form

$$x^\mu = x^\mu(y^p)\,, \tag{C.1}$$

where x are coordinates on $\mathcal{M}$ and y^p, $p = 1, \dots d$ are intrinsic coordinates on Σ. Alternatively, the hypersurface can be defined by implicit equations

$$\Phi(x^\mu) = 0\,. \tag{C.2}$$

Exercise C.1. *As in standard Euclidean geometry, show that the vector $\partial_\mu \Phi$ is always normal to the hypersurface.*

A hypersurface is null if $g^{\mu\nu}\partial_\mu \Phi \partial_\nu \Phi = 0$ and space/time-like if the vectors in the tangent space at each point are space/time-like. When the hypersurface is not null, we can introduce the unit normal vector, defined by

$$n^\mu n_\mu = \epsilon = \begin{cases} +1\,, & \text{if } \Sigma \text{ is space-like,} \\ -1 & \text{if } \Sigma \text{ is time-like.} \end{cases} \tag{C.3}$$

When the hypersurface is defined implicitly, the normal is proportional to $\partial_\mu \Phi$. By definition, the normal is pointed in the direction of increasing Φ, i.e. $n^\mu \partial_\mu \Phi > 0$.

Exercise C.2. *Show that the normal can be written as*

$$n_\mu = \frac{\epsilon\, \partial_\mu \Phi}{|g^{\mu\nu}\partial_\mu \Phi \partial_\nu \Phi|}\,. \tag{C.4}$$

Define now the tangent vectors

$$E^\mu{}_p = \frac{\partial x^\mu}{\partial y^a}\,, \qquad n_\mu E^\mu{}_p = 0\,. \tag{C.5}$$

The pull-back of the metric to Σ is then given by

$$\mathrm{d}s^2\big|_\Sigma = g_{\mu\nu} E^\mu{}_p E^\nu{}_q \mathrm{d}y^p \mathrm{d}y^q = -\epsilon h_{pq} \mathrm{d}y^p \mathrm{d}y^q\,. \tag{C.6}$$

This defines the so-called induced metric, or first fundamental form, h_{pq}. For non-null surfaces we can then define the surface element as

$$\mathrm{d}\Sigma = |h|^{1/2}\mathrm{d}^d y\,, \qquad \mathrm{d}\Sigma_\mu = \epsilon n_\mu \mathrm{d}\Sigma\,. \tag{C.7}$$

One then has the following Lorentzian (or Pseudo-Riemannian) version of Stokes's theorem,

$$\int_{\mathcal{M}} \mathrm{d}^{d+1}x\sqrt{|g|}\nabla_\mu A^\mu = \int_{\partial\mathcal{M}} \mathrm{d}\Sigma_\mu A^\mu\,. \tag{C.8}$$

Exercise C.3. *Show that the ambient metric $g^{\mu\nu}$, when restricted to Σ, can be decomposed as*

$$g^{\mu\nu} = \epsilon\left(n^\mu n^\nu - h^{pq}E^\mu{}_p E^\nu{}_q\right)\,. \tag{C.9}$$

The second fundamental form, or extrinsic curvature, is defined as

$$K_{pq} = \nabla_\nu n_\mu E^\mu{}_p E^\nu{}_q\,. \tag{C.10}$$

Exercise C.4. *Show that K_{pq} defined in* (C.10) *is a symmetric tensor and can furthermore be written as*

$$K_{pq} = \frac{1}{2}\mathcal{L}_n g_{\mu\nu}E^\mu{}_p E^\nu{}_q\,, \tag{C.11}$$

where $\mathcal{L}_n$ is the Lie derivative along the normal vector n.

The trace of the extrinsic curvature is given by

$$K = h^{pq}K_{pq} = (n^\mu n^\nu - \epsilon g^{\mu\nu})\nabla_\nu n_\mu = h^{pq}E^\mu{}_p E^\nu{}_q \nabla_\nu n_\mu\,. \tag{C.12}$$

Exercise C.5. *As an example, and to get familiar with the concepts introduced above, consider the spacetime $\mathcal{M}$ with metric,*

$$\mathrm{d}s^2 = V(r)\mathrm{d}t^2 - V(r)^{-1}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2_{d-1}\,. \tag{C.13}$$

and consider the hypersurface defined by $r =$ constant.

1. *Compute the tangent and normal vectors.*
2. *Compute the induced metric and extrinsic curvature.*

Appendix D

Variational calculus

To keep the discussion in these notes self-contained, this appendix includes a short discussion of variational calculus, in particular as applied to the Einstein-Hilbert action. Even though we do not discuss dynamical gravity in these notes this will be useful to revisit the general principles.

The Einstein-Hilbert action for the gravitational field in $d+1$ dimensions is

$$S_{\mathrm{EH}} = \frac{1}{16\pi G_N} \int \mathrm{d}^{d+1}x \sqrt{|g|} R \,. \tag{D.1}$$

Its variation leads to the vacuum Einstein equations. After including additional matter fields these give rise to a non-trivial stress-tensor. The derivation of Einstein's equations is standard and can be found in many textbooks. However, the standard derivations often do not carefully include the contributions due to boundary terms. Such contributions are not so important if your main interest is the study of the solutions to Einstein's equations in classical GR. However, in a quantum theory, the action becomes a crucial object since it gives the weight of a field configuration to the path integral. For this reason let us revisit this derivation with particular care paid to the boundary terms.

Let us consider the gravitational action in a region $\mathcal{M}$ of space-time, with boundary $\partial\mathcal{M}$ and analyze the variation of the action as we vary the metric, with the condition that the metric variation vanishes at the boundary:

$$\delta g_{\mu\nu}\Big|_{\partial\mathcal{M}} = 0 \,. \tag{D.2}$$

To study the variation of the Einstein Hilbert action, note that

$$\delta\sqrt{|g|} = -\frac{1}{2}\sqrt{|g|} g_{\mu\nu} \delta g^{\mu\nu} \,, \tag{D.3}$$

such that

$$\delta\left(\sqrt{|g|}R\right) = \sqrt{|g|} G_{\mu\nu} \delta g^{\mu\nu} + \sqrt{|g|} g^{\mu\nu} \delta R_{\mu\nu} \,, \tag{D.4}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ is the Einstein tensor. The vanishing of the first term leads to Einstein's equations in empty space, while the last term is usually neglected by arguing that it vanishes on a boundary at infinity. However, let us have a closer look at this term. Using a local frame where the Christoffel symbols vanish, i.e. using Gauss's normal coordinates, we have

$$\delta R_{\mu\nu} = \delta R^{\rho}{}_{\mu\rho\nu} = \delta\left(\partial_\nu \Gamma^\rho_{\nu\rho} - \partial_\rho \Gamma^\rho_{\mu\nu}\right) = \nabla_\nu \delta\Gamma^\rho_{\mu\rho} - \nabla_\rho \delta\Gamma^\rho_{\mu\nu} \,. \tag{D.5}$$

This last expression is covariant and hence applies in any coordinate system. We can therefore write

$$g^{\mu\nu}\delta R_{\mu\nu} = \nabla_\mu \delta v^\mu \,, \qquad \delta v^\mu = g^{\mu\rho}\delta\Gamma^\nu_{\rho\nu} - g^{\rho\nu}\delta\Gamma^\mu_{\rho\nu} \,. \tag{D.6}$$

Using Stokes' theorem we can then write

$$\int_{\mathcal{M}} \mathrm{d}^{d+1}x\sqrt{|g|}g^{\mu\nu}\delta R_{\mu\nu} = \int_{\mathcal{M}} \mathrm{d}^{d+1}x\sqrt{|g|}\nabla_\mu \delta v^\mu = \int_{\partial\mathcal{M}} \mathrm{d}\Sigma_\mu \delta v^\mu \,. \tag{D.7}$$

Let us now proceed to compute the variation of v. Since the variation of the metric, but not of its derivatives, vanishes at the boundary, we find

$$\delta\Gamma^\mu_{\rho\sigma} = \frac{1}{2}g^{\mu\nu}\left(\delta\partial_\sigma g_{\nu\rho} + \delta\partial_\rho g_{\nu\sigma} - \delta\partial_\nu g_{\rho\sigma}\right), \tag{D.8}$$

and therefore we have

$$\delta v_\mu = g^{\rho\sigma}\left(\delta\partial_\mu g_{\rho\sigma} - \delta\partial_\rho g_{\mu\sigma}\right). \tag{D.9}$$

Finally, before we can plug this into (D.7), we need to contract this with the vector n normal to the boundary,

$$n^\mu \delta v_\mu = -n^\mu \epsilon\left(n^\rho n^\sigma - h^{pq}E^\rho{}_p E^\sigma{}_q\right)\left(\delta\partial_\sigma g_{\mu\rho} - \delta\partial_\mu g_{\sigma\rho}\right). \tag{D.10}$$

Now notice that the last bracket is anti-symmetric in μ and σ, while $n^\mu n^\sigma$ is symmetric under the interchange of indices. Therefore, the part of the metric which involves the normal vectors drops out. Next, we note that $\delta\partial_\sigma g_{\nu\rho}E^\sigma{}_p = 0$, since the variation of the metric vanishes everywhere on the boundary and therefore the variation on its tangential derivatives has to vanish as well. Summarizing, we have

$$n^\mu \delta v_\mu = -\epsilon h^{\rho\sigma}\delta\partial_\mu g_{\rho\sigma}n^\mu \,. \tag{D.11}$$

This involves the variation of the derivative of the metric along the normal direction to the boundary so it is in general non-vanishing.

Putting everything together, we find

$$16\pi G_N \delta S_{EH} = \int_{\mathcal{M}} \mathrm{d}^{d+1}x\sqrt{|g|}G_{\mu\nu}\delta g^{\mu\nu} - \int_{\partial\mathcal{M}} \mathrm{d}^d y\sqrt{|h|}h^{\mu\nu}\delta\partial_\rho g_{\mu\nu}n^\rho \,. \tag{D.12}$$

Therefore, in the presence of a boundary, Einstein's equations are not sufficient to guarantee the vanishing of the variation of the action, due to the second term in (D.12). To remedy this, we must add an explicit boundary term to the action,

$$S_{\mathrm{GH}} = \frac{1}{8\pi G_N}\int_{\partial\mathcal{M}} \mathrm{d}^d y\sqrt{|h|}K \,, \tag{D.13}$$

called the Gibbons-Hawking counterterm. To see that this counterterm has the correct properties, let's have a quick look at its variation. Since the metric is fixed at the boundary the variation of h vanishes and the only variation comes from the extrinsic curvature K. Computing its variation we find

$$\delta K = h^{\mu\nu}\delta\partial_\nu n_\mu = \frac{1}{2}h^{\mu\nu}\delta\partial_\rho g_{\mu\nu}n^\mu \,. \tag{D.14}$$

Varying the Gibbons-Hawking counterterm we then find

$$16\pi G_N \delta S_{\mathrm{GH}} = \int_{\mathcal{M}} \mathrm{d}^d y \sqrt{|h|} h^{\mu\nu} \delta \partial_\rho g_{\mu\nu} n^\rho \,, \tag{D.15}$$

which exactly cancels the boundary term in the original variation and thus rendering the variational principle well-defined.

In the context of black hole physics and AdS/CFT this is not always enough. In general, the action, supplied with the Gibbons-Hawking counterterm, will lead to divergences when evaluated, even in flat space. In order to obtain finite values for the on-shell action, it is usually necessary to add additional counterterms removing the divergences. In the context of AdS/CFT the prescription to remove said divergences goes under the name of holographic renormalisation.

Appendix E

Ingredients from general relativity

In this appendix we review a variety of useful facts from general relativity that will come in handy in this course.

E.1 Maximally symmetric spaces

Let us start with the maximally symmetric manifolds. In Euclidean signature these are given by the sphere S^d, flat Euclidean space $\mathbb{R}^d$ and hyperbolic space H^d, which are respectively positively curved, flat or negatively curved. We will sometimes collectively denote them by $M_{k,d}$, where $k = 0, \pm 1$ and write their metrics as

$$\mathrm{d}s^2_{M_{k,d}} = \begin{cases} \mathrm{d}\Omega^2_d\,, & k=1\,, \\ \sum_{i=1}^d \mathrm{d}x_i^2\,, & k=0\,, \\ \mathrm{d}\Xi^2_d\,, & k=-1\,, \end{cases} \tag{E.1}$$

where $\mathrm{d}\Omega^2_d$ and $\mathrm{d}\Xi^2_d$ are the standard metrics on the d-sphere and d-dimensional hyperbolic space respectively,

$$\mathrm{d}\Omega^2_d = \sum_{i=1}^d \left(\prod_{j=1}^{i-1} \sin^2\phi_j \right) \mathrm{d}\phi_i^2\,, \qquad \mathrm{d}\Xi^2_d = \frac{1}{x_d^2}\sum_{i=1}^d \mathrm{d}x_i^2\,. \tag{E.2}$$

There are many other coordinate choices but unless explicitly stated otherwise we will always use the metrics above.

In most of this course we are interested in spaces with Minkowskian signature. The maximally symmetric Lorentzian spacetimes are de Sitter space, Minkowski space and anti-de Sitter space, which are respectively positively curved, flat or negatively curved.

de Sitter space

The de Sitter space is the maximally symmetric spacetime of positive curvature

$$R = \frac{d(d+1)}{L^2}\,, \tag{E.3}$$

where L is the characteristic length scale of the space. This space describes a exponentially expanding universe. For this reason, there is an observer-dependent horizon, called the ‘cosmological horizon’ beyond which spacetime is expanding faster than the speed of light. This is a null surface beyond which the observer can never receive a signal.

In $d+1$ dimensions de Sitter space can be described by a hypersurface in $d+2$ dimensional Minkowski space. consider the embedding space $\mathbb{R}^{1,d+1}$ with metric,

$$\mathrm{d}s^2 = -\mathrm{d}X_0^2 + \sum_{i=1}^{d+1} \mathrm{d}X_i^2 \,. \tag{E.4}$$

The de Sitter space of radius L is then defined as the hyperboloid

$$X_\mu X^\mu = -X_0^2 + \sum_{i=1}^{d+1} X_i^2 = L^2 \,, \tag{E.5}$$

In the same way that the two sphere embedded in $\mathbb{R}^3$ inherits the O(3) symmetry of its ambient space, de Sitter space inherits an $\mathrm{O}(1,d)$ symmetry from the ambient Minkowski space. There are various useful coordinate systems to describe the de Sitter space of which we list a few below:

- **Global coordinates.** These coordinates $\{\tau, \omega_i\}$ are defined by

  $$X_0 = L \sinh \frac{\tau}{L} \,, \qquad X_i = L\omega_i \cosh \frac{\tau}{L} \,, \tag{E.6}$$

  where $i = 1, \dots d+1$ and the ω_i are constrained coordinates on a round unit sphere S^d such that $\sum_i \omega_i^2 = 1$. In these coordinates, the metric on the de Sitter space reads,

  $$\mathrm{d}s^2 = -\mathrm{d}\tau^2 + L^2 \cosh^2 \frac{\tau}{L} \, \mathrm{d}\Omega_d^2 \,. \tag{E.7}$$

  Global coordinates are sometimes also called the closed slicing of de Sitter, especially in the context of cosmology. This terminology comes from the FLRW metric (see below) since for these coordinates the space-like slices are closed.

- **Planar coordinates.** These coordinates $\{t, \mathbf{x}\}$ are defined as

  $$\begin{aligned} X_0 =& L \sinh \frac{t}{L} + \frac{\mathbf{x}^2}{2L} \mathrm{e}^{\frac{t}{L}} \,, \\ X_{d+1} =& L \cosh \frac{t}{L} - \frac{\mathbf{x}^2}{2L} \mathrm{e}^{\frac{t}{L}} \,, \\ X_i =& x_i \mathrm{e}^{\frac{t}{L}} \,, \end{aligned} \tag{E.8}$$

  where $i = 1, \dots, d$. These coordinates do not cover the full de Sitter space but only the patch

  $$X_0 + X_{d+1} = L\mathrm{e}^{\frac{t}{L}} > 0 \,. \tag{E.9}$$

  In these coordinates, the metric reads,

  $$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{e}^{\frac{2t}{L}} \sum_{i=1}^{d} \mathrm{d}x_i^2 \,. \tag{E.10}$$

- **Static coordinates.** de Sitter space enjoys various time-like isometries inherited from the boosts in embedding space. Yet, the metrics considered so far are time dependent. Since there is a time-like Killing vector, there must exist coordinates such that time does not appear explicitly in the metric. These are static coordinates and are defined as

$$\begin{aligned}X_0 =& \sqrt{L^2-r^2}\sinh\frac{t}{L}\,,\\ X_{d+1} =& \sqrt{L^2-r^2}\cosh\frac{t}{L}\,,\\ X_0 =& r\,\omega_i\,,\end{aligned} \tag{E.11}$$

with $i=1,\ldots,d$ and $0 \leq r < L$. They only cover the region

$$X_{d+1} > 0\,, \qquad \sum_i X_i < L^2\,. \tag{E.12}$$

The resulting metric reads

$$\mathrm{d}s^2 = -\left(1-\frac{r^2}{L^2}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1-\frac{r^2}{L^2}} + r^2\mathrm{d}\Omega_{d-1}^2\,. \tag{E.13}$$

This metric is manifestly static. In static coordinates the cosmological horizon is located at $r = L$ Therefore these coordinates cover precisely the patch that is accessible to a single observer, in the sense that the observer can both send and receive signals to/from this entire region.

- **Hyperbolic coordinates.** Global coordinates foliate de Sitter with spheres, while planar coordinates foliate with planes. To cover the third possibility we introduce hyperbolic coordinates which foliate de Sitter by hyperbolic spaces. The embedding coordinates are defined by

$$\begin{aligned}X_0 =& \sinh\tau\cosh\psi\,,\\ X_{d+1} =& \cosh\tau\,,\\ X_i =& \sinh\tau\sinh\psi\,\omega_i\,,\end{aligned} \tag{E.14}$$

such that the metric takes the form

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 + \sinh^2\tau\,\mathrm{d}\Xi_d^2\,. \tag{E.15}$$

- **Conformal coordinates.** Finally, to obtain the conformal coordinates we start from planar coordinates and perform a coordinate transformation to conformal time

$$\eta = \int_\infty^t \frac{\mathrm{d}t'}{a(t')} = -L\mathrm{e}^{-\frac{t}{L}}\,, \tag{E.16}$$

so that $-\infty < \eta < 0$ and $\eta \to -\infty$ corresponds to the infinite past $t \to -\infty$. However, one can extend this spacetime to almost all of the de Sitter by extending the range of η to the full real line. These coordinates can be parameterised as

$$\begin{aligned}
X_0 =& \frac{1}{2\eta}\left(\eta^2 - \mathbf{x}^2 - L^2\right) , \\
X_{d+1} =& -\frac{1}{2\eta}\left(\eta^2 - \mathbf{x}^2 + L^2\right) , \\
X_0 =& -\frac{L}{\eta} x_i ,
\end{aligned} \tag{E.17}$$

so that the metric is given by

$$\mathrm{d}s^2 = \frac{L^2}{\eta^2}\left(-\mathrm{d}\eta^2 + \sum_{i=1}^{d} \mathrm{d}x_i^2\right) \tag{E.18}$$

These coordinates cover only the submanifold $X_0 + X_{d+1} = 0$. When written in these coordinates, the metric of the de Sitter space, is manifestly conformally flat.

An important quantity in the geometry of the de Sitter space is the geodesic distance between two points $\zeta(X, X')$. As in the case of the sphere or the hyperbolic plane in two dimensions, the distance between two points of the de Sitter space is closely related to the distance defined in the embedding space. Therefore, let us define

$$P(X, X') = H^2 \eta_{ab} X^a X^b . \tag{E.19}$$

Notice that, if $X = X'$ are identical, we have $P = 1$. However, if X and X' are antipodal, i.e. $X' = -X$, one has $P = \frac{1}{L^2}\eta_{ab}X^a X^b = -1$. Plugging in the explicit parameterisation, we find an expression of the geodesic distance in conformal coordinates:

$$P(X, X') = 1 + \frac{(\eta - \eta')^2 - (\mathbf{x} - \mathbf{x}')^2}{2\eta\eta'} . \tag{E.20}$$

One important property of $P(X, X')$ is that it is a manifestly $\mathrm{O}(1, d)$ invariant function on de Sitter space, since it is constructed out of the Lorentz invariant product in $\mathbb{R}^{1,d+1}$. Depending on the causal relationship between X and X', we have the following behaviour for $\zeta(X, X')$:

- If X and X' are joined by a time-like geodesic $P(X, X') > 1$ and the geodesic distance is given by

$$\zeta(X, X') = \frac{1}{H}\cosh^{-1}(P) . \tag{E.21}$$

- If X and X' are space-like separated, $|P(X, X')| < 1$ and

$$\zeta(X, X') = \frac{1}{H}\cos^{-1}(P) . \tag{E.22}$$

- If X and X' are light-like separated, $P(X, X') = 1$ and $\zeta(X, X') = 0$.

Notice that there are points of de Sitter space which cannot be joined by geodesics to a given point X. These are the points in the interior of the past and future light cones of $-X$, the antipodal point of X.

For these points, we have that $P(X,X') < -1$. The results listed above can be obtained by an explicit analysis of geodesics in de Sitter space.

Exercise E.1. *The geodesics in the de Sitter space can be obtained by minimizing the distance in the embedding space subject to the constraint* (E.5)*. Employ an appropriate Lagrange multiplier to solve this minimisation problem and explicitly find the geodesics in the de Sitter space.*

Anti-de Sitter space

The anti-de Sitter space is the maximally symmetric spacetime with negative curvature. Its scalar curvature is given by

$$R = -\frac{d(d+1)}{L^2}\,. \tag{E.23}$$

Analogous to the de Sitter space we can describe Anti-de Sitter space in $d+1$ dimensions by the hyperboloid

$$X_0^2 + X_1^2 - \sum_{i=1}^{d} X_i^2 = L^2\,, \tag{E.24}$$

embedded in a $(d+2)$ dimensional ambient space with metric

$$\mathrm{d}s^2 = -\mathrm{d}X_0^2 - \mathrm{d}X_1^2 + \sum_{i=2}^{d+1} \mathrm{d}X_i^2\,. \tag{E.25}$$

The constant L is the AdS length parameterising the characteristic scale of the anti-de Sitter space. The anti-de Sitter space inherits an $O(2,d)$ symmetry from the ambient space. There are a variety of useful coordinates on the Anti-de Sitter space, similar to the de Sitter space.

- **Global coordinates.** These coordinates $\{\tau, \rho, \omega_i\}$ are defined by

$$\begin{aligned} X_0 &= L\cosh\rho\cos\tau\,,\\ X_1 &= L\cosh\rho\sin\tau\,,\\ X_i &= L\sinh\rho\,\omega_i\,, \end{aligned} \tag{E.26}$$

  where $i = 2, \ldots d+1$ and the ω_i are embedding coordinates on a round unit sphere S^{d-1} such that $\sum_i \omega_i^2 = 1$. In these coordinates, the metric on the Anti-de Sitter space reads,

$$\mathrm{d}s^2 = L^2\left(-\cosh^2\rho\,\mathrm{d}\tau^2 + \mathrm{d}\rho^2 + \sinh^2\rho\,\mathrm{d}\Omega_{d-1}^2\right)\,. \tag{E.27}$$

  In the limit $\rho \to \infty$ one approaches the conformal boundary which in global coordinates is given by $\mathbb{R} \times S^{d-1}$.

- **Poincaré coordinates.** These coordinates $\{z, x\}$ are defined as

$$\begin{aligned}
X_0 &= \frac{L^2 - t^2 + \mathbf{x}^2 + z^2}{2z}\,, \\
X_1 &= \frac{Lt}{z}\,, \\
X_i &= \frac{Lx_i}{z}\,, \\
x_{d+1} &= \frac{-L^2 - t^2 + \mathbf{x}^2 + z^2}{2z}\,,
\end{aligned} \tag{E.28}$$

where $i = 2, \ldots, d$. These coordinates do not cover the full anti-de Sitter space but only the patch

$$X_0 - X_{d+1} = \frac{L^2}{z} > 0\,. \tag{E.29}$$

In these coordinates, the metric reads,

$$\mathrm{d}s^2 = \frac{L^2}{z^2}\left(\mathrm{d}z^2 - \mathrm{d}t^2 + \sum_i \mathrm{d}x_i^2\right). \tag{E.30}$$

In these coordinates the conformal boundary is located at $z \to 0$ and the geometry of the boundary is that of d-dimensional Minkowski space.

- **Static coordinates.** These coordinates are given by

$$\begin{aligned}
X_0 &= L\sqrt{1 + \frac{r^2}{L^2}}\sin\frac{t}{L}\,, \\
X_1 &= L\sqrt{1 + \frac{r^2}{L^2}}\cos\frac{t}{L}\,, \\
X_i &= r\,\omega_i\,,
\end{aligned} \tag{E.31}$$

with $i = 2, \ldots, d+1$. In these coordinates the metric reads

$$\mathrm{d}s^2 = -\left(1 + \frac{r^2}{L^2}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 + \frac{r^2}{L^2}} + r^2\mathrm{d}\Omega_{d-1}^2\,. \tag{E.32}$$

The conformal boundary is located at $r \to 0$ and its geometry is given by $\mathbb{R} \times S^{d-1}$.

- **de Sitter slicing.** Finally, we can slice the anti-de Sitter space with de Sitter slices. The embedding coordinates are given by

$$\begin{aligned}
X_0 &= L\sinh\rho\sinh t\cosh\xi\,, \\
X_1 &= L\cosh\rho\,, \\
X_2 &= L\sinh\rho\cosh t\,, \\
X_i &= L\sinh\rho\sinh t\sinh\xi\,\omega_i
\end{aligned} \tag{E.33}$$

so that the metric is given by

$$ds^2 = L^2 \left(d\rho^2 + \sinh^2 \rho \, ds^2_{dS_{d-1}} \right) . \tag{E.34}$$

where $ds^2_{dS_{d-1}}$ is a metric on de Sitter space with Hubble scale $H = 1$.

- **Hyperspherical coordinates.** In these coordinates the embedding is parametrised as

$$\begin{aligned} X_0 &= L \sec \rho \cos \tau \, , \\ X_1 &= L \sec \rho \sin \tau \, , \\ X_i &= L \tan \rho \, \omega_i \, , \end{aligned} \tag{E.35}$$

with ω_i again parametrising a unit $d-1$ sphere. The metric in these coordinates is given by

$$ds^2 = L^2 \sec^2 \rho \left(-d\tau^2 + d\rho^2 + \sin^2 \rho \, d\Omega^2_{d-1} \right) . \tag{E.36}$$

In these coordinates the boundary of (the universal covering of) AdS is the Einstein static universe.

An important quantity in the geometry of the anti-de Sitter space is the geodesic distance between two points,

$$P(X, X') = \frac{1}{L^2} \eta_{ab} X^a X'^b \, . \tag{E.37}$$

where η_{ab} is the metric in the ambient space. If $X = X'$ are identical, we have $P = 1$. However, if X and X' are antipodal one has $P = -1$.

One important property of $P(X, X')$ is that it is a manifestly $O(2, d)$ invariant function on anti-de Sitter space, since it is constructed out of the Lorentz invariant product in $R^{2,d}$. Depending on the causal relationship between X and X', we have the following behaviour for $P(X, X')$:

- If X and X' are joined by a time-like geodesic $|P(X, X')| < 1$ and the geodesic distance is given by

$$d(X, X') = L \cos^{-1}(P) \, . \tag{E.38}$$

- If X and X' are space-like separated, $P(X, X') > 1$ and

$$d(X, X') = L \cosh^{-1}(P) \, . \tag{E.39}$$

- If X and X' are light-like separated, $P(X, X') = 1$ and $d(X, X') = 0$.

Notice that this is opposite of the de Sitter case. Furthermore, notice that in the anti-de Sitter space it is possible to reach the conformal boundary in a finite time. I.e. there exists a time-like geodesic connecting any point X with the conformal boundary. This is why AdS is often thought of as a finite 'box'.

Exercise E.2. *The geodesics in the anti-de Sitter space can be obtained by minimizing the distance in the embedding space subject to the constraint* (E.24). *Employ an appropriate Lagrange multiplier to solve this minimisation problem and explicitly find the geodesics in the de Sitter space.*

Euclidean AdS

In some situations it is more convenient to perform computations in Euclidean signature and after Euclidean AdS spacetime is the hyperboloid

$$-X_0^2 + \sum_{i=2}^{d+1} X_i^2 = -R^2 \, , \qquad X^0 > 0 \, , \tag{E.40}$$

embedded in $\mathbb{R}^{d+1,1}$. From this definition it is clear that Euclidean AdS is invariant under SO$(d+1,1)$. Let us be more explicit it this case and write out the symmetry generators as

$$J_{AB} = -i\left(X_A \frac{\partial}{\partial X^B} - X_B \frac{\partial}{\partial X^A}\right) \, . \tag{E.41}$$

Analogous to AdS , we define the Poincaré coordinates by

$$\begin{aligned} X_0 &= R\frac{1+x^2+z^2}{2z} \\ X_\mu &= R\frac{x^\mu}{z} \\ X_{d+1} &= R\frac{1-x^2-z^2}{2z} \end{aligned} \tag{E.42}$$

where $x^\mu \in \mathbb{R}^d$ and $z > 0$. In these coordinates the metric reads

$$ds^2 = R^2 \frac{dz^2 + \delta_{\mu\nu} dx^\mu dx^\nu}{z^2} \, . \tag{E.43}$$

This shows that EAdS is conformal to $\mathbb{R}^+ \times \mathbb{R}^d$ whose boundary at $z = 0$ is just $\mathbb{R}^d$. These coordinates make explicit the subgroup SO$(1,1) \times$ ISO(d) of the full isometry group of EAdS. These correspond to dilatation and Poincaré symmetries inside the $d-$dimensional conformal group. In particular, the dilatation generator is

$$D = -i\, J_{0,d+1} = -X_0 \frac{\partial}{\partial X^{d+1}} + X_{d+1} \frac{\partial}{\partial X^0} = -z\frac{\partial}{\partial z} - x^\mu \frac{\partial}{\partial x^\mu} \, . \tag{E.44}$$

Global coordinates in Euclidean AdS can simply be obtained from the global coordinates in AdS by analytically continuing $\tau \to \mathrm{i}\tau$ such that the metric is given by

$$ds^2 = R^2 \left[\cosh^2\rho \, \mathrm{d}\tau^2 + \mathrm{d}\rho^2 + \sinh^2\rho \, \mathrm{d}\Omega_{d-1}^2\right] . \tag{E.45}$$

To understand the global structure of this spacetime it is convenient to change the radial coordinate via $\tanh\rho = \sin r$ so that $r \in [0, \frac{\pi}{2}[$. Then, the metric becomes

$$ds^2 = \frac{R^2}{\cos^2 r}\left[d\tau^2 + dr^2 + \sin^2 r \, d\Omega_{d-1}^2\right] , \tag{E.46}$$

which is conformal to a solid cylinder whose boundary at $r = \frac{\pi}{2}$ is $\mathbb{R} \times S^{d-1}$. In these coordinates, the dilatation generator $D = -i\, J_{0,d+1} = -\frac{\partial}{\partial \tau}$ is the Hamiltonian conjugate to global time.

Exercise E.3. *Explicitly write out the symmetry generators for (Lorentzian) (A)dS spacetime, analogous to the discussion in this last subsection.*

E.2 Warped product manifolds and FLRW spaces

Apart from the maximally symmetric spacetimes, our second most loved example is given by warped product manifolds of the form $\mathbb{R} \times \mathcal{M}$ with metric

$$ds^2 = -dt^2 + a(t)^2 ds^2_{\mathcal{M}}, \tag{E.47}$$

When $\mathcal{M}$ is a maximally symmetric Euclidean manifold, i.e. $\mathcal{M} = M_{k,d}$ these are the FLRW manifolds introduced in the main text. When $a(t)$ is a constant function these represent static spacetimes. In the case $k = +1$ this spacetime is often called the static Einstein universe.

On the other hand, when $a(t)$ is a non-trivial function of time, these manifold provide an excellent toy model for cosmology. The spatial section of the universe contracts or expands according to the scale factor $a(t)$. It is often useful to define the conformal time coordinate,

$$\eta = \int_{-\infty}^{t} \frac{dt'}{a(t')}, \tag{E.48}$$

in terms of which the FLRW metric becomes

$$ds^2 = a^2(\eta)\left(-d\eta^2 + ds^2_{M_{k,d}}\right). \tag{E.49}$$

Note that the de Sitter space can be thought of as a FLRW space.

In maximally symmetric spacetimes, such as Minkowski or (A)dS, there can be no beginning or end of time. There cannot be any history because every time is equivalent. The simplest way to introduce some time dependence is to consider FLRW spacetimes. For this reason they are natural toy models in cosmology. The coordinate t introduced above, corresponds to the proper time of an observer at rest with respect to the co-moving spatial coordinates. The spatial manifold in an FLRW space is maximally symmetric hence such spacetimes describe a homogeneous and isotropic universe, i.e. a universe that looks the same at every point in space. This metric is simple enough that it allows for a very explicit, often exact, analysis. A most remarkable fact that should blow your mind is that this most simple spacetime for $k = 0$ is in fact a very good description of our own universe on distances much larger than average distance between galaxies, about a few Megaparsec (Mpc). Of course there are small deviations from perfect homogeneity in our universe but on large enough scales this gives an excellent description.

In contrast to the maximally symmetric spacetimes, in an FLRW universe time translation and Lorentz boost fail to be isometries because of the time dependence of the scale factor $a(t)$. A useful way to capture this dependence is by defining the Hubble parameter

$$H(t) = \frac{\dot{a}}{a}. \tag{E.50}$$

The absence of time translations has profound implications for constructing QFTs in these backgrounds, as energy is not preserved.

Next, let us quickly review the Einstein equations in such backgrounds. In order to have any hope to solve them, we need to impose some particularly symmetric stress tensor. The most general stress tensor consistent with the symmetries of FLRW spaces is given by

$$T_{\mu\nu} = \mathrm{diag}\,(-\rho, p, \cdots, p)\,, \tag{E.51}$$

where the energy density ρ and pressure p are functions of time only. We can interpret this as the energy-momentum tensor of a homogeneous perfect fluid in its rest frame,

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - g_{\mu\nu}p\,, \tag{E.52}$$

where u_μ is the normalised fluid velocity, $|u|^2 = 1$, which in rest frame would be $u_\mu = \delta_{\mu t}$. Einsteins equation imply that the energy momentum tensor is covariantly conserved,

$$\nabla_\nu T^{\mu\nu} = \partial_\nu T^{\mu\nu} + \Gamma^\mu_{\alpha\nu} T^{\alpha\nu} + \Gamma^\nu_{\alpha\nu} T^{\mu\nu} = 0\,. \tag{E.53}$$

Plugging in the FLRW solution this reduces to the so-called continuity equation,

$$\dot{\rho} + d\,H(\rho + p) = 0\,. \tag{E.54}$$

This equation tells us that the energy density changes only if the universe expands or contracts, i.e. if $H \neq 0$. The Einstein equations however will not tell us what kind of matter permeates the universe. For that we need to specify an equation of state giving a relation between the pressure, density and possibly other thermodynamic variables. Most systems of interest in cosmology can be described to a good approximation by the very simple equation of state,

$$p = w\rho\,, \tag{E.55}$$

with a single parameter w. For this equation of state we can immediately solve the continuity equation giving us

$$\rho(t) = \rho_0 a(t)^{-d(1+w)}\,. \tag{E.56}$$

- Non-relativistic matter, a.k.a. dust, has a velocity much smaller than the speed of light. For this type of matter, the pressure is negligible compared to the energy density, $p \ll \rho$, or $0 < w \ll 1$. Therefore in an expanding universe, dust dilutes as $\rho \propto a^{-d}$.

- Relativistic matter, a.k.a. radiation, on the other hand has pressure and energy density of the same order. A statistical mechanics analysis furthermore predicts that $p = \rho/d$ and so $w = 1/d$. This is precisely the proportionality constant to make the matter conformal. Hence we find $\rho \propto a^{-(d+1)}$.

- Finally, a cosmological constant, or vacuum energy has $T_{\mu\nu} = -\Lambda g_{\mu\nu}$ and hence $p = -\rho = -\Lambda$, or $w = -1$. Since in this case we have $p + \rho = 0$, the continuity equation teaches us that the cosmological constant does not dilute, $\rho \propto a^0$.

Solving the Einstein equations for an FLRW metric results in the Friedmann equations, which can be written as

$$H^2 = \frac{16\pi G_N}{d(d-1)}\rho - \frac{k}{a^2}\,, \qquad \dot{H} = -\frac{8\pi G_N}{d-1}(\rho+p) + \frac{k}{a^2}\,. \tag{E.57}$$

Here p and ρ can be thought of as the effective energy density and pressure build from the combination of all the matter present in the universe, together with the cosmological constant,

$$p = \sum_m p_m - \frac{\Lambda}{8\pi G_N}\,, \qquad \rho = \sum_m \rho_m + \frac{\Lambda}{8\pi G_N}\,. \tag{E.58}$$

The first Friedmann equation can be used to estimate the age of the universe, while the second encodes the acceleration of the universe. Since most cosmological matter respects the null energy condition, which in this case reads $\rho + p > 0$, we find that H typically decreases during the expansion of the universe.

E.3 Black holes

Another important set of spacetimes that play a key role in this course are black holes. contrary to the above examples, such spacetimes are singular and hence not complete.

In this course we restrict ourselves to the simplest of black holes, the Schwarzschild black hole in four dimension, as it will suffice to illustrate the relevant phenomena. This is the unique four-dimensional non-rotating neutral, asymptotically flat black hole. More generally, black holes can have mass and/or charge. For such more general solutions as well as black holes in other dimensions we refer the reader to the course General Relativity II.

The Schwarzschild black hole

The Schwarzschild black hole, with metric

$$\mathrm{d}s^2 = -\left(1-\frac{2M}{r}\right)\mathrm{d}t^2 + \left(1-\frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + \mathrm{d}\Omega_2^2\,, \tag{E.59}$$

is the unique non-rotating asymptotically flat black hole. This metric has a singularity at $r = 2M$, the location of the event horizon, and therefore only describes the exterior of the black hole. To see that the singularity at the event horizon is not a physical singularity it is useful to introduce the tortoise or Regge-Wheeler coordinate r^*, which is defined such that massless free falling observers follow the path $t = r^* + \text{constant}$. For such an observer one has,

$$\begin{aligned} \mathrm{d}s^2 = 0 \quad &\to \quad \mathrm{d}t = \frac{1}{1-\frac{2M}{r}}\mathrm{d}r \equiv \mathrm{d}r^* \\ &\to \quad r^* = r + 2M\log\left(\frac{r}{2M}-1\right)\,, \end{aligned} \tag{E.60}$$

where we put the arbitrary additive constant to zero. In these coordinates the metric takes the form,

$$\mathrm{d}s^2 = \left(1-\frac{2M}{r}\right)\left(-\mathrm{d}t^2 + \mathrm{d}r^{*2}\right) + r^2\mathrm{d}\Omega_2^2\,. \tag{E.61}$$

From this line element we see that the two-dimensional metric is conformally equivalent to Minkowski space. Next we introduce the retarded and advanced Eddington-Finkelstein coordinates

$$u = t + r^*, \qquad v = t - r^*, \tag{E.62}$$

which places the horizon at $(u, v) \to (\infty, -\infty)$. In these coordinates the metric becomes

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}u\mathrm{d}v + r^2\mathrm{d}s^2_{S^2}, \tag{E.63}$$

where r can be expressed as a complicated function of $u - v$. In these coordinates the metric is still singular at $r = 2M$ but we can introduce one more coordinate transformation

$$U = -4M\exp\left(-\frac{u}{4M}\right), \qquad V = 4M\exp\left(\frac{v}{4M}\right), \tag{E.64}$$

called Kruskal-Szekeres coordinates. The final metric is then given by

$$\mathrm{d}s^2 = -\frac{2M}{r}\mathrm{e}^{-\frac{r}{2M}}\mathrm{d}U\mathrm{d}V + r^2\mathrm{d}\Omega_2^2. \tag{E.65}$$

which makes it clear that there is no singularity at $r = 2M$. In these coordinates the event horizon is at $U = 0$ or $V = 0$ and the original Schwarzschild metric only covers the patch $U < 0$ and $V > 0$. However, there is no obstruction to extend U, V to the full real line. The fully extended metric covers both the inside and outside of the Schwarzschild black hole and is the maximal extension of this spacetime. Finally, an explicit map between U, V and r, t is given by

$$UV = \mathrm{e}^{\frac{r}{2M}}\left(1 - \frac{r}{2M}\right), \qquad \frac{U}{V} = \mathrm{e}^{\frac{t}{2M}}. \tag{E.66}$$

We see in fact that $r = 2M$ is a null hypersurface ruled by outgoing null geodesics. The presence of the event horizon means that not all light rays escape to infinity. For $r > 2M$, light rays with $\dot{r} > 0$ can and do escape. However, for $r < 2M$, all causal geodesics have future end point at the true singularity at $r = 0$. More precisely, we can define the event horizon is as

Definition E.1. The event horizon is the boundary of the past of $\mathscr{I}^+$.

this can easily be seen from the metric using the retarded Eddington-Finkelstein coordinate

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + r^2\mathrm{d}\Omega_2^2, \tag{E.67}$$

as the boundary is at $u \to \infty$, which corresponds to $r \to 2M$. There is a corresponding time-reversed picture using coordinates (v, r). However, now we have that $r = 2M$ is the past horizon being the boundary $v = -\infty$ of the future of $\mathscr{I}^-$. The full causal structure of the Schwarzschild black hole can be summarised in its Penrose diagram, see Figure E.1.

Similar diagrams can be drawn for Reissner-Nordstrom, Kerr and the Kerr-Newman, see GR II, although the latter have the novelty of having Cauchy horizons, hypersurfaces beyond which neither fields nor space-time itself are determined by Cauchy data essentially as a consequence of naked singularities, singularities in the past of observers. However, these cannot be seen from infinity. These

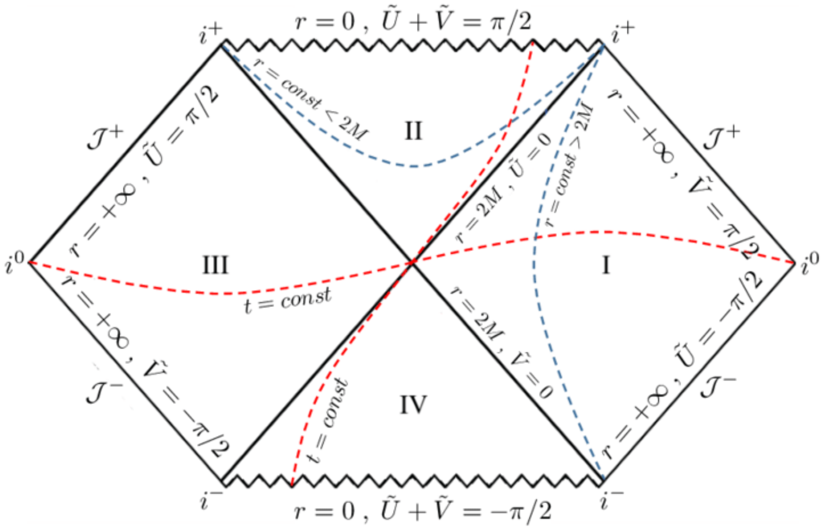

Figure E.1: Penrose diagram for the Kruskal extension of the Schwarzschild spacetime. The singularity at $r = 0$ (which is a genuine curvature singularity) is a black hole to the future of every observer that crosses the future event horizon (or a white hole in the past).

black hole solutions are unique subject to various assumptions (like the existence of a stationary Killing vector that looks like a time translation at large distances). Similarly, all of these spacetimes have generalisations with non-vanishing cosmological constant.

E.4 Killing horizons and surface gravity

Consider a null hypersurface Σ defined implicitly through the equation $\Phi(x) = 0$. In other words, the tangent vector n satisfies $n_\mu n^\mu = 0$ on Σ. Hence, n is normal and tangent to the surface![1] A null hypersurface is said to be a Killing horizon if there exists a Killing vector field that is normal to Σ.

A black hole is a spacetime that contains a region which is not in the backward lightcone of future timelike infinity. The boundary of such a region is called an event horizon. Moreover, the event horizon of a stationary asymptotically flat black hole is typically a Killing horizon.[2] The Killing vector field associated with a Killing event horizon is a combination of the Killing vector field generating $K = \partial_t$ generating time translations at infinity , and of the rotational Killing vector $\widetilde{K} = \partial_\phi$, and can be written as

$$n = \partial_t + \Omega_H \partial_\phi \,, \tag{E.68}$$

where Ω_H is a constant called the angular velocity of the horizon. In the static case we simply have $n = \partial_t$.

To every Killing horizon, we can associate a surface gravity. Since $n_\mu n^\mu = 0$ on Σ, the gradient $\nabla_\mu(n^\nu n_\nu)$ is normal to Σ and therefore proportional to n at each point on Σ. It follows that there

[1] Remember that a tangent vector v to a hypersurface satisfies $v_\mu n^\mu = 0$.

[2] The converse is not true, see for example the Killing horizon associated to the Killing vector $n = x\partial_t + t\partial_x$ that we considered in our discussion of the Unruh effect. This is clearly not an event horizon.

exists a function κ, called the surface gravity, such that on Σ

$$\nabla_\mu(n^\nu n_\nu) = -2\kappa n_\mu \,. \tag{E.69}$$

Using the Killing equation this can be rewritten as

$$n^\nu \nabla_\nu n^\mu = \kappa n^\mu \,, \tag{E.70}$$

This is nothing but the geodesic equation, where κ measures the failure of the integral curves of n to be affinely parametrised. Another useful formula for the surface gravity is

$$\kappa^2 = -\frac{1}{2}\nabla^\mu n^\nu \nabla_\mu n_\nu \,, \tag{E.71}$$

evaluated on Σ.

Exercise E.4. *Derive the formula* (E.71).

Let us show that κ is constant on orbits of n. Take a vector v tangent to Σ. Since (E.71) holds everywhere on Σ, we can write on Σ,

$$v^\rho \nabla_\rho \kappa^2 = -\nabla^\mu n^\nu v^\rho \nabla_\rho \nabla_\mu n_\nu = -\nabla^\mu n^\nu v^\rho R_{\nu\mu\rho\sigma} n^\sigma \,, \tag{E.72}$$

where in the second step we used that n is a Killing vector. Since n is also tangent, we can choose $v = n$, which gives

$$n^\rho \nabla_\rho \kappa^2 = -\nabla^\mu n^\nu R_{\nu\mu\rho\sigma} n^\rho n^\sigma = 0 \,. \tag{E.73}$$

One can actually show that $v^\rho \nabla_\rho \kappa = 0$ for every tangent vector, namely that κ is constant over the horizon. See [Wal84] for a proof.

Remark. Note that if Σ is a Killing horizon for a Killing vector field n with surface gravity κ, then it is also a Killing horizon for cn with surface gravity $c\kappa$, where c is any non-zero constant. This shows that the surface gravity is not an intrinsic property of the Killing horizon, it also depends on the normalization of n. While there is no natural normalization of n on Σ (since it is null), in a stationary, asymptotically flat spacetime we conventionally normalize the generator of time translations K so that $K_\mu K^\mu = -1$ at spatial infinity. The sign is fixed by requiring that K is future-directed. This completely fixes the the normalization of $n = K + \Omega_H \widetilde{K}$.

The main reason we are interested in the surface gravity is that it provides the Hawking temperature of a black hole. However, even in classical GR the surface gravity has a physical meaning. In a static, asymptotically at spacetime, the surface gravity is the acceleration of a particle at rest on the horizon, as measured by a static observer at infinity. The acceleration of a static observer near the horizon tends to infinity, but the redshift factor that relates this to the acceleration measured from infinity goes to zero. The surface gravity arises from the limiting value of the product of these two quantities, with the result typically being finite. When the spacetime is not static, this physical interpretation does not hold, but the surface gravity is still well-defined.

To see this in more detail, consider a static particle in a spacetime containing a static black hole. By definition, a static particle has four-velocity U proportional to the time-translation Killing vector field, $K = V(x)U$, for some function $V(x)$. This function is called the redshift factor. Recalling that the four-velocity satisfies $U^\mu U_\mu = 1$, clearly we have $V = \sqrt{-K_\mu K^\mu}$. This ranges from 0 at the horizon to 1 at infinity. Now consider the four-acceleration $a^\mu = U^\nu \nabla_\nu U^\mu$. Explicit computation results in

$$a_\mu = \frac{\nabla_\mu V}{V}\,, \tag{E.74}$$

and thus its magnitude is $a = \sqrt{a_\mu a^\mu} = V^{-1}\sqrt{\nabla^\mu V \nabla_\mu V}$. This is infinite at the horizon, as V vanishes there. But the acceleration as measured at infinity is redshifted by a factor of V, and reads

$$aV = \sqrt{\nabla_\mu V \nabla^\mu V}\,, \tag{E.75}$$

which is generically finite. One can check that the square of this evaluated on the horizon agrees with our expression for the surface gravity. Hence $\kappa = aV$ evaluated at the horizon.

Exercise E.5. *Show that* (E.74) *holds by using the Killing equation.*

Exercise E.6. *Apply this to the Schwarzschild black hole and evaluate its surface gravity. Notice that the surface gravity is inversely proportional to the mass, so it is large for small black holes, and vice-versa.*

Appendix F

Hypergeometric functions

In this appendix we review the definition and various properties of hypergeometric functions. For further reference see for example [AAR99].

The hypergeometric function is a solution of Euler's hypergeometric differential equation,

$$z(1-z)\partial_z^2 F + [\gamma - (\alpha+\beta+1)z]\,\partial_z F - \alpha\beta F = 0\,. \tag{F.1}$$

which has three regular singular points at $z = 0, 1$ and ∞. Any second order linear equation with three regular singular points can be converted to this equation through a change of variables.

For $|z| < 1$, the hypergeometric function can be defined through the following series expansion

$${}_2F_1(\alpha,\beta;\gamma|z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\frac{z^n}{n!}\,, \tag{F.2}$$

where we introduced the (rising) Pochhammer symbol

$$(q)_n = \frac{\Gamma(q+n)}{\Gamma(q)}\,. \tag{F.3}$$

when either α or β is a non-positive integer this series terminates in which case the hypergeometric function reduces to a polynomial. For complex $|z| \geq 1$ it can be analytically continued along any path that avoids the branch points at $z = 1$ and $z = \infty$.

Depending on the sign of $\mathrm{Re}(\gamma-\alpha-\beta)$ we find the following behaviour near $z = 1$,

$${}_2F_1(\alpha,\beta;\gamma|z) \overset{z\to 1}{\simeq} \begin{cases} \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}\,, & \text{for} \quad \mathrm{Re}(\gamma-\alpha-\beta) > 0\,, \\ \frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}(1-z)^{\gamma-\alpha-\beta}\,, & \text{for} \quad \mathrm{Re}(\gamma-\alpha-\beta) < 0\,. \end{cases} \tag{F.4}$$

In addition,we have the following identities for the analytic continuation of the hypergeometric functions,

$$\begin{aligned}{}_2F_1(\alpha,\beta;\gamma|z) =& \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}{}_2F_1(\alpha,\beta;1+\alpha+\beta-\gamma|1-z) \\ &+ \frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}(1-z)^{\gamma-\alpha-\beta}{}_2F_1(\gamma-\alpha,\gamma-\beta;1_\gamma-\alpha-\beta|1-z)\,.\end{aligned} \tag{F.5}$$

and

$${}_2F_1(\alpha,\beta;\gamma|z) = (1-z)^{\gamma-\alpha-\beta}{}_2F_1(\gamma-\alpha,\gamma-\beta;\gamma|z)\,. \tag{F.6}$$

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