

BO1.1 History of Mathematics

Lecture IV

The beginnings of calculus, part 2: quadrature

MT24 Week 2

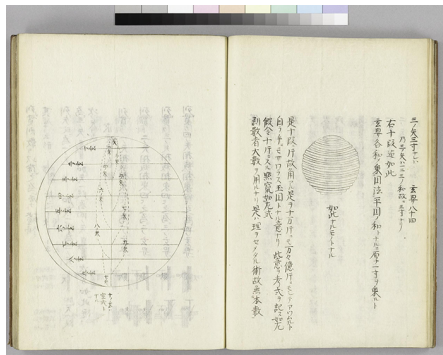
Summary

- ▶ *Enri*: a non-Western prelude
- ▶ Quadrature (finding areas)
- ▶ Indivisibles
- ▶ Infinitesimals
- ▶ The contributions of Newton & Leibniz

Seki and *enri*

The development of calculus is a largely Western story, but similar ideas did appear elsewhere

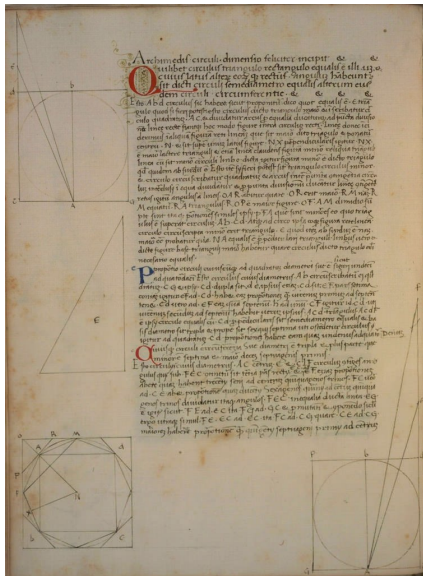
For example, in the late 17th century, Seki Takakazu and his school developed *enri* 円理 ('circle principles'), which concerned the calculation of arc lengths, areas, and volumes



One result of *enri* was the determination of the volume of a sphere via 'slicing'

But *enri* was much narrower in scope than calculus

Archimedes: Κύκλου μέτρησις (Measurement of a circle)

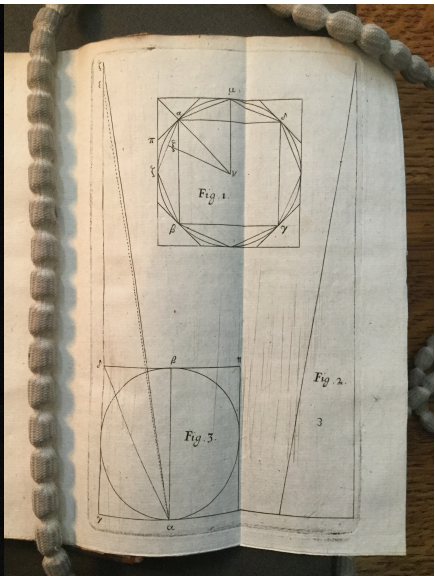
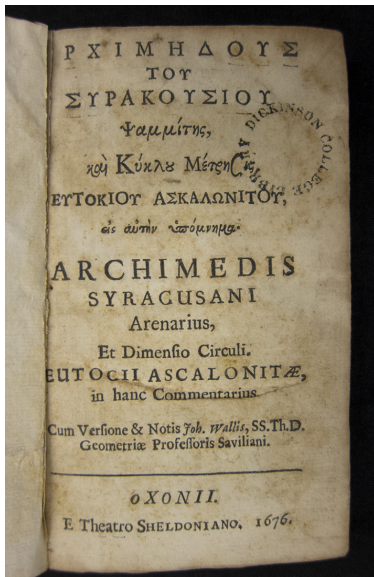


Translated into Latin as *Dimensio circoli* by Jacobus Cremonensis, c. 1450–1460

Illustrated by Piero della Francesca

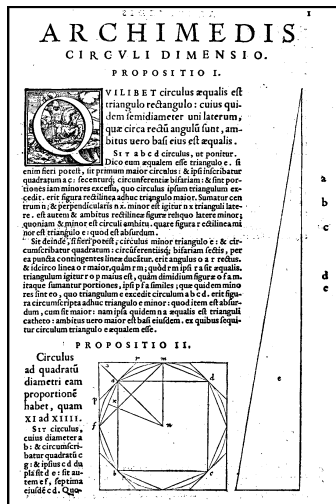
Available online with other texts by Archimedes

Archimedes: Κύκλου μέτρησις (Measurement of a circle)



Edition by John Wallis, Oxford, 1676

Archimedes: Κύκλου μέτρησις (*Measurement of a circle*)



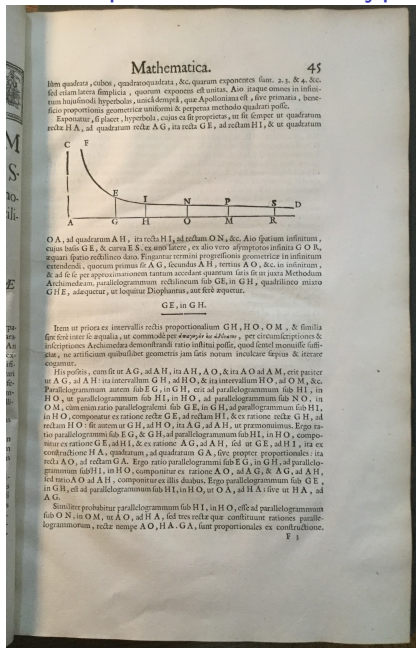
A circle is equal to a right-angled triangle with height equal to the circumference of the circle and base equal to the radius.

Proof by exhaustion and double contradiction

Later: the ratio of the circumference to the diameter is greater than $3\frac{10}{71}$ and less than $3\frac{1}{7}$.

(*Archimedis opera*, edited by Commandino, 1558) — see *Mathematics emerging*, §1.2.3

Fermat's quadrature of a hyperbola (c. 1636)



Worked out c. 1636, but only published posthumously in *Varia opera mathematica*, 1679.

In modern terms, this is the curve described by $y = \frac{1}{x^2}$.

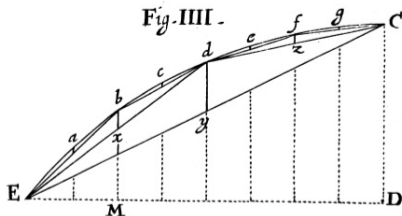
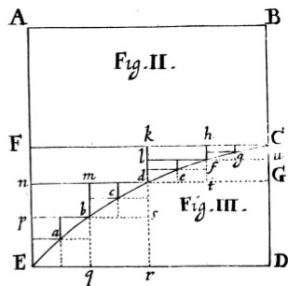
See *Mathematics emerging*, §3.2.1.

The rectangular (or 'Apollonian') hyperbola

In modern notation, $y = \frac{1}{x}$

- ▶ Quadrature evaded Fermat
- ▶ Partial results obtained by Grégoire de Saint Vincent, c. 1625, published in *Opus geometricum*, 1647
- ▶ Empirical observation that if $A(x)$ is the area under the hyperbola from 1 to x , then $A(\alpha\beta) = A(\alpha) + A(\beta)$ (cf. logarithms)
- ▶ Problem solved in early 1650s by William Brouncker; published in 1668 in volume 3 of *Philosophical Transactions of the Royal Society*

Brouncker's quadrature of the hyperbola (1668)



To put this into modern terms, take A as the origin, and AB , AE as the x - and y -axes, respectively. Then the diagram represents the area under $\frac{1}{1+x}$ from $x = 0$ to $x = 1$.

(See *Mathematics emerging*, §3.2.2.)

(645)

Numb. 34.

PHILOSOPHICAL TRANSACTIONS.

Monday, April 13, 1668

The Contents.

The Squaring of the Hyperbola by an infinite series of Rational Numbers, together with its Demonstration, by the Right Honourable the Lord Viscount Brouncker. An Extract of a Letter sent from Danzick, touching some Chymical, Medicinal and Anatomical particulars. Two Letters, written by Dr. John Wallis to the Publisher; One, concerning the Variety of the Annual High-Tides in respect to several places: the other, concerning some Mistakes of a Book entitled SPECIMINA MATHEMATICÆ Francisci Dulaurens, especially touching a certain Probleme, affirm'd to have been proposed by Dr. Wallis to the Mathematicians of all Europe, for a solution. An Account of some Observations concerning the true Time of the Tydes, by Mr. Hen. Philips. An Account of three Books: I. W. SENGWERMUS PH.D. de Tarantula, II. REGNERI de GRAEF M.D. Epistola de nonnullis circa Partes Genitales Inventis Novis, III. JOHANNIS van HORNE M.D. Observationum suarum circa Partes Genitales in utroque sexu, PRODROMUS.

The Squaring of the Hyperbola, by an infinite series of Rational Numbers, together with its Demonstration, by that Eminent Mathematician, the Right Honourable the Lord Viscount Brouncker.

What the Acute Dr. John Wallis had intimated, some years since, in the Dedication of his Answer to M. Meibomius de proportionibus, vid. That the World one day would learn from the Noble Lord Brouncker, the Quadrature of the Hyperbole; the Ingenious Reader may see performed in the subjoynd operation, which its Excellent Author w's now pleas'd to communicate, as followeth in his own words;

Z z z

M v

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My Method for Squaring the Hyperbola is this:

Let AB be one Asymptote of the Hyperbola E d C; and let AE and BC be parallel to th'other: Let also AE be to BC as 2 to 1; and let the Parallelogram ABDE equal 1. See Fig. 1. And note, that the Lett. x every where stands for Multiplication.

Supposing the Reader knows, that EA. a. z. KH. g. v. d. f. y. x. s. a. e. p. C. B. &c. are in an Harmonic series, or a serie reciproca primariorum seu arithmetice proportionatum (otherwise he is refer'd for satisfaction to the 87, 88, 89, 90, 91, 92, 93, 94, 95, prop. Arithm. Infinitor. & allis):

$$\left. \begin{aligned} \text{I say } ABCdEA &= \frac{1}{1 \times 2} + \frac{1}{3 \times 4} + \frac{1}{5 \times 6} + \frac{1}{7 \times 8} + \frac{1}{9 \times 10} \&c. \\ EdCDE &= \frac{1}{2 \times 3} + \frac{1}{4 \times 5} + \frac{1}{6 \times 7} + \frac{1}{8 \times 9} + \frac{1}{10 \times 11} \&c. \\ EdCyE &= \frac{1}{2 \times 3 \times 4} + \frac{1}{4 \times 5 \times 6} + \frac{1}{6 \times 7 \times 8} + \frac{1}{8 \times 9 \times 10} \&c. \end{aligned} \right\} \text{in infinitum.}$$

For (in Fig. 2 &c.) the Parallelog.

And (in Fig. 4.) the Triangl.

$CA = \frac{1}{1 \times 2}$	$EdC = \frac{1}{2 \times 3 \times 4} = \frac{EdD - EdF}{2}$	Note. CA = dD + dF dD = b r + b n dF = f G + f k b r = a q + a p b n = c s + c m f G = e t + e l f k = g u + g h e c.
$dD = \frac{1}{2 \times 3}$ $dF = \frac{1}{3 \times 4}$	$Ebd = \frac{1}{4 \times 5 \times 6} = \frac{Edr - Edn}{2}$	
$br = \frac{1}{4 \times 5}$ $bn = \frac{1}{5 \times 6}$	$d f C = \frac{1}{6 \times 7 \times 8} = \frac{Edg - Edk}{2}$	
$fG = \frac{1}{6 \times 7}$ $f k = \frac{1}{7 \times 8}$	$Eab = \frac{1}{8 \times 9 \times 10} = \frac{Eaq - Eap}{2}$	
$a q = \frac{1}{8 \times 9}$ $a p = \frac{1}{9 \times 10}$	$bed = \frac{1}{10 \times 11 \times 12} = \frac{Ecs - Ecm}{2}$	
$cs = \frac{1}{10 \times 11}$ $cm = \frac{1}{11 \times 12}$	$def = \frac{1}{12 \times 13 \times 14} = \frac{Edt - Edl}{2}$	
$et = \frac{1}{12 \times 13}$ $el = \frac{1}{13 \times 14}$	$fGC = \frac{1}{14 \times 15 \times 16} = \frac{Egu - Egh}{2}$	
$gu = \frac{1}{14 \times 15}$ $gh = \frac{1}{15 \times 16}$	e c.	

And

New methods: indivisibles and infinitesimals

Indivisibles: geometric objects making up a higher-dimensional object (e.g., points $\rightarrow$ line, lines $\rightarrow$ plane)

Infinitesimal: arbitrarily small but nonzero quantity

But distinction often blurred

During the 17th century, both concepts saw much use — despite the fact that they appeared to contradict Euclidean principles

Indivisibles

Early treatments by de Saint Vincent in c. 1623 (but not published until 1647) and Roberval in c. 1628–34 (but not published until 1693).

First **published** treatment by Bonaventura Cavalieri (1598–1647) in *Geometria indivisibilibus continuorum nova quadam ratione promota* [*Geometry advanced in a new way by the indivisibles of the continua*] (1635).

Used by Evangelista Torricelli (1608–1647) in 1644 to calculate the volume of an infinite hyperboloid of revolution.

Developed by John Wallis (1616–1703) and others.

Cavalieri's *Geometria*

GEOMETRIÆ CAVALERII

LIBER SECVNDVS.

In quo de Triangulo præcipuè, & Parallelogrammo, ac Solidis ab eisdem generis plura demonstrantur, necnon alia quedam Propositiones lemmaticæ pro sequentibus Libris ostenduntur.

DEFINITIONES.

I.



Super oppositas tangentes circuli, & datæ planæ figuræ, ducantur duo plana inuicem parallela, recta, siue inclinata ad planum datæ figuræ, hinc inde indefinitè producta; quorum alterius moueatur, versus reliquum eisdem, semper æquidistans, donec illi congruerit; singulæ rectæ lineæ, quæ in toto motu sunt communes

A

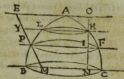
secundo.

LIBER II.

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aliæ scilicet transversæ, qui in tali inclinatione sit.

Intelligamus nunc $\triangle ABC$, esse solidum, cuius duo opposita plana tangentia



sint, quæ transeunt per, EO, BC , moueatur autem adhuc planum, per, EO , extensum, versus planum per, BC , semper illi æquidistans, igitur huius plani moti, siue fluxus et recessus concepti in solido, ABC , figura, quæ in toto motu fieri intelliguntur, voco: Omnia plana solidi, ABC , sumpta regula eorum uno, quarum aliqua representare possunt plana, LH, PF, TC .

Alterius duo rectæ lineæ, ON, EM , occurrant plexis per, EO, BC , transeuntibus iam dictis in punctis, O, N, E, M , quarum, ON , perpendiculariter, EM , vero oblique illi incidat, puncta igitur, quæ sunt communes sectiones omnium planorum solidi, ABC , producuntur, si opus sit. Et rectæ, ON , vocantur ipsius omnia puncta rectis transversis, quarum aliqua sunt puncta, H, I, N , quæ inter ipsa, & extremum punctum, O , continentur, ut ipse, OH, OI, ON , dicuntur abscissæ, quæ inter eadem puncta, & aliud extremum, quod est, N , continentur, ut ipse, NI, NH, NO , residuæ omnium abscissarum; et æquales ipsi, ON , quæ sunt omnes abscissæ, siue residuæ omnium abscissarum, ON , dicuntur maxime abscissarum, siue omnium abscissarum, ON , quibus si adiungatur aliqua recta linea, dicuntur abscissæ, residuæ, siue maxime

Def. Tert.
huius.

Def. Qua.
huius.

Def. Qui.
huius.

Def. Scia.
huius.

Def. A.
huius.

Def. Sept.
huius.

B

aditu.

Def. Sept.
huius.

Torricelli's hyperbolic solid (*Opera geometrica*, 1644)



(See *Mathematics emerging*, §3.3.1.)

John Wallis (1616–1703)

Studied at Emmanuel College,
Cambridge (BA 1637, MA 1640)

1643–1649: scribe for Westminster
Assembly

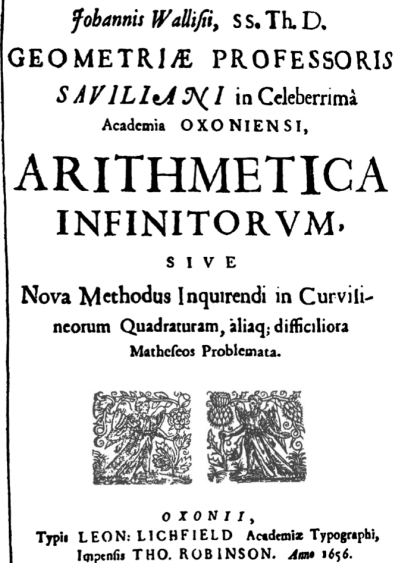
1644–1645: Fellow of Queens'
College, Cambridge

1643–1689: cryptographer to
Parliament, then to the Crown

1649–1703: Savilian Professor of
Geometry in Oxford



Arithmetica infinitorum



John Wallis,
Arithmetica infinitorum
(*The arithmetic of infinitesimals*)
Oxford, 1656

Translation by
Jacqueline A. Stedall
Springer, 2004

Arithmetica infinitorum

- ▶ **Arithmetical** methods rather than **geometrical**, but repeatedly appealed to geometry for justification
- ▶ Investigation of sums of sequences of powers (or ratios of these to a known fixed quantity) — usually decreasing
- ▶ Fixed an endpoint, dividing interval into infinite number of arbitrarily small subintervals — these are the ‘infinitesimals’ of Wallis’ title

Wallis and indivisibles

2

Arithmetica Infinitorum.

Prop. 2, 3.

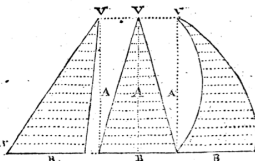
PROP. II. *Theorema.*

Si sumatur series quantitatum Arithmetice proportionalium (sive juxta naturalem numerorum consecutionem) continue crescentium, a puncto vel o inchoatarum, & numero quidem vel finitarum vel infinitarum (nulla enim discriminis causa erit,) erit illa ad seriem totidem maximæ æqualium, ut 1 ad 2.

Nempe, si primus terminus sit o, secundus 1, (nam si secus, moderatio adhibenda erit,) & ultimus l erit summa $\frac{l+1}{2} l$. (erit enim, eo casu, numerus terminorum $l+1$.) vel, (posito numero terminorum a , quantuscunq; sit terminus secundus) $\frac{a+1}{2} a$.

PROP. III. *Corollarium.*

Ergo, Triangulum ad Parallelogrammum (super æquali base, æque altum,) est ut 1 ad 2.



Triangulum

For the triangle . . . consists of an infinite number of parallel lines in arithmetic proportion . . .

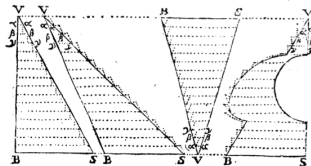
(See Mathematics emerging, §2.4.2.)

Wallis and indivisibles?

Prop. 14. *Arithmetica Infinitorum.*

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ralis initium. Quamvis enim Sectorum illorum numero infinitorum aggregatum, ipsi figuræ lineis recta & Spirali terminatæ, (juxta methodum Indivisibilium) æquale ponatur; non tamen illud de omnium Arcubus cum ipsa Spirali (proprie dicta) comparatis obtinebit. Tantundem enim esset, ac si quis, dum infinita numero parallelogramma triangulo inscripta (aut etiam circumscripta) toti triangulo VBS æqualia videat, inde



concluderet eorum omnium latera rectæ VS adjacentia (rectæ VB parallela) ipsi VS simul æqualia esse, vel quæ rectæ VB adjacent (ipsi VS parallela) æqualia simul esse toti VB. (Quod si quando verum esse contingat, puta in triangulo isosceli, non tamen id universaliter concludendum erit.) Atque hoc quidem eo potius admonendum duxi, quod viderim etiam viros doctos nonnunquam speciosa ejusmodi verisimilitudine in lapsum proclives esse. Cur autem omissa Spirali genuina, spuriam hanc peripheriæ comparaverim; causa est, quod huic possum, non autem illi, æqualem peripheriam assignare.

PROP. XIV. *Corollarium.*

ET propterea etiam segmenta spiralis, a principio spiralis exorsa, sunt ad rectas conterminas, sicut Parabolæ Diametri interceptæ, ad ordinatim-applicatas.

D d 2

Nempe

For it amounts to the same thing as if, when an infinite number of parallelograms are inscribed in (or circumscribed about) a triangle, it seems that they equal to complete triangle ...

(See *Mathematics emerging*, §2.4.2.)

Sums of powers

Wallis' method depended upon the summation rule

$$\sum_{a=0}^A a^n \approx \frac{A^{n+1}}{n+1}$$

A version of this was developed by Ibn al-Haytham (Alhazen) in 11th-century Egypt, and it was certainly known to Fermat, Roberval, and Cavalieri in the 1630s for positive integers n , but in the 1650s Wallis extended it to negative and fractional n .

See: [Victor J. Katz, 'Ideas of calculus in Islam and India', *Mathematics Magazine* 68\(3\) \(1995\), 163–174](#)

Simple 'integrals'

Using the summation rule we can 'integrate'

$$x^2, \quad x^3, \quad \dots, \quad x^{1/3}, \quad \dots, \quad x^{-4}, \quad \dots$$

and

$$(1+x)^3 \quad \text{or} \quad (1+x^2)^5 \quad \text{or} \quad \dots$$

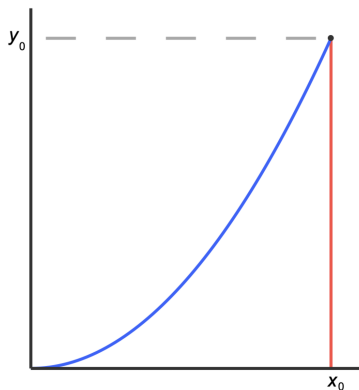
but what about

$$(1-x^2)^{1/2} \quad [\text{for a circle}]$$

or

$$(1+x)^{-1} \quad [\text{for a hyperbola}] ?$$

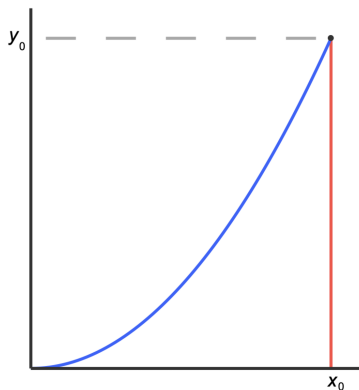
Wallis and the quadrature of a parabola



Wallis sought the area under the parabola $y = x^2$ between $x = 0$ and $x = x_0$

He used the language of ratio, hence sought to calculate the ratio of the area A under the curve to that of the corresponding rectangle $(x_0 y_0)$, which we may think of as the fraction $\frac{A}{x_0 y_0}$

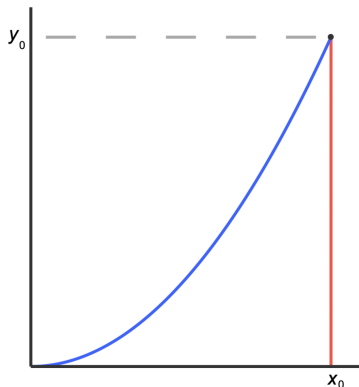
Wallis and the quadrature of a parabola



Wallis considered an area to be the sum of the lengths of the lines contained within it (makes sense?), so

- ▶ A is the sum of the values of x^2 as x ranges from 0 to x_0
- ▶ $x_0 y_0$ is the sum of as many copies of x_0^2 (?)

Wallis and the quadrature of a parabola



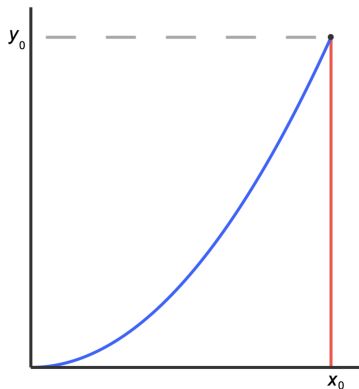
Break $(0, x_0)$ into n subintervals, suppose that x only takes the values at the endpoints of these, and consider the ratio

$$R = \frac{0^2 + 1^2 + 2^2 + \cdots + n^2}{n^2 + n^2 + n^2 + \cdots + n^2}$$

As we make n larger, this ratio will become a closer approximation to $\frac{A}{x_0 y_0}$

[Note that we are deliberately avoiding the terminology of **limits**, and that some x_0^2 s have been cancelled, thanks to the use of ratios]

Wallis and the quadrature of a parabola

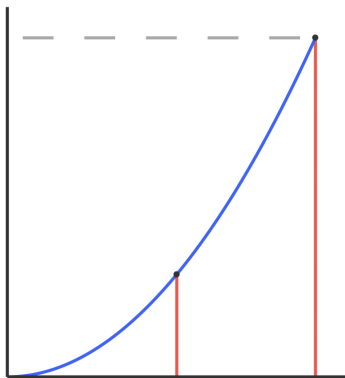


Wallis investigated the cases of small n

For $n = 1$ (one red line),

$$R = \frac{0^2 + 1^2}{1^2 + 1^2} = \frac{1}{2} = \frac{1}{3} + \frac{1}{6}$$

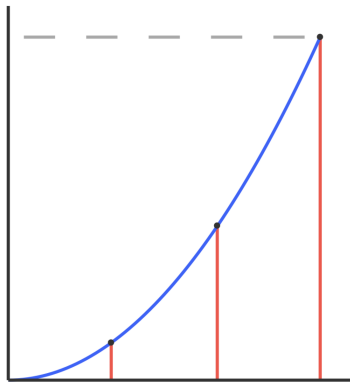
Wallis and the quadrature of a parabola



For $n = 2$ (two red lines),

$$R = \frac{0^2 + 1^2 + 2^2}{2^2 + 2^2 + 2^2} = \frac{5}{12} = \frac{1}{3} + \frac{1}{12}$$

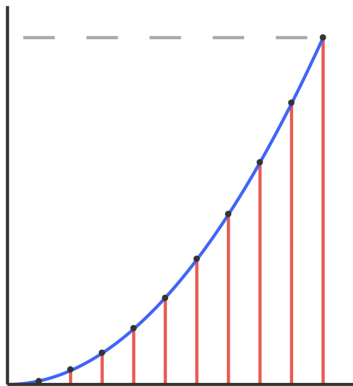
Wallis and the quadrature of a parabola



For $n = 3$ (three red lines),

$$\begin{aligned} R &= \frac{0^2 + 1^2 + 2^2 + 3^2}{3^2 + 3^2 + 3^2 + 3^2} \\ &= \frac{14}{36} = \frac{1}{3} + \frac{1}{18} \end{aligned}$$

Wallis and the quadrature of a parabola



So as n increases $\frac{A}{x_0 y_0}$ approaches $\frac{1}{3}$, hence $A = \frac{1}{3}x_0^3$, as we'd expect

Wallis called this method of spotting and extending a pattern 'induction' — it was criticised at the time (for example, by Pascal)

Enter Newton...

In his own words:

In the winter between the years 1664 and 1665 upon reading Dr Wallis's Arithmetica infinitorum and trying to interpolate his progressions for squaring the circle, I found out first an infinite series for squaring the circle and then another infinite series for squaring the Hyperbola ...

Newton extended Wallis' method of interpolation...

Newton's integration of $(1+x)^{-1}$

	$(1+x)^{-1}$	$(1+x)^0$	$(1+x)^1$	$(1+x)^2$	$(1+x)^3$	$(1+x)^4$	...
x	?	1	1	1	1	1	...
$\frac{x^2}{2}$	?	0	1	2	3	4	...
$\frac{x^3}{3}$	?	0	0	1	3	6	...
$\frac{x^4}{4}$	?	0	0	0	1	4	...
$\frac{x^5}{5}$	?	0	0	0	0	1	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

The entry in the row labelled $\frac{x^m}{m}$ and the column labelled $(1+x)^n$ is the coefficient of $\frac{x^m}{m}$ in

$\int(1+x)^n dx$. (NB. Newton did **not** use the notation $\int(1+x)^n dx$.)

Newton's integration of $(1+x)^{-1}$

	$(1+x)^{-1}$	$(1+x)^0$	$(1+x)^1$	$(1+x)^2$	$(1+x)^3$	$(1+x)^4$	...
x	1	1	1	1	1	1	...
$\frac{x^2}{2}$	-1	0	1	2	3	4	...
$\frac{x^3}{3}$	1	0	0	1	3	6	...
$\frac{x^4}{4}$	-1	0	0	0	1	4	...
$\frac{x^5}{5}$	1	0	0	0	0	1	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

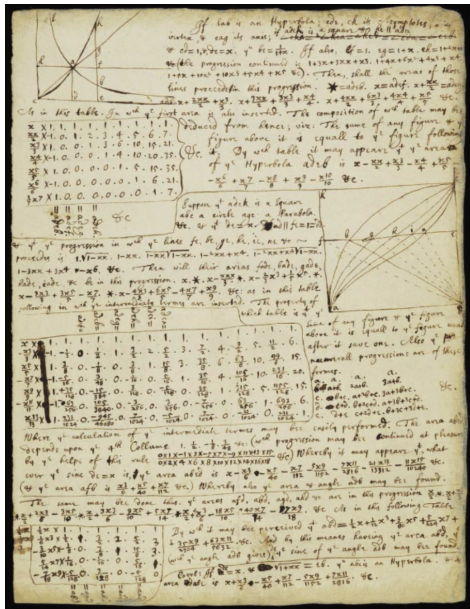
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The general binomial theorem

CUL Add. MS 3958.3, f. 72

See *Mathematics emerging*,
§8.1.1



Newton's calculus: 1664–5

- ▶ rules for quadrature (influenced by Wallis's ideas of interpolation)
- ▶ rules for tangents (influenced by Descartes' double root method)
- ▶ recognition that these are inverse processes

Newton's vocabulary and notation

Newton's calculus 1664–5:

- ▶ fluents $x, y, \dots$ (quantities that vary with time t)
- ▶ fluxions $\dot{x}, \dot{y}, \dots$ (rate of change of those quantities)
- ▶ moments o (infinitesimal time in which x increases by $\dot{x}o$)

Newton's calculus in action (*The method of fluxions*, 1736)

12. Ex. 5. As if the Equation $xx + axz - y^2 = 0$ were propos'd to express the Relation between x and y , as also $\sqrt{ax - xx} = BD$, for determining a Curve, which therefore will be a Circle. The Equation $xx + axz - y^2 = 0$, as before, will give $2xz + ax\dot{x} + ax\dot{z} - 2y\dot{y} = 0$, for the Relation of the Celerities $\dot{x}$, $\dot{y}$, and $\dot{z}$. And therefore since it is $z = x \times BD$ or $z = \sqrt{ax - xx}$, substitute this Value instead of it, and there will arise the Equation $2xz + ax\dot{x} \sqrt{ax - xx} + ax\dot{z} - 2y\dot{y} = 0$, which determines the Relation of the Celerities $\dot{x}$ and $\dot{y}$.

DEMONSTRATION of the Solution.

13. The Moments of flowing Quantities, (that is, their indefinitely small Parts, by the accession of which, in indefinitely small portions of Time, they are continually increas'd,) are as the Velocities of their Flowing or Increasing.

14. Wherefore if the Moment of any one, as x , be represented by the Product of its Celerity $\dot{x}$ into an indefinitely small Quantity o (that is, by $\dot{x}o$), the Moments of the others y , z , will be represented by $\dot{y}o$, $\dot{z}o$; because $\dot{y}o$, $\dot{x}o$, $\dot{z}o$, and zo , are to each other as $\dot{y}$, $\dot{x}$, $\dot{z}$, and z .

15. Now since the Moments, as $\dot{x}o$ and $\dot{y}o$, are the indefinitely little accessions of the flowing Quantities x and y , by which those Quantities are increas'd through the several indefinitely little intervals of Time; it follows, that those Quantities x and y , after any indefinitely small interval of Time, become $x + \dot{x}o$ and $y + \dot{y}o$. And therefore the Equation, which at all times indifferently expresses the Relation of the flowing Quantities, will as well express the Relation between $x + \dot{x}o$ and $y + \dot{y}o$, as between x and y : So that $x + \dot{x}o$ and $y + \dot{y}o$ may be substituted in the same Equation for those Quantities, instead of x and y .

16. Therefore let any Equation $x^2 - ax^2 + axy - y^2 = 0$ be given, and substitute $x + \dot{x}o$ for x , and $y + \dot{y}o$ for y , and there will arise

$$\left. \begin{aligned} &x^2 + 3x\dot{x}o + 3\dot{x}^2oo + \dot{x}^3o^3 \\ &- ax^2 - 2ax\dot{x}o - ax^2oo \\ &+ axy + ax\dot{y}o + a\dot{y}xx + a\dot{x}yoo \\ &- y^2 - 3y\dot{y}o - 3\dot{y}^2oo - y^3o^3 \end{aligned} \right\} = 0.$$

17.

17. Now by Supposition $x^2 - ax^2 + axy - y^2 = 0$, which therefore being expunged, and the remaining Terms being divided by o , there will remain $3x\dot{x} + 3\dot{x}^2\dot{x}o + \dot{x}^3\dot{o} - 2ax\dot{x} - ax^2o + a\dot{y}x + a\dot{x}y + a\dot{y}oo - 3y\dot{y} - 3\dot{y}^2\dot{o} - y^3\dot{o} = 0$. But whereas o is suppos'd to be infinitely little, that it may represent the Moments of Quantities; the Terms that are multiply'd by it will be nothing in respect of the rest. Therefore I reject them, and there remains $3x\dot{x} - 2ax\dot{x} + axy + a\dot{y}x - 3y\dot{y} = 0$, as above in Examp. 1.

18. Here we may observe, that the Terms that are not multiply'd by o will always vanish, as also those Terms that are multiply'd by o of more than one Dimension. And that the rest of the Terms being divided by o , will always acquire the form that they ought to have by the foregoing Rule: Which was the thing to be proved.

19. And this being now shewn, the other things included in the Rule will easily follow. As that in the propos'd Equation several flowing Quantities may be involved; and that the Terms may be multiply'd, not only by the Number of the Dimensions of the flowing Quantities, but also by any other Arithmetical Progressions; so that in the Operation there may be the same difference of the Terms according to any of the flowing Quantities, and the Progression be dispos'd according to the same order of the Dimensions of each of them. And these things being allow'd, what is taught besides in Examp. 3, 4, and 5, will be plain enough of itself.

PROB. II.

An Equation being propos'd, including the Fluxions of Quantities, to find the Relations of those Quantities to one another.

A PARTICULAR SOLUTION.

1. As this Problem is the Converse of the foregoing, it must be solved by proceeding in a contrary manner. That is, the Terms multiply'd by $\dot{x}$ being dispos'd according to the Dimensions of $\dot{x}$, they must be divided by $\frac{\dot{x}}{x}$, and then by the number of their Dimensions, or perhaps by some other Arithmetical Progression. Then the same work must be repeated with the Terms multiply'd by $\dot{y}$, $\dot{z}$, or

E

*See the Analysis des
infiniment petites of
the M. de l'Hospital.*

Newton's calculus in action (*The method of fluxions*, 1736)

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$$\left. \begin{aligned} x^2 &+ 2x\dot{x}o + 3x^2o\dot{x} + \dot{x}^2o^2 \\ - ax^2 &- 2a\dot{x}xo - a\dot{x}^2o \\ + axy &+ a\dot{x}y + a\dot{y}xo + a\dot{x}y\dot{y}o \\ - y^2 &- 2y\dot{y}o - 2y\dot{y}o\dot{y} - \dot{y}^2o^2 \end{aligned} \right\} = 0.$$

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Leibniz's calculus

Independently, ten years later than Newton...

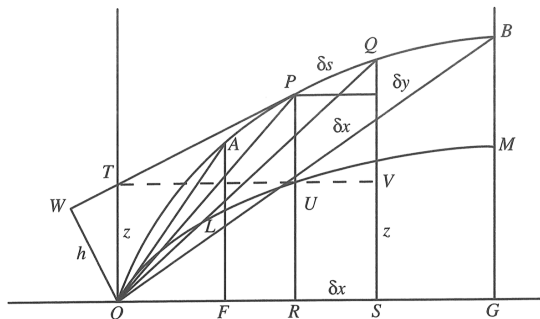
Leibniz's calculus, 1673–76:

- ▶ rules for quadrature — especially the transformation theorem (a.k.a. the transmutation theorem)
- ▶ rules for tangents — by characteristic (or differential) triangles
- ▶ recognition that these are inverse processes

Differentials: du , dv ;

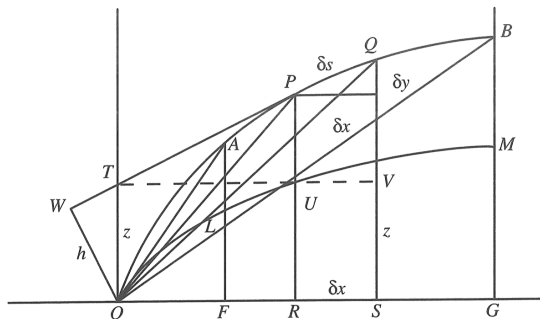
integrals: $\int$, later between S and I

Leibniz's Transmutation Theorem



$$OABG = \sum RPQS = \frac{1}{2} OG \cdot GB + \sum OPQ$$

Leibniz's Transmutation Theorem



$$\begin{aligned} OABG &= \sum RPQS = \frac{1}{2} OG \cdot GB + \sum OPQ \\ &= \frac{1}{2} OG \cdot GB + \frac{1}{2} OLMG \end{aligned}$$

Supplementum Geometriae Dimensoriae ... (1693)

No. IX.

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ACTA ERUDITORUM

publicata Lipsiae,

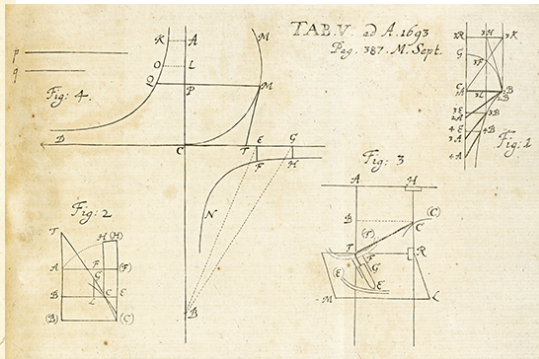
Calendis Septembris, Anno M DC XCIH.

G. G. L. SUPPLEMENTUM GEOMETRIAE Dimensoriae, seu generalissima omnium Tetragonismorum effectio per motum: Similiterque multiplex constructio lineae ex data tangentium conditione.

Dimensiones linearum, superficierum & solidorum plerumque, ut & inventiones centrorum gravitatis, reducuntur ad tetragonismos figurarum planarum; & hinc nascitur Geometria Dimensoria, toto ut sic dicam genere diversa a Determinatrice, quam rectorum tantum magnitudines ingrediuntur, atque hinc quaesita puncta ex punctis datis determinantur. Et Geometria quidem determinatrix reduci potest regulariter ad aequationes Algebraicas, in quibus scilicet incognita ad certum assurgit gradum. Sed dimensionaria sua natura ab Algebra, non pendet; etsi aliquando eveniat (in casu scilicet quadraturarum ordinariarum) ut ad Algebraicas quantitates revocetur; uti Geometria determinatrix ab Arithmetica non pendet; etsi aliquando eveniat (in casu scilicet commensurabilitatis) ut ad numeros seu rationales quantitates revocatur. Unde triplices habemus quantitates: racionales, Algebraicas, & transcendentes. Est autem fons irrationalium Algebraicarum, ambiguitas problematis seu multiplicitas; neque enim possibile foret, plures valores eidem problemati satisfaciens eodem calculo exprimere, nisi per quantitates radicales; ex vero non nisi in casibus specialibus ad rationalitates revocari possunt. Sed fons transcendentium quantitarum est infinitudo. Ita ut Geometria transcendentium (cujus pars dimensionaria est) respondens Analysis, sit ipsissima scientia infiniti. Porro quemadmodum ad construendas quantitates Algebraicas, certi adhibentur motus,

Ccc

motus,



A Supplement to the Geometry of Measurements, or the Most General of all Quadratures to be Effected by a Motion: and likewise the various constructions of a curve from a given condition of the tangent (Acta eruditorum, 1693)

Supplementum Geometriae Dimensoriae ... (1693)

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ACTA ERUDITORUM.

occasione defungi tandem praestet, ne intercedant, & satis diu ista, ultra, Horatiani limitis duplum pressa, Lucinam expectarunt.

Ostendam autem problema generale *Quadraturarum reduci ad inventionem lineae datam habentis legem declivitatem*, sive in qua latera Trianguli characteristici assignabilis datam inter se habeant relationem, deinde ostendam hanc lineam per motum a nobis excogitatum describi posse. Nimirum in omni curva $C(C)$ (*figur. 2*) intelligo *triangulum characteristicum* duplex: assignabile TBC , & inassignabile GLC , similia inter se. Et quidem *inassignabile* comprehendit ipsis GL LC , elementis coordinatarum CB , CF , tanquam crucibus, & GC , elemento arcus, tanquam basi seu hypotenusa. Sed *Assignabile* TBC comprehenditur inter axem, ordinatam, & tangentem, exprimitque adeo angulum, quem directio curvae (seu ejus tangens) ad axem vel basin facit, hoc est curvae declivitatem in proposito puncto C . Sit jam zona quadranda $F(H)$ comprehensa inter curvam $H(H)$, duas rectas parallelas FH & $(F)(H)$ & axem $F(F)$ in hoc Axe sumto puncto fixo A , per A ducatur ad AF normalis AB tanquam axis conjugatus, & in quavis HF (producta prout opus) sumatur punctum C : seu fiat linea nova $C(C)$ cujus haec sit natura, ut ex puncto C ducta ad axem conjugatum AB (si opus productum) tam ordinata conjugata CB , (æquali AF) quam tangente CT , sit portio hujus axis inter eas comprehensa TB , ad BC , ut HF ad constantem a , seu a in BT æquetur rectangulo AFH (circumscripto circa trilineum $AFHA$). His positis ajo rectangulum sub a & sub $E(C)$ (discrimine inter FC & $(F)(C)$ ordinatas curvae) æquari zonæ $F(H)$; adeoque si linea $H(H)$ producta incidat in A , trilineum $AFHA$ figuræ quadrandæ, æquari rectangulo sub a constante, & FC ordinata figuræ quadratricis. Rem noster calculus statim ostendit, sit enim $AF y$; & FH, z ; & BT, t ; & FC, x ; erit $t = zy : a$, ex hypothese: rursus $t = y dx : dy$ ex natura tangentium nostro calculo expressa. Ergo $a dx = z dy$, adeoque $ax = \int x dy = AFHA$. Linea igitur $C(C)$ est *quadratrix* respectu lineæ $H(H)$, cum ipsius $C(C)$ ordinata FC , ducta in a constantem, faciat rectangulum æquale arcæ seu summæ ordinatarum ipsius $H(H)$ ad abscissas debitas AF applicatarum. Hinc cum BT sic ad AF ut FH ad a (ex hypothese) deturque relatio ipsius FH ad AF (naturam exhibens figuræ quadrandæ) dabitur ergo & relatio BT ad

"I shall now show the general problem of quadratures to be reduced to the invention of a line having a given law of declivity"

i.e., integration is reduced to the finding of a curve with a particular tangent — in modern terms, the antiderivative

For Latin-readers: [full paper available online](#)

See also: Niccolò Guicciardini, Newton's method and Leibniz's calculus, *A history of analysis* (ed. Hans Niels Jahnke), AMS/LMS, 2003, pp. 73–103

Newton's calculus and Leibniz's calculus compared

Newton (1664–65):

rules for quadrature
rules for tangents
'fundamental theorem'

dot notation

physical intuition:
rates of change

PROBLEM:
vanishing quantities o

Leibniz (1673–76):

rules for quadrature
rules for tangents
'fundamental theorem'

'modern' notation

algebraic intuition
rules and procedures

PROBLEM:
vanishing quantities $du, dv, \dots$

An elementary introduction to the development of calculus

