BO1.1. History of Mathematics
Lecture III
Analytic geometry and the beginnings of calculus, part 1

MT24 Week 2

Summary

- Brief overview of the 17th century
- A cautionary tale
- Development of notation
- Use of algebra in geometry
- The beginnings of calculus

The 17th century

The main mathematical innovations of the 17th century:

- symbolic notation
- analytic (algebraic) geometry
- calculus
- infinite series [to be treated in later lectures]
- mathematics of the physical world [to be treated in later lectures]

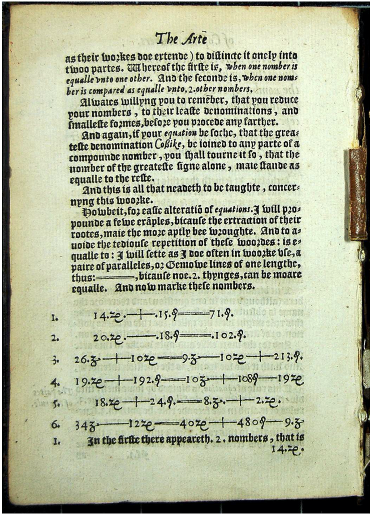
Symbolic notation

Symbolic notation makes mathematics easier

- to read
- to write
- to communicate (though perhaps not orally)
- to think about — and thus stimulates mathematical advances?
- BUT it took a long time to develop
- why did it develop when it did?

Early European notation (abbreviation)

Johannes Widman, *Behende und hüpsche Rechenung auff allen Kauffmanschafft* (1489)



Robert Recorde, *The Whetstone of Witte* (1557)

The communication of mathematics

Initially entirely verbal — but usually using a set form of words

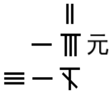
Scribal abbreviations often used

- e.g., later editors of Diophantus (3rd-century Egypt) used ς as an abbreviation for an unknown quantity
- e.g., Bhāskara II (12th-century India) used the initial letters of *yāvattāvat* (*unknown*) and *rūpa* (*unit*) as shorthand: '*yā* 1 *rū* 1' denoted '$x + 1$'

But these were not symbols that could be manipulated algebraically

Arrangement of signs on the page could carry information

- e.g., *tiān yuán shù* 天元術 (13th-century China):



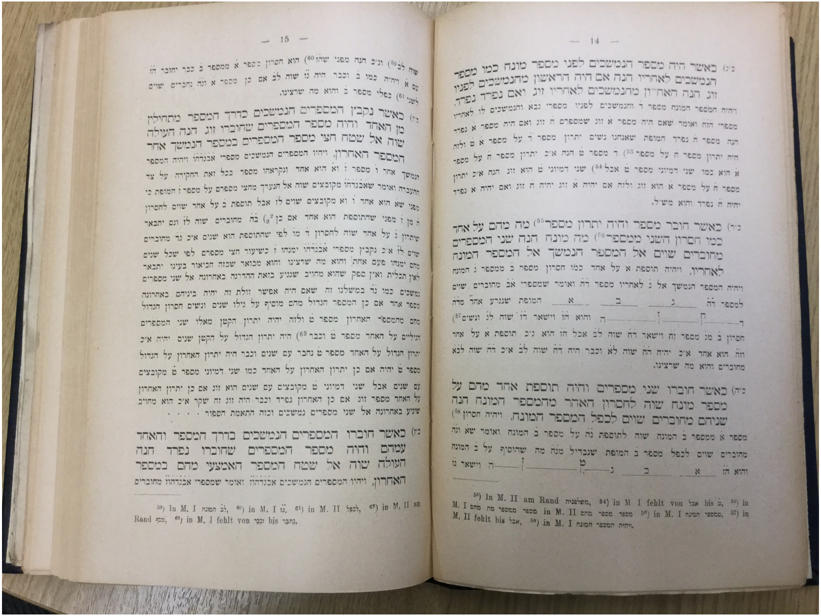
$(2x^2 + 18x - 316)$

Algebraic symbolism of the form that we use came later

A cautionary tale: Levi Ben Gerson and sums of integers

Levi Ben Gerson (Gersonides), *Ma'aseh Hoshev* (*The Work of the Calculator*), 1321 [picture is of a version printed in Venice in 1716]

A cautionary tale: Levi Ben Gerson and sums of integers



A cautionary tale: Levi Ben Gerson and sums of integers

Book I, Proposition 26:

> *If we add all consecutive numbers from one to any given number and the given number is even, then the addition equals the product of half the number of numbers that are added up times the number that follows the given even number.*

Book I, Proposition 27:

> *If we add all consecutive numbers from one to any given number and the given number is odd, then the addition equals the product of the number at half way times the last number that is added.*

(Translations from Hebrew by Leo Corry.)

A cautionary tale: Levi Ben Gerson and sums of integers

Converting these into modern notation, we get:

Book I, Proposition 26:

If n is an even number, then $1 + 2 + 3 + \cdots + n = \frac{n}{2}(n+1)$.

Book I, Proposition 27:

If n is an odd number, then $1 + 2 + 3 + \cdots + n = \frac{n+1}{2}n$.

The formulae are clearly the same, so why are these treated as separate propositions? The answer lies in the proofs, which, like the results themselves, are entirely verbal.

A cautionary tale: Levi Ben Gerson and sums of integers

A fundamental problem here lies in the difficulty of expressing the notion of 'any given number' (our 'n').

A commonly adopted solution was to outline the proof for a specific example, on the understanding that the reader should then be able to adapt the method to any other instance.

Ben Gerson's proof of Proposition 26 takes this approach, and is based on the idea of forming pairs of numbers with equal sums.*

*You might have heard a story about the young Gauss doing the same thing.

A cautionary tale: Levi Ben Gerson and sums of integers

Proof of Proposition 26:

Take the example of 6. If we add 1 and 6, we get 7 ('the number that follows the given even number'). Notice that 2 is obtained from 1 by adding 1, and that 5 is obtained from 6 by subtracting 1, so 2 added to 5 is the same as 1 added to 6, namely 7. The only remaining pair is 3 and 4, which also add to give 7. The number of pairs is half the given even number, hence the total sum is half the number of numbers that are added up times the number that follows the given even number.

This proof is clearly not valid when the given number is odd, since Ben Gerson would have been required to halve it — but he was working only with (positive) integers

A cautionary tale: Levi Ben Gerson and sums of integers

Proposition 27 therefore needs a separate proof, which similarly does not apply when the given number is even (see Leo Corry, *A brief history of numbers*, OUP, 2015, p. 119)

As Corry notes:

> *For Gersonides, the two cases were really different, and there was no way he could realize that the two situations . . . were one and the same as they are for us.*

Moral: take care when converting historical mathematics into modern terms!

Notation: compare Cardano (*Ars magna*, 1545)...

Having raised a third part of the number of things to a cube, to which you add the square of half the number in the equation and take the root of the total, consider the square [root], which you will take twice; and to one of them you add half of the same, and you will have the binome with its apotome, whence taking the cube root of the apotome from the cube root of its binome, the difference that comes from this, is the value of the thing.

(*Mathematics emerging*, p. 327)

... with Viète (c. 1590)...

François Viète
(Francisci Vieta)
Opera mathematica
1646, p. 130

II.

Si A quad. — B in A 2, æquetur Z plano. A — B eſto E. Igitur E quad, æquabitur Z plano + B quad.

Conſectarium.

Itaque $\sqrt{\text{Z plani} + \text{B quad.}}$ + B fit A, de qua primum quærebatur.

Sit B 1. Z planum 20. A 1 N. 1 Q — 2 N, æquabitur 20. & fit 1 N. √ 21 + 1.

III.

Si D 2 in A — A quad., æquetur Z plano. D — E, vel D + E eſto A. E quad., æquabitur D quad. — Z plano.

Conſectarium.

Itaque, D minus, pluſve $\sqrt{\text{D quad.} - \text{Z plano}}$ fit A, de qua primum quærebatur.

Sit D 5. Z planum 20. A 1 N. 10 N — 1 Q, æquatur 20. & fit 1 N. 5 — √ 5, *vel* 5 + √ 5.

De reductione cuborum ſimpliciter adfectorum ſub quadrato, ad cubos ſimpliciter adfectos ſub latere.

Formulæ tres.

I.

Si A cubus + B 3 in A quad., æquetur Z ſolido. A + B eſto E. E cubus — B quad. 3 in E, æquabitur Z ſolido — B cubo 2.

1 C + 6 Q, *æquatur* 1600. *eſt* 1 N 10. 1 C — 12 N, *æquatur* 1584. *eſt* 1 N 12.

Ad Arithmetica non incongrue σημεῖον aliquod ſuperimponitur notis alteratæ radicis, ad differentiam notarum ejus, de qua primum quærebatur.

II.

Si A cubus — B 3 in A quad., æquetur Z ſolido. A — B eſto E. E cubus — B quad. 3 in E, æquabitur Z ſolido + B cubo 2.

1 C — 6 Q, *æquetur* 400. *eſt* 1 N 10. 1 C — 12 N, *æquatur* 416. *eſt* 1 N 8.

III.

Si B 3 in A quad. — A cubo, æquetur Z ſolido. A — B eſto E. B quad. 3 in E. — E cubo, æquabitur Z ſolido — B cubo 2. Vel B — A eſto E. B quad. 3 in E. — E cubo, æquabitur B cubo 2 — Z ſolido.

21 Q — 1 C, *æquetur* 972. *& eſt* 1 N 9, *vel* 18. 147 N — 1 C, *æquatur* 286. *& eſt* 1 N 2, *vel* 11.

9 Q — 1 C, *æquetur* 28. *& eſt* 1 N 2. 27 N — 1 C, *æquatur* 26. *& eſt* 1 N 1.

De reductione cuborum adfectorum tam ſub quadrato quam latere, ad cubos adfectos ſimpliciter ſub latere.

Formulæ ſeptem.

I.

Si A cubus + B 3 in A quad. + D plano in A, æquetur Z ſolido. A + B eſto E. E cubus $\overline{+\ \text{D plano} - \text{B quad. } 3}$ in E æquabitur Z ſolido + D plano in B — B cubo 2.

1 C + 30 Q + 330 N, *æquetur* 788. *& eſt* 1 N 2. 1 C + 30 N, *æquatur* 2088. *& eſt* 1 N 12.

1 C +

... with Viète (c. 1590)...

II.

SI A quad. — B in A 2, æquetur Z plano. A — B eſto E. Igitur E quad, æquabitur Z plano + B quad.

Conſectarium.

Itaque $\sqrt{\text{Z plani} + \text{B quad.}}$ + B fit A, de qua primum quærebatur.

Sit B 1. Z planum 20. A 1 N. 1 Q — 2 N, æquabitur 20. & fit 1 N. $\sqrt{21}$ + 1.

III.

SI D 2 in A — A quad., æquetur Z plano. D — E, vel D + E eſto A. E quad., æquabitur D quad. — Z plano.

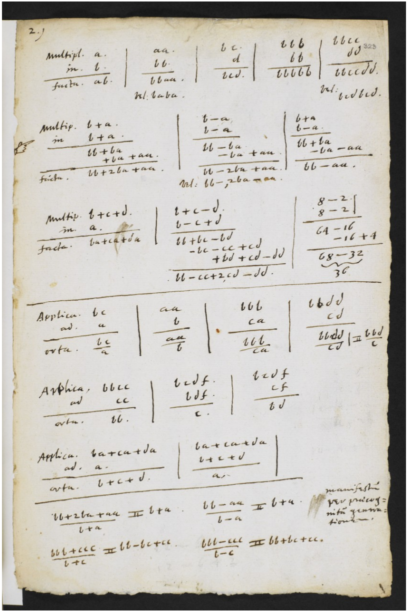
Conſectarium.

Itaque, D minus, pluſve $\sqrt{\text{D quad.} - \text{Z plano}}$ fit A, de qua primum quærebatur.

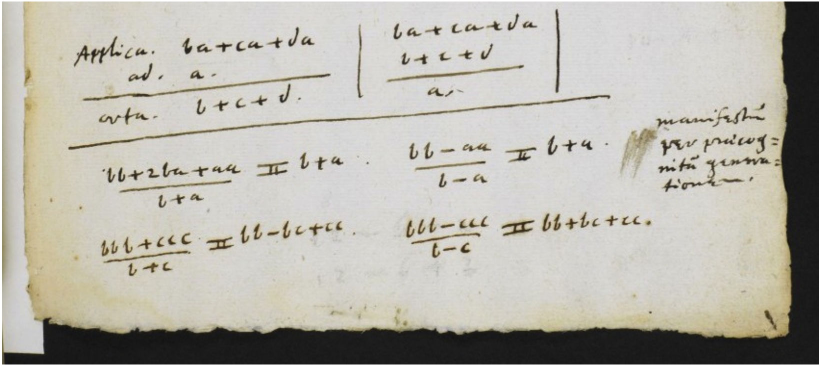
Sit D 5. Z planum 20. A 1 N. 10 N — 1 Q, æquatur 20. & fit 1 N. 5 — $\sqrt{5}$, vel 5 + $\sqrt{5}$.

… and with Harriot (c. 1600)

British Library
Add MS 6784 f. 323
available at
Thomas Harriot Online



... and with Harriot (c. 1600)



Applica. $ba+ca+da$
ad. a.
orta. $b+c+d$.

$$\frac{ba+ca+da}{b+c+d} \quad a.$$

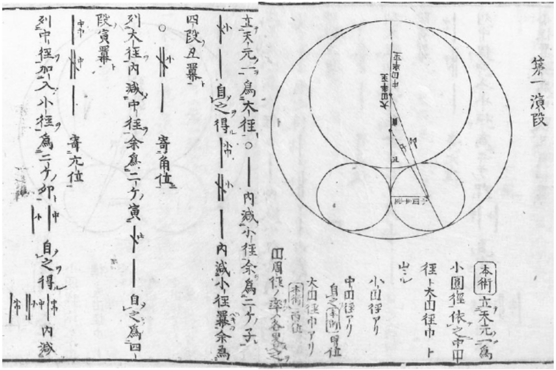
$$\frac{bb+2ba+aa}{b+a} = b+a. \qquad \frac{bb-aa}{b-a} = b+a.$$

$$\frac{bbb+ccc}{b+c} = bb-bc+cc. \qquad \frac{bbb-ccc}{b-c} = bb+bc+cc.$$

manifestum per praecognitam generationem.

Elsewhere in the world

Seki Takakazu, *Hatsubi Sanpō* 発微算法 (1674), concerning the solution of equations in several variables:



Equations written using the technique of *bōshohō* 傍書法 ('side-writing'; a.k.a. *tenzan jutsu* 点竄術)

Notation: Viète (Tours, c. 1590)

François Viète (1540–1603, France):

A, E, ... (i.e., vowels) for unknowns

B, C, D, ... (i.e., consonants) for known or given quantities

symbols $+$, $-$

but otherwise verbal descriptions and connections: quadratum (squared), cubus (cubed), aequatur (be equal), ...

Notation: Harriot (London, c. 1600)

Thomas Harriot (1560–1621, England):

a, e, ... for unknowns

b, c, d, ... for known or given quantities

$+$, $-$

ab, aa, aaa

and many symbols: $=$, $>$, ...

(For another example of Harriot's use of notation, see *Mathematics emerging*, §2.2.1.)

Harriot papers online: http://echo.mpiwg-berlin.mpg.de/content/scientific_revolution/harriot

Notation: Descartes (Netherlands, 1637)

René Descartes (1596–1650, France and Holland):

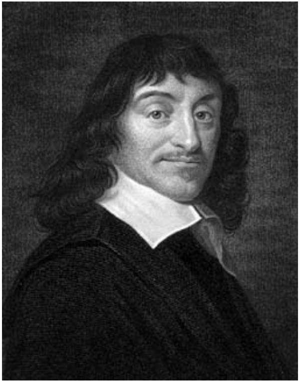
x, y, ... for unknowns

a, b, c, ... for known or given quantities

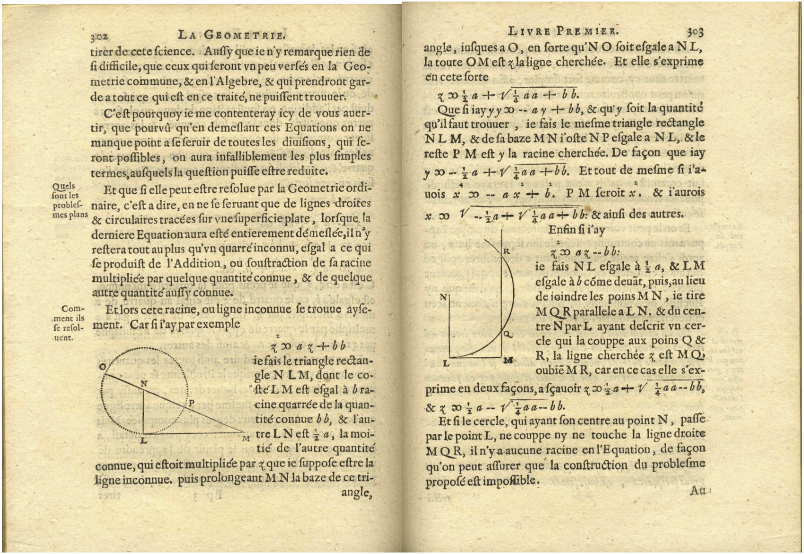
$+$, $-$

xx, x^3, x^4, ...

Descartes' notation was widely adopted, although his '$\propto$' for equality eventually gave way to '$=$', and his '$\surd C$' to '$\sqrt[3]{\ }$'.



Descartes' notation



302 LA GEOMETRIE.

tirer de cete ſcience. Auſſy que ie n'y remarque rien de ſi difficile, que ceux qui ſeront vn peu verſés en la Geometrie commune, & en l'Algebre, & qui prendront garde a tout ce qui eſt en ce traité, ne puiſſent trouuer.

C'eſt pourquoy ie me contenteray icy de vous auertir, que pourvû qu'en demeſlant ces Equations on ne manque point a ſe ſeruir de toutes les diuiſions, qui ſeront poſſibles, on aura infalliblement les plus ſimples termes, auſquels la queſtion puiſſe eſtre reduite.

Quels ſont les problesmes plans

Et que ſi elle peut eſtre reſolue par la Geometrie ordinaire, c'eſt a dire, en ne ſe ſeruant que de lignes droites & circulaires tracées ſur vne ſuperficie plate, lorſque la derniere Equation aura eſté entierement démeſlée, il n'y reſtera tout au plus qu'vn quarré inconnu, eſgal a ce qui ſe produiſt de l'Addition, ou ſouſtraction de ſa racine multipliée par quelque quantité connue, & de quelque autre quantité auſſy connue.

Comment ils ſe reſoluent.

Et lors cete racine, ou ligne inconnue ſe trouue ayſement. Car ſi i'ay par exemple

$$z^2 \infty a z + b b$$

ie fais le triangle rectangle N L M, dont le coſté L M eſt eſgal à b racine quarrée de la quantité connue $b b$, & l'autre L N eſt $\frac{1}{2} a$, la moitié de l'autre quantité connue, qui eſtoit multipliée par z que ie ſuppoſe eſtre la ligne inconnue. puis prolongeant M N la baze de ce triangle,

LIVRE PREMIER. 303

angle, iuſques a O, en ſorte qu'N O ſoit eſgale a N L, la toute O M eſt z la ligne cherchée. Et elle s'exprime en cete ſorte

$$z \infty \tfrac{1}{2} a + \sqrt{\tfrac{1}{4} a a + b b}.$$

Que ſi iay $y y \infty - a y + b b$, & qu'y ſoit la quantité qu'il faut trouuer, ie fais le meſme triangle rectangle N L M, & de ſa baze M N i'oſte N P eſgale a N L, & le reſte P M eſt y la racine cherchée. De façon que iay $y \infty - \frac{1}{2} a + \sqrt{\frac{1}{4} a a + b b}$. Et tout de meſme ſi i'auois $x^4 \infty - a x^2 + b^2$. P M ſeroit x^2. & i'aurois $x \infty \sqrt{- \frac{1}{2} a + \sqrt{\frac{1}{4} a a + b b}}$: & ainſi des autres.

Enfin ſi i'ay

$$z^2 \infty a z - b b:$$

ie fais N L eſgale à $\frac{1}{2} a$, & L M eſgale à b cóme deuãt, puis, au lieu de ioindre les poins M N, ie tire M Q R parallele a L N. & du centre N par L ayant deſcrit vn cercle qui la couppe aux poins Q & R, la ligne cherchée z eſt M Q, oubiẽ M R, car en ce cas elle s'exprime en deux façons, a ſçauoir $z \infty \frac{1}{2} a + \sqrt{\frac{1}{4} a a - b b}$, & $z \infty \frac{1}{2} a - \sqrt{\frac{1}{4} a a - b b}$.

Et ſi le cercle, qui ayant ſon centre au point N, paſſe par le point L, ne couppe ny ne touche la ligne droite M Q R, il n'y a aucune racine en l'Equation, de façon qu'on peut aſſurer que la conſtruction du probleſme propoſé eſt impoſſible.

Au

Symbolism established in algebra

Frontispiece to: Johannes Faulhaber, *Ingenieurs-Schul, Anderer Theil*, Ulm, 1633 (on fortification)

See: Volker Remmert, 'Antiquity, nobility, and utility: picturing the Early Modern mathematical sciences', in *The Oxford handbook of the history of mathematics* (Eleanor Robson & Jacqueline Stedall, eds.), OUP, 2009, pp. 537–563

‘Analysis’ vs ‘synthesis’

Viète (and others) sought to ‘restore’ ancient Greek mathematical ideas — in particular, those found in the recently rediscovered *Collection* (or *Synagoge*: Συναγωγοή) of Pappus of Alexandria (4th century AD) [published in Latin by Federico Commandino in 1588]

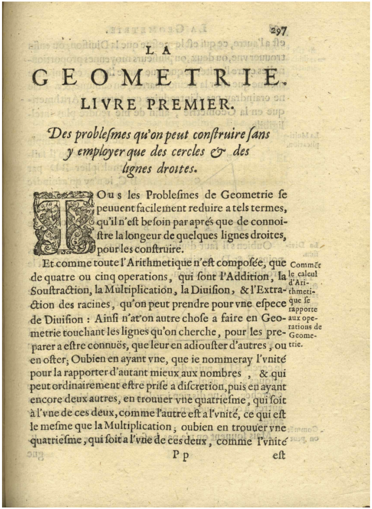
Book VII of Pappus’s *Collection* outlines the methods of analysis and synthesis:

Synthesis: starting from what is known, we make a sequence of deductions until we arrive at what is sought (constructive method, as, e.g., in Euclid’s *Elements*)

Analysis: starting from what is sought, as if it has already been established, we work backwards until we arrive at what is known (method of discovery or problem-solving, preliminary to synthesis)

“Analysis was thus the working tool of the geometer, but it was with synthesis that one could demonstrate things in an indisputable way.” (Niccolò Guicciardini, ‘Analysis and synthesis in Newton’s mathematical work’, *The Cambridge Companion to Newton* (ed. I. Bernard Cohen and George E. Smith), CUP, 2002, pp. 308–328 at p. 308)

Analytic (algebraic) geometry



LA GEOMETRIE. 297

LA

GEOMETRIE.

LIVRE PREMIER.

Des probleſmes qu'on peut conſtruire ſans y employer que des cercles & des lignes droites.

Tous les Probleſmes de Geometrie ſe peuuent facilement reduire a tels termes, qu'il n'eſt beſoin par aprés que de connoiſtre la longeur de quelques lignes droites, pour les conſtruire.

Et comme toute l'Arithmetique n'eſt compoſée, que de quatre ou cinq operations, qui ſont l'Addition, la Souſtraction, la Multiplication, la Diuiſion, & l'Extraction des racines, qu'on peut prendre pour vne eſpece de Diuiſion: Ainſi n'a-t'on autre choſe a faire en Geometrie touchant les lignes qu'on cherche, pour les preparer a eſtre connuës, que leur en adiouſter d'autres, ou en oſter; Oubien en ayant vne, que ie nommeray l'vnité pour la rapporter d'autant mieux aux nombres, & qui peut ordinairement eſtre priſe a diſcretion, puis en ayant encore deux autres, en trouuer vne quatrieſme, qui ſoit à l'vne de ces deux, comme l'autre eſt a l'vnité, ce qui eſt le meſme que la Multiplication; oubien en trouuer vne quatrieſme, qui ſoit a l'vne de ces deux, comme l'vnité

Comment le calcul d'Arithmetique ſe rapporte aux operations de Geometrie.

Pp   eſt

La géométrie (1637)

Solution of geometric problems by algebraic methods

Appendix to
Discours de la méthode

"by commencing with objects the simplest and easiest to know, I might ascend by little and little"

Descartes' analytic geometry

We may label lines (line segments) with letters a, b, c, ...

Then $a + b$, $a - b$, ab, a/b, $\sqrt{a}$ may be constructed by ruler and compass.

Descartes' method

- represent all lines by letters
- use the conditions of the problem to form equations
- reduce the equations to a single equation
- solve
- construct the solution geometrically

For examples, see Katz (3rd ed.), §14.2

Algebraic methods in geometry: some objections

Pierre de Fermat (1656, France):

> *I do not know why he has preferred this method with algebraic notation to the older way which is both more convincing and more elegant ...*

Thomas Hobbes (1656, England):

> *... a scab of symbols ...*

The beginnings of calculus: tangent methods

Calculus:

- finding tangents;
- finding areas.

Descartes' method for finding tangents (1637)

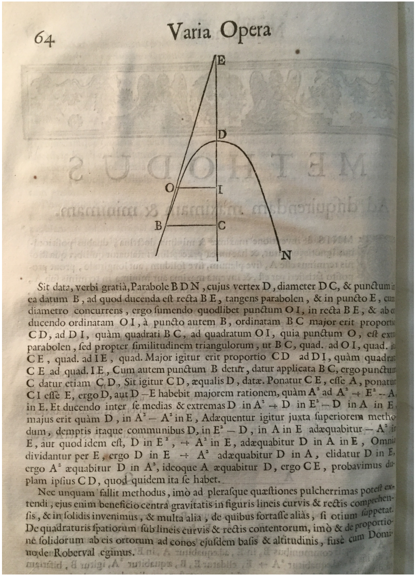
- based on finding a circle that touches the curve at the given point — a tangent to the circle is then a tangent to the curve
- used his algebraic approach geometry to find double roots to equation of intersection
- was in principle a general method — but laborious

Fermat's method for finding tangents

Pierre de Fermat (1601–1665):

- steeped in classical mathematics
- like Descartes, investigated problems of Pappus
- devised a tangent method (1629) quite different from that of Descartes

Fermat's tangent method (1629)



64 Varia Opera

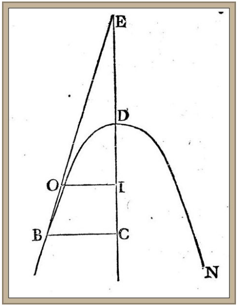
Sit data, verbi gratià, Parabole B D N, cujus vertex D, diameter D C, & punctum in ea datum B, ad quod ducenda eſt recta B E, tangens parabolen, & in puncto E, cum diametro concurrens, ergo ſumendo quodlibet punctum O I, in recta B E, & ab eo ducendo ordinatam O I, à puncto autem B, ordinatam B C major erit proportio C D, ad D I, quàm quadrati B C, ad quadratum O I, quia punctum O, eſt extra parabolen, ſed propter ſimilitudinem triangulorum, ut B C, quad. ad O I, quad. ita C E, quad. ad I E, quad. Major igitur erit proportio C D ad D I, quàm quadrati C E ad quad. I E, Cum autem punctum B detur, datur applicata B C, ergo punctum C datur etiam C D, Sit igitur C D, æqualis D, datæ. Ponatur C E, eſſe A, ponatur C I eſſe E, ergo D, aut D – E habebit majorem rationem, quàm A^2 ad A^2 + E^2 -- A, in E. Et ducendo inter ſe medias & extremas D in A^2 + D in E^2 – D in A in E, majus erit quàm D, in A^2 – A^2 in E, Adæquentur igitur juxta ſuperiorem methodum, demptis itaque communibus D, in E^2 – D, in A in E adæquabitur – A^2 in E, aut quod idem eſt, D in E^2, + A^2 in E, adæquabitur D in A in E, Omnia dividantur per E, ergo D in E + A^2 adæquabitur D in A, elidatur D in E, ergo A^2 æquabitur D in A^2, ideoque A æquabitur D, ergo C E, probavimus duplam ipſius C D, quod quidem ita ſe habet.

Nec unquam fallit methodus, imò ad pleraſque quæſtiones pulcherrimas poteſt extendi, ejus enim beneficio centra gravitatis in figuris lineis curvis & rectis comprehenſis, & in ſolidis invenimus, & multa alia, de quibus fortaſſe aliàs, ſi otium ſuppetat. De quadraturis ſpatiorum ſub lineis curvis & rectis contentorum, imò & de proportione ſolidorum ab eis ortorum ad conos ejuſdem baſis & altitudinis, fusè cum Domino de Roberval egimus.

Worked out c. 1629, but only published posthumously in *Varia opera mathematica*, 1679.

See *Mathematics emerging*, §3.1.1.

Fermat's tangent method (1629)



Choose an arbitrary point B on the parabola.

Suppose that the tangent at B exists, and that it crosses the axis of the parabola at E.

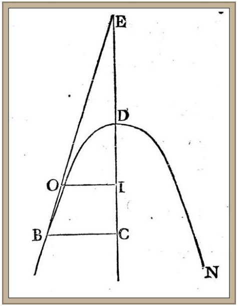
Choose any point O on the line BE.

Draw horizontals OI and BC.

Since O is outside the parabola, we have

$$\frac{CD}{DI} > \frac{(BC)^2}{(OI)^2}.$$

Fermat's tangent method (1629)



Since O is outside the parabola, we have

$$\frac{CD}{DI} > \frac{(BC)^2}{(OI)^2}.$$

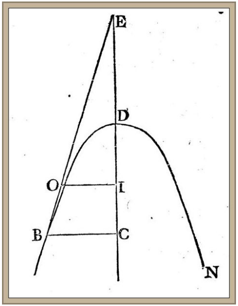
By similarity of triangles,

$$\frac{(BC)^2}{(OI)^2} = \frac{(CE)^2}{(IE)^2}.$$

Therefore

$$\frac{CD}{DI} > \frac{(CE)^2}{(IE)^2}.$$

Fermat's tangent method (1629)



Therefore

$$\frac{CD}{DI} > \frac{(CE)^2}{(IE)^2}.$$

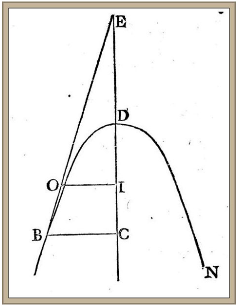
Put $CD = d$, $CE = a$, $CI = e$, so that

$$\frac{d}{d-e} > \frac{a^2}{(a-e)^2}.$$

Now (Fermat says), we obtain equality as e decreases (as OI becomes BC):

$$\frac{d}{d-e} = \frac{a^2}{(a-e)^2}.$$

Fermat's tangent method (1629)



We solve the equality

$$\frac{d}{d-e} = \frac{a^2}{(a-e)^2}.$$

Rearranging gives $de^2 + a^2e = 2ade.$

Cancel e: $de + a^2 = 2ad.$

Now e will be small, so we can neglect it, leaving us with $a^2 = 2ad.$

Hence $a = 2d.$

Or $CE = 2 \times CD.$