B8.3: Mathematical Modelling of Financial Derivatives —Exercises—

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Exercise Sheet 1

The exercises in all sheets draw from previous years (Sam Cohen) and from "The Mathematics of Financial Derivatives —A Student Introduction", by Paul Wilmott, Sam Howison and Jeff Dewynne.

Part A

1. If $dS_t = \mu\, S_t\, dt + \sigma\, S_t\, dW_t$, where S_t denotes the time t price of the stock, dW_t denotes the increments of a standard Wiener process, and μ, σ, A and n are constants, find the stochastic equations satisfied by
(a) $f(S) = A\, S$,
(b) $f(S) = S^n$,
(c) Show that $\mathbb{E}_t[S_T] = S_t\, e^{\mu\,(T-t)}$.

Solution

Use Itô's Lemma to write

$$df = \left(\frac{\partial f}{\partial t} + \mu S \frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}\right) dt + \sigma S \frac{\partial f}{\partial S} dZ. \quad (1)$$

(a) Replacing $f(S) = AS$ into (1) we obtain

$$\begin{aligned} df &= \mu AS dt + \sigma AS dZ \\ \frac{df}{f} &= \mu dt + \sigma dZ. \end{aligned} \quad (2)$$

(b) Replacing $f(S) = S^n$ into (1) we obtain

$$\begin{aligned} df &= \left(n\mu + \frac{\sigma^2}{2}n(n-1)\right) S^n dt + \sigma n S^n dZ \\ \frac{df}{f} &= \hat{\mu} dt + \hat{\sigma} dZ, \end{aligned} \tag{3}$$

where $\hat{\mu} = n\mu + \frac{\sigma^2}{2}n(n-1)$ and $\hat{\sigma} = \sigma n$.

Part B

1. An Ornstein–Uhlenbeck process X satisfies the stochastic differential equation

$$dX_t = \kappa(\theta - X_t)dt + \sigma dW_t$$

where $\kappa > 0$ and $\theta \in \mathbb{R}$.

(a) By using Itô's lemma applied to $e^{\kappa t} X_t$, show that for a given initial value X_0, the value of X_t is given by

$$X_t = \theta + \left((X_0 - \theta) + \sigma \int_0^t e^{\kappa s} dW_s\right) e^{-\kappa t}$$

With $Y_t = e^{\kappa t} X_t$ we apply Itô's lemma to get

$$dY_t = \left(\kappa Y_t + e^{\kappa t}\kappa(\theta - X_t)\right) dt + e^{\kappa t}\sigma dW_t = \kappa e^{\kappa t}\theta dt + e^{\kappa t}\sigma dW_t$$

Integrating we observe

$$e^{\kappa t} X_t = Y_t = Y_0 + \int_0^t \kappa e^{\kappa s}\theta ds + \int_0^t e^{\kappa s}\sigma dW_s = Y_0 + \theta(e^{\kappa t} - 1) + \sigma \int_0^t e^{\kappa s} dW_s$$

and rearrangement gives the stated formula.

(b) Show that this implies that, for any deterministic initial value X_0, X_t has a Gaussian distribution, with mean and variance you should determine.

We know (from lectures) that the integral of a *deterministic* function against a Brownian motion is Gaussian, so in particular

$$\sigma e^{-\kappa t} \int_0^t e^{\kappa s} dW_s \quad \sim \quad N\left(0, \sigma^2 e^{-2\kappa t} \int_0^t e^{2\kappa s} ds\right) = N\left(0, \frac{1 - e^{-2\kappa t}}{2\kappa}\sigma^2\right).$$

Substituting, we get

$$X_t \sim N\left(\theta + (X_0 - \theta)e^{-\kappa t}, \frac{1 - e^{-2\kappa t}}{2\kappa}\sigma^2\right)$$

(c) Calculate $f(x,t) = \mathbb{E}[X_T^2|X_t = x]$, and check explicitly that this is a solution to the corresponding PDE:

$$\frac{\partial f}{\partial t} + \kappa(\theta - x)\frac{\partial f}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2} = 0.$$

Similarly to above, we have

$$X_T|X_t \sim N\Big(\theta + (X_t - \theta)e^{-\kappa(T-t)}, \frac{1-e^{-2\kappa(T-t)}}{2\kappa}\sigma^2\Big).$$

Therefore,

$$\begin{aligned} f(x,t) &= \mathbb{E}[X_T^2|X_t = x] \\ &= \mathbb{E}[X_T|X_t = x]^2 + \mathrm{var}[X_T|X_t = x] \\ &= \Big(\theta + (x-\theta)e^{-\kappa(T-t)}\Big)^2 + \frac{1-e^{-2\kappa(T-t)}}{2\kappa}\sigma^2 \\ &= \theta^2 + 2\theta(x-\theta)e^{-\kappa(T-t)} + (x-\theta)^2 e^{-2\kappa(T-t)} + \frac{1-e^{-2\kappa(T-t)}}{2\kappa}\sigma^2 \end{aligned}$$

Calculating derivatives, we have

$$\begin{aligned} \frac{\partial f}{\partial t} &= 2\kappa\theta(x-\theta)e^{-\kappa(T-t)} + 2\kappa(x-\theta)^2 e^{-2\kappa(T-t)} - e^{-2\kappa(T-t)}\sigma^2 \\ \frac{\partial f}{\partial x} &= 2\theta e^{-\kappa(T-t)} + 2(x-\theta)e^{-2\kappa(T-t)} \\ \frac{\partial^2 f}{\partial x^2} &= 2e^{-2\kappa(T-t)} \end{aligned}$$

and the result follows by substitution.

2. Let the price S_t of an asset satisfy

$$dS_t = \alpha\ (\mu - \ln S_t)\ S_t\, dt + \sigma\, S_t\, dW_t\,, \tag{4}$$

where α and σ are non-negative constants, μ is a constant, and W a standard Brownian motion.

(a) Show that

$$x_T = x_t\, e^{-b\,(T-t)} + \frac{a}{b}\,\left(1 - e^{-b\,(T-t)}\right) + \sigma\, e^{-b\,T} \int_t^T e^{b\,s}\, dW_s\,,$$

where $x_t = \ln S_t$, $a = \alpha\,\hat{\mu}$ (for a choice of $\hat{\mu}$), and $b = \alpha$.

Here, the spot price is mean reverting to the long-term level $\overline{S} = e^{\mu - \frac{\sigma^2}{4\alpha}}$ at a speed given by the mean reversion rate $\alpha \geq 0$, μ is the drift and σ the volatility.

Let $\tilde{\mu} \equiv \alpha\,(\mu - \ln S_t)$ and rewrite (4) as

$$dS_t = \tilde{\mu}\, S_t\, dt + \sigma\, S_t\, dW_t\,. \tag{5}$$

Now, let us write the process followed by the logarithm of the underlying by applying Itô's Lemma to $f = \ln S$.
Itô's Lemma for the SDE defined in (5) reads

$$df = \left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + \tilde{\mu} S \frac{\partial f}{\partial S}\right) dt + \sigma S \frac{\partial f}{\partial S} dW. \tag{6}$$

Substituting $f = \ln S$ in the above, we arrive to

$$df = \left(\tilde{\mu} - \frac{\sigma^2}{2}\right) dt + \sigma dW. \tag{7}$$

Replacing back $\tilde{\mu} \equiv \alpha\,(\mu - \ln S)$ we get

$$\begin{aligned} df &= \left(\alpha\,(\mu - \ln S) - \frac{\sigma^2}{2}\right) dt + \sigma dW \\ &= \alpha\left(\mu - \frac{\sigma^2}{2\alpha} - \ln S\right) dt + \sigma dW. \end{aligned} \tag{8}$$

Finally, calling $x \equiv f = \ln S$ and renaming $\hat{\mu} \equiv \mu - \frac{\sigma^2}{2\alpha}$ we obtain

$$dx = \alpha\,(\hat{\mu} - x)\, dt + \sigma dW. \tag{9}$$

Thus we arrive to an OU process for $x_t = \ln S_t$.
Let us first rewrite (9) in the more familiar way

$$dx = (a - bx)\, dt + \sigma dW, \tag{10}$$

where $a = \alpha\hat{\mu}$ and $b = \alpha$. Then we may write

$$dx + bxdt = adt + \sigma dW, \tag{11}$$

and multiplying the above by the integrating factor e^{bt} we obtain

$$e^{bt}dx + xbe^{bt}dt = ae^{bt}dt + \sigma e^{bt}dW, \tag{12}$$

and noting that $d(xe^{bt}) = e^{bt}dx + xbe^{bt}dt$ we replace the LHS on (12) to obtain

$$\begin{aligned} d(xe^{bt}) &= ae^{bt}dt + \sigma e^{bt}dW \\ \int_t^T \mathrm{d}(xe^{bs}) &= a\int_t^T e^{bs}\mathrm{d}s + \sigma\int_t^T e^{bs}\mathrm{d}W \\ x_Te^{bT} - x_te^{bt} &= a\frac{e^{bs}}{b}\bigg|_t^T + \sigma\int_t^T e^{bs}\mathrm{d}W \\ x_Te^{bT} &= x_te^{bt} + \frac{a}{b}\left(e^{bT} - e^{bt}\right) + \sigma\int_t^T e^{bs}\mathrm{d}W \\ x_T &= x_te^{bt}e^{-bT} + \frac{a}{b}e^{-bT}\left(e^{bT} - e^{bt}\right) + \sigma e^{-bT}\int_t^T e^{bs}\mathrm{d}W \\ &= x_te^{-b(T-t)} + \frac{a}{b}\left(1 - e^{-b(T-t)}\right) + \sigma e^{-bT}\int_t^T e^{bs}\mathrm{d}W. \end{aligned} \tag{13}$$

Note, that we also could have directly applied Itô's Lemma with $f(t,x) = e^{bt}\ln x$ to arrive at this result.

(b) Use the above to calculate $\mathbb{E}_t[S_T]$.
Using the above, we have that

$$S_T = \exp\ln S_T = \exp\left(\ln S_te^{-b(T-t)} + \frac{a}{b}\left(1 - e^{-b(T-t)}\right) + \sigma e^{-bT}\int_t^T e^{bs}dW_s\right). \tag{14}$$

Thus,

$$\mathbb{E}_t[S_T] = \exp\left(\ln S_te^{-b(T-t)} + \frac{a}{b}\left(1 - e^{-b(T-t)}\right)\right)\mathbb{E}_t\left[\exp\left(\sigma e^{-bT}\int_t^T e^{bs}dW_s\right)\right]. \tag{15}$$

As $\sigma e^{-bT}\int_t^T e^{bs}dW_s \sim \mathcal{N}\left(0, \sigma^2e^{-2bT}\int_t^T e^{2bs}ds\right) = \mathcal{N}\left(0, \frac{\sigma^2}{2b}(1 - e^{-2b(T-t)})\right)$, we can use the expectation of a lognormal distribution to get

$$\mathbb{E}_t\left[\exp\left(\sigma e^{-bT}\int_t^T e^{bs}dW_s\right)\right] = \exp\left(\frac{\sigma^2}{4b}(1 - e^{-2b(T-t)})\right). \tag{16}$$

Therefore,

$$\mathbb{E}_t[S_T] = \exp\left(\ln S_te^{-b(T-t)} + \frac{a}{b}\left(1 - e^{-b(T-t)}\right) + \frac{\sigma^2}{4b}(1 - e^{-2b(T-t)})\right). \tag{17}$$

Note that (by substituting $a = \alpha\hat{\mu} = \alpha(\mu - \frac{\sigma^2}{2\alpha})$, $b = \alpha$) we get the long-term level

$$\overline{S} = \lim_{T\to\infty}\mathbb{E}[S_T] = \exp\left(\frac{a}{b} + \frac{\sigma^2}{4b}\right) = \exp\left(\mu - \frac{\sigma^2}{2\alpha} + \frac{\sigma^2}{4\alpha}\right) = \exp\left(\mu - \frac{\sigma^2}{4\alpha}\right). \tag{18}$$

3. In the following $(W_t)_{t\geq 0}$ denotes a standard Brownian motion and $t \geq 0$ denotes time. A partition Π of the interval $[0, t]$ is a sequence of points $0 = t_0 < t_1 < t_2 < \cdots < t_n = t$ and $|\Pi| = \max_k(t_{k+1} - t_k)$. On a given partition $W_k \equiv W_{t_k}$, $\delta W_k \equiv W_{k+1} - W_k$, $\delta t_k \equiv t_{k+1} - t_k$ and if f is a function on $[0, t]$, $f_k \equiv f(t_k)$ and $\delta f_k \equiv f_{k+1} - f_k$.

(a) Show that if $t, s \geq 0$ then $\mathbb{E}[W_s W_t] = \min(s, t)$.

If $s = t \geq 0$ then we have

$$\mathbb{E}[W_s W_t] = \mathbb{E}[W_t^2] = t.$$

If not, assume $0 \leq s < t$ and write $W_t = W_s + W_t - W_s$. Then

$$\begin{aligned}\mathbb{E}[W_s W_t] &= \mathbb{E}\big[W_s W_s + W_s \left(W_t - W_s\right)\big] \\ &= \mathbb{E}[W_s W_s] + \mathbb{E}\big[W_s \left(W_t - W_s\right)\big] \\ &= \mathbb{E}[W_s^2] \;=\; s \;=\; \min(s, t).\end{aligned}$$

(b) Assuming that both the integral and its variance exist, show that

$$\mathrm{Var}\Big[\int_0^t f(W_s, s)\, dW_s\Big] = \int_0^t \mathbb{E}\left[f(W_s, s)^2\right]\, ds\,.$$

Is it generally the case that $\int_0^t f(W_s, s) dW_s$ has a Gaussian distribution?
[Note: if the integral and its variance exist then it is legitimate to interchange the order of expectation and dt-integration.]

Let's write $Y_t = \int_0^t f(W_s, s) dW_s$. As Y_t has a finite variance at every time it is a martingale, and its quadratic variation is

$$[Y]_t = \int_0^t f(W_s, s)^2 ds.$$

As Y is a martingale, we have

$$\mathbb{E}[Y_t] = 0$$

and in particular

$$\mathrm{var}[Y_t] = \mathbb{E}[Y_t^2] = \mathbb{E}\big[[Y]_t\big].$$

Substituting in the quadratic variation and exchanging the order of integration and expectation we have the desired identity.

Note that we could have directly used Itô's Isommetry to prove this.

It is not generally the case that $\int_0^t f(W_s, s) dW_s$ is Gaussian – this would only usually be the case if $f(W_s, s)$ is deterministic (so f does not depend on W).

(c) Use the differential version of Itô's lemma to show that

i. $\displaystyle\int_0^t W_s\,ds = t\,W_t - \int_0^t s\,dW_s = \int_0^t (t-s)\,dW_s,$

We have

$$d\big(t\,W_t\big) = W_t\,dt + t\,dW_t$$

which integrates to show

$$t\,W_t = \int_0^t W_s\,ds + \int_0^t s\,dW_s.$$

Rearranging gives

$$\int_0^t W_s\,ds = t\,W_t - \int_0^t s\,dW_s = \int_0^t (t-s)\,dW_s.$$

ii. $\displaystyle\int_0^t W_s^2\,dW_s = \tfrac{1}{3}W_t^3 - \int_0^t W_s\,ds,$

This time

$$d\big(W_t^3\big) = 3\,W_t^2\,dW_t + 3\,W_t\,dt$$

which integrates to give

$$W_t^3 = 3\int_0^t W_s^2\,dW_s + 3\int_0^t W_s\,ds.$$

Dividing by 3 and rearranging gives

$$\int_0^t W_s^2\,dW_s = \tfrac{1}{3}\,W_t^3 - \int_0^t W_s\,ds.$$

Part C

1. Define X_t to be the 'area under a Brownian motion', $X_0 = 0$ and $X_t = \int_0^t W_u\,du$ for $t > 0$. Show that X_t is normally distributed with

$$\mathbb{E}\big[X_t\big] = 0, \quad \mathbb{E}\big[X_t^2\big] = \tfrac{1}{3}\,t^3.$$

From Question 3(c) we have

$$\int_0^t W_s\,ds = \int_0^t (t-s)\,dW_s.$$

As $t-s$ does not depend on W_s, the integral is normally distributed with

$$\mathbb{E}\Big[\int_0^t W_s\,ds\Big] = \mathbb{E}\Big[\int_0^t (t-s)\,dW_s\Big] = 0$$

and

$$\begin{aligned}\operatorname{var}\Big[\int_0^t W_s\,ds\Big] &= \operatorname{var}\Big[\int_0^t (t-s)\,dW_s\Big]\\ &= \int_0^t (t-s)^2\,ds\\ &= \tfrac{1}{3}\,t^3.\end{aligned}$$

Note that X_t is continuously differentiable for $t>0$, with $\dot{X}_t = W_t$ (recall W_t is continuous in t).

Now define Y_t as the 'average area under a Brownian motion',

$$Y_t = \begin{cases} 0 & \text{if } t=0,\\ X_t/t & \text{if } t>0.\end{cases}$$

Show that Y_t has $\mathbb{E}[Y_t]=0$, $\mathbb{E}\big[Y_t^2\big]=t/3$ and that Y_t is continuous for all $t\geq 0$.

Is $\sqrt{3}\,Y_t$ a Brownian motion? Give reasons for your answer.

For any $t>0$, Y_t is normal because X_t is, with

$$\mathrm{E}\big[Y_t\big] = \mathrm{E}\big[X_t\big]/t = 0$$

and

$$\operatorname{var}\big[Y_t\big] = \operatorname{var}\big[X_t/t\big] = \operatorname{var}\big[X_t\big]/t^2 = \tfrac{1}{3}\,t.$$

For $t>0$ we have $Y_t = X_t/t$ which is the ratio of two differentiable functions and as the denominator is never zero it follows that Y_t is differentiable for $t>0$, which implies continuous. Moreover, it means we can use l'Hopital's rule to show

$$\lim_{t\to 0^+} Y_t = \lim_{t\to 0^+} \frac{W_t}{1} = 0,$$

which shows that Y_t is continuous at $t=0$.

The function $\sqrt{3}\,Y_t$ is not only continuous but *differentiable* for $t>0$ and therefore it can't be a Brownian motion. (The hard way to do this part of the question is to show that the increments over disjoint intervals are not independent.)

2. Consider the general stochastic differential equation

$$dG = A(G,t)\,dt + B(G,t)\,dW_t\,,$$

where G and B are functions and W is Brownian motion. Use Itô's Lemma to show that it is theoretically possible to find a function $f(G,t)$ which itself follows a random walk with zero drift.

Solution

For the SDE

$$dG = A(G,t)dt + B(G,t)dW \tag{19}$$

we can write Itô's Lemma as

$$df = \left(\frac{\partial f}{\partial t} + \frac{1}{2}B^2\frac{\partial^2 f}{\partial G^2} + A\frac{\partial f}{\partial G}\right)dt + B\frac{\partial f}{\partial G}dZ. \tag{20}$$

Consequently, any function f which satisfies, with appropriate boundary conditions, the differential equation

$$\frac{\partial f}{\partial t} + \frac{1}{2}B^2\frac{\partial^2 f}{\partial G^2} + A\frac{\partial f}{\partial G} = 0 \tag{21}$$

will be such that f follows itself a random walk with no drift, namely

$$df = B\frac{\partial f}{\partial G}\,dW\,. \tag{22}$$

Exercise Sheet 2

Part A

1. Draw the expiry payoff diagrams for each of the following portfolios:

   (a) Short one share, long two calls with exercise price K.

   (b) Long one call and one put, both with exercise price K.

   (c) Long one call and two puts, all with exercise price K.

   (d) Long one put and two calls, all with exercise price K.

   (e) Long one call with exercise price K_1 and one put with exercise price K_2. Compare the three cases: $K_1 > K_2$, $K_1 = K_2$ and $K_1 < K_2$.

   (f) As (e), but also short one call and one put with exercise price K for $K_1 < K < K_2$.

Solution (a) *Short one share, long two calls with exercise price K.*

$S < K : -S + 0 + 0 = -S$

$S \geq K : -S + (S - K) + (S - K) = S - 2\,K$, hence

$$V(S,T) = \begin{cases} -S & \text{for} \quad S < K \\ S - 2\,K & \text{for} \quad S \geq K \end{cases} \tag{23}$$

(b) *Long one call and one put, both with exercise price K.*

$S < K : 0 + (K - S) = K - S$

$S \geq K : S - K + 0 = S - K$, hence

$$V(S,T) = \begin{cases} -(S-K) & \text{for} \quad S < K \\ S - K & \text{for} \quad S \geq K \end{cases} \tag{24}$$

(c) *Long one call and two puts, all with exercise price K.*

$S < K : 0 + (K - S) + (K - S) = 2\,(K - S)$

$S \geq K : S - K + 0 + 0$, hence

$$V(S,T) = \begin{cases} 2(K-S) & \text{for} \quad S < K\,, \\ S - K & \text{for} \quad S \geq K. \end{cases} \tag{25}$$

(d) *Long one put and two calls, all with exercise price K.*

$S < K : (K - S) + 0 + 0 = (K - S)$

$S \geq K : 0 + (S - K) + (S - K) = 2\,(S - K)$, hence

$$V(S,T) = \begin{cases} K - S & \text{for} \quad S < K\,, \\ 2\,(S - K) & \text{for} \quad S \geq K\,. \end{cases} \tag{26}$$

(e) *Long one call with exercise price K_1 and one put with exercise price K_2. Compare the three cases: $K_1 > K_2$, $K_1 = K_2$ and $K_1 < K_2$.*

Case $K_1 > K_2$

$S < K_2 : 0 + (K_2 - S) = (K_2 - S)$

$K_2 \leq S < K_1 : 0$

$S \geq K_1 : S - K_1 + 0 = S - K_1$

$$V(S,T) = \begin{cases} K_2 - S & \text{for} \quad S < K_2\,, \\ 0 & \text{for} \quad K_2 \leq S < K_1\,, \\ S - K_1 & \text{for} \quad S \geq K_1\,. \end{cases} \tag{27}$$

Case $K_1 < K_2$

$S < K_1 : 0 + (K_2 - S) = (K_2 - S)$

$K_1 \leq S < K_2 : (S - K_1) + (K_2 - S) = K_2 - K_1$

$S \geq K_2 : S - K_1 + 0 = S - K_1$

$$V(S,T) = \begin{cases} K_2 - S & \text{for} \quad S < K_1\,, \\ K_2 - K_1 & \text{for} \quad K_1 \leq S < K_2\,, \\ S - K_1 & \text{for} \quad S \geq K_2\,. \end{cases} \tag{28}$$

Case $K_1 = K_2 = K$

$S < K : 0 + (K - S) = (K - S)$

$S \geq K : S - K + 0 = S - K$

$$V(S,T) = \begin{cases} K - S & \text{for} \quad S < K\,, \\ S - K & \text{for} \quad S \geq K\,. \end{cases} \tag{29}$$

(f) As (e), but also short one call and one put with exercise price K for $K_1 < K < K_2$.

$S < K_1 : 0 + (K_2 - S) + 0 + S - K = K_2 - K$

$K_1 \leq S < K : (S - K_1) + (K_2 - S) + 0 + (S - K) = S - (K + K_1 - K_2)$

$K \leq S < K_2 : (S - K_1) + (K_2 - S) + (K - S) + 0 = (K_2 - K_1 + K) - S$

$S \geq K_2 : S - K_1 + 0 + K - S + 0 = K - K_1$

$$V(S,T) = \begin{cases} K_2 - K & \text{for} \quad S < K_1 \\ S - (K + K_1 - K_2) & \text{for} \quad K_1 \leq S < K\,, \\ (K_2 - K_1 + K) - S & \text{for} \quad K \leq S < K_2\,, \\ K - K_1 & \text{for} \quad S \geq K_2\,. \end{cases} \tag{30}$$

Part B

1. There are n assets satisfying the following stochastic differential equations

$$dS_i = \mu_i\, S_i\, dt + \sigma_i\, S_i\, dZ_i \qquad \text{for } i = 1, \cdots, n\,.$$

Here, Z_i is a standard Brownian motion and the quadratic variation

$$[Z_i,\, Z_j] = \rho_{ij}\, t\,,$$

where $-1 \leq \rho_{ij} = \rho_{ji} \leq 1$ is the correlation between Z_i and Z_j.

Derive Itô's Lemma for a function $f(S_1, \ldots, S_n)$ of the n assets $S_1, \ldots, S_n$. (Hint: use $dZ_i\, dZ_j = \rho_{ij}\, dt$.)

Solution

There are n assets satisfying the following stochastic differential equations

$$dS_i = \mu_i S_i dt + \sigma_i S_i dZ_i \qquad \text{for } i = 1, \ldots, n. \tag{31}$$

where the Wiener process dZ_i is such that it satisfies

$$\mathbb{E}[dZ_i] = 0, \qquad dZ_i^2 = dt. \tag{32}$$

However, the asset price changes are correlated with

$$dZ_i dZ_j = \rho_{ij} dt \tag{33}$$

where $-1 \leq \rho_{ij} = \rho_{ji} \leq 1$.

Let us then derive Itô's multivariate Lemma. We can write the system of stochastic differential equation defined in (31) subject to the constraint in (32) and (33) as

$$\begin{pmatrix} dS_1 \\ dS_2 \\ \vdots \\ dS_n \end{pmatrix} = \begin{pmatrix} \mu_1 S_1 \\ \mu_2 S_2 \\ \vdots \\ \mu_n S_n \end{pmatrix} dt + \begin{pmatrix} \sigma_{11} S_1 & \sigma_{12} S_2 & \cdots & \sigma_{1n} S_1 \\ \sigma_{21} S_2 & \sigma_{22} S_2 & \cdots & \sigma_{2n} S_2 \\ \vdots & \vdots & \vdots & \vdots \\ \sigma_{n1} S_n & \sigma_{n2} S_n & \cdots & \sigma_{nn} S_n \end{pmatrix} \begin{pmatrix} d\tilde{Z}1 \\ d\tilde{Z}_2 \\ \vdots \\ d\tilde{Z}_n \end{pmatrix} \tag{34}$$

where the Wiener increments are now independent, such that $d\tilde{Z}_i \, d\tilde{Z}_j = 0$. The correlation is now present in the $n \times n$ matrix defined by the σ_{ij} components.

However, we may also write the system of correlated differential equations as

$$\begin{pmatrix} dS_1 \\ dS_2 \\ \vdots \\ dS_n \end{pmatrix} = \begin{pmatrix} \mu_1 S_1 \\ \mu_2 S_2 \\ \vdots \\ \mu_n S_n \end{pmatrix} dt + \begin{pmatrix} \sigma_1 S_1 \\ \sigma_2 S_2 \\ \vdots \\ \sigma_n S_n \end{pmatrix} \begin{pmatrix} dZ_1 & dZ_2 & \ldots & dZ_n \end{pmatrix} \tag{35}$$

where the asset price changes are correlated with $dZ_i \, dZ_j = \rho_{ij} \, dt$.

Assume we can expand in Taylor series the function $f(S_1, \cdots, S_n)$ and write

$$\begin{aligned} df &= \frac{\partial f}{\partial t} dt + \left(\frac{\partial f}{\partial S_1} dS_1 + \frac{\partial f}{\partial S_2} dS_2 + \ldots + \frac{\partial f}{\partial S_n} dS_n \right) \\ &+ \left(\frac{\partial f}{\partial S_1} dS_1 + \frac{\partial f}{\partial S_2} dS_2 + \ldots + \frac{\partial f}{\partial S_n} dS_n \right)^2 + \mathcal{O}_{m>2}(dS^m). \end{aligned} \tag{36}$$

Rewrite (36), up to second order, in a more compact form as

$$df = \frac{\partial f}{\partial t} dt + \sum_{i=1}^{n} \frac{\partial f}{\partial S_i} dS_i + \sum_{i=1}^{n} \frac{\partial^2 f}{\partial S_i^2} (dS_i)^2 + \sum_{i \neq j}^{n} \frac{\partial^2 f}{\partial S_i S_j} dS_i dS_j, \tag{37}$$

and replacing into (37) dS_i as given by (31), with (32) and (33) we obtain (up to second order in dt)

$$(dS_i)^2 = \sigma_i^2 S_i^2 dt \tag{38}$$

and

$$\begin{aligned} dS_i dSj &= \sigma_i \sigma_j S_i S_j dZ_i dZ_j \\ &= \sigma_i \sigma_j S_i S_j \rho_{ij} dt. \end{aligned} \tag{39}$$

Finally, regrouping terms we may write Itô's Lemma for a function $f(S_1, \ldots, S_n)$ of the n assets $S_1, \ldots, S_n$ as

$$df = \frac{\partial f}{\partial t} dt + \sum_{i=1}^{n} \left(\frac{\partial f}{\partial S_i} \mu_i S_i + \frac{\partial^2 f}{\partial S_i^2} \sigma_i^2 S_i^2 \right) dt + \sum_{i \neq j}^{n} \left(\frac{\partial^2 f}{\partial S_i S_j} \sigma_i \sigma_j S_i S_j \rho_{ij} \right) dt + \sum_{i=1}^{n} \frac{\partial f}{\partial S_i} \sigma_i S_i dZ_i. \tag{40}$$

2. For $t > 0$, let

$$p(y; x, t) = \frac{1}{\sqrt{2\,\pi\,t}} e^{-(x-y)^2/2t}.$$

This can be interpreted as the probability density function for a normal random variable Y which has mean x and variance t. Show, by direct calculation, that $p(y; x, t)$ also satisfies the heat equation

$$\frac{\partial p}{\partial t} = \tfrac{1}{2} \frac{\partial^2 p}{\partial x^2}, \quad \text{for } t > 0,\ x \in \mathbb{R}.$$

Solution

Direct calculation gives

$$\begin{aligned}
\frac{\partial p}{\partial x} &= -\frac{(x-y)}{t}\, p(y; x, t), \\
\frac{\partial^2 p}{\partial x^2} &= -\frac{1}{t}\, p(y; x, t) - \frac{(x-y)}{t}\, \frac{\partial p}{\partial x} \\
&= -\frac{1}{t}\, p(y; x, t) + \frac{(x-y)^2}{t^2}\, p(y; x, t), \\
\frac{\partial p}{\partial t} &= \frac{-1}{2\sqrt{2\,\pi\,t^3}}\, e^{-(x-y)^2/2t} + \frac{-(x-y)^2}{-2t^2}\, p(y; x, t) \\
&= \tfrac{1}{2} \left(-\frac{1}{t}\, p(y; x, t) + \frac{(x-y)^2}{t^2}\, p(y; x, t) \right) \\
&= \tfrac{1}{2}\, \frac{\partial^2 p}{\partial x^2}.
\end{aligned}$$

Hence deduce that

$$u(x, t) = \mathbb{E}\big[\, f(y) \,\big] = \frac{1}{\sqrt{2\,\pi\,t}} \int_{-\infty}^{\infty} f(y)\, e^{-(x-y)^2/2t}\, dy$$

satisfies the heat equation

$$\frac{\partial u}{\partial t} = \tfrac{1}{2}\,\frac{\partial^2 u}{\partial x^2}\,, \qquad \text{for } t > 0,\ x \in \mathbb{R}\,,$$

provided the integral converges absolutely. [Hint: you can assume that the absolute convergence means you can swap the order of partial differentiation and integration.]

Solution

Write the solution in the form

$$u(x,t) = \int_{-\infty}^{\infty} f(y)\,p(y;x,t)\,dy$$

and assume that the integral is absolutely convergent, which means that

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial}{\partial t}\int_{-\infty}^{\infty} f(y)\,p(y;x,t)\,dy \\ &= \int_{-\infty}^{\infty} \frac{\partial}{\partial t}\left(f(y)\,p(y;x,t)\right)dy \\ &= \int_{-\infty}^{\infty} f(y)\,\frac{\partial p}{\partial t}(y;x,t)\,dy\end{aligned}$$

and, similarly,

$$\frac{\partial^2 u}{\partial x^2} = \int_{-\infty}^{\infty} f(y)\,\frac{\partial^2 p}{\partial x^2}(y;x,t)\,dy.$$

Thus

$$\frac{\partial u}{\partial t} - \tfrac{1}{2}\,\frac{\partial^2 u}{\partial x^2} = \int_{-\infty}^{\infty} f(y)\left(\frac{\partial p}{\partial t} - \tfrac{1}{2}\,\frac{\partial^2 p}{\partial x^2}\right)dy = 0$$

since

$$\frac{\partial p}{\partial t} - \tfrac{1}{2}\,\frac{\partial^2 p}{\partial x^2} = 0$$

identically for $t > 0$.

Assuming that the integral converges absolutely and f is continuous at all points in $\mathbb{R}$, show that

$$\lim_{t\to 0^+} u(x,t) = \lim_{t\to 0^+} \frac{1}{\sqrt{2\,\pi\,t}}\int_{-\infty}^{\infty} f(y)e^{-(x-y)^2/2t}\,dy = f(x)$$

for each $x \in \mathbb{R}$. [Hint: change variables to $s = (y-x)/\sqrt{t}$ and assume that the absolute convergence allows you to interchange the order of limit and integration.]

Solution

We have

$$\begin{aligned}
\lim_{t\to 0^+} u(x,t) &= \lim_{t\to 0^+} \frac{1}{\sqrt{2\,\pi\,t}} \int_{-\infty}^{\infty} f(y) e^{-(x-y)^2/2t}\, dy \\
&= \lim_{t\to 0^+} \frac{1}{\sqrt{2\,\pi}} \int_{-\infty}^{\infty} f\big(\,x + s\sqrt{t}\,\big)\, e^{-s^2/2}\, ds \quad (s = (y-x)/\sqrt{t}) \\
&= \frac{1}{\sqrt{2\,\pi}} \int_{-\infty}^{\infty} \lim_{t\to 0^+} f\big(\,x + s\sqrt{t}\,\big)\, e^{-s^2/2}\, ds \\
&= \frac{1}{\sqrt{2\,\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-s^2/2}\, ds \\
&= \frac{f(x)}{\sqrt{2\,\pi}} \int_{-\infty}^{\infty} e^{-s^2/2}\, ds \\
&= f(x)
\end{aligned}$$

The reason we can write

$$\lim_{t\to 0^+} f\big(x + s\sqrt{t}\big) = f(x)$$

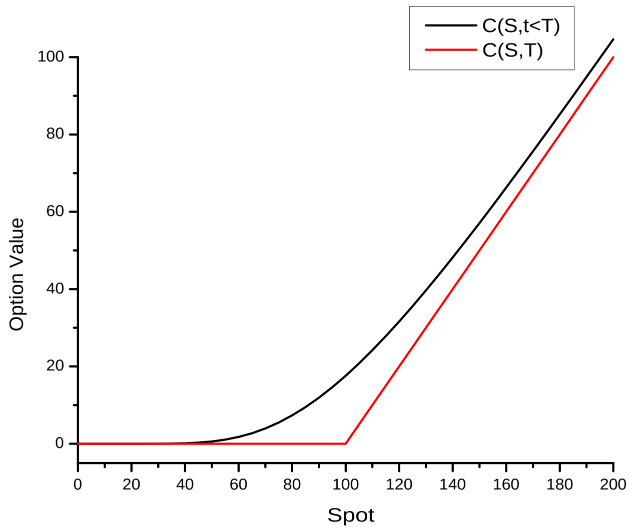
is that we assume that $f(y)$ is continuous for all $y \in \mathbb{R}$. If there is a point x where f is not continuous then, in general, the result

$$\lim_{t\to 0^+} u(x,t) = f(x)$$

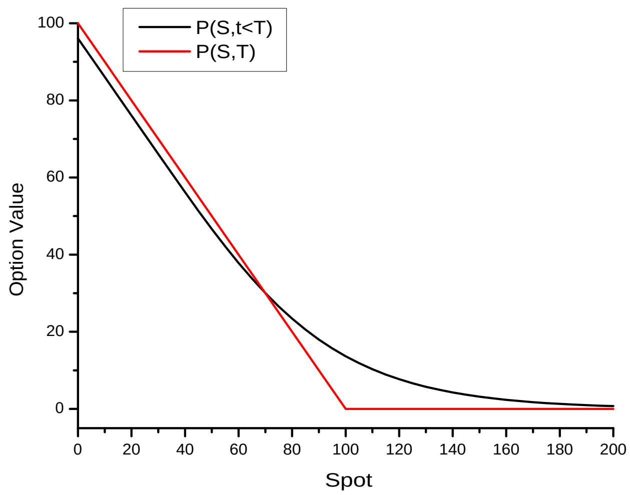
does *not* apply at that point.

3. * Sketch the graphs of the Δ for the European call and put. Suppose that the asset price now is $S = K$ (each of these options is at-the-money). Convince yourself that it is plausible that the delta-hedging strategy is self-financing for each option, in the two cases that the option expires in-the-money and out-of-the-money; look at the contract from the point of views of the writer.

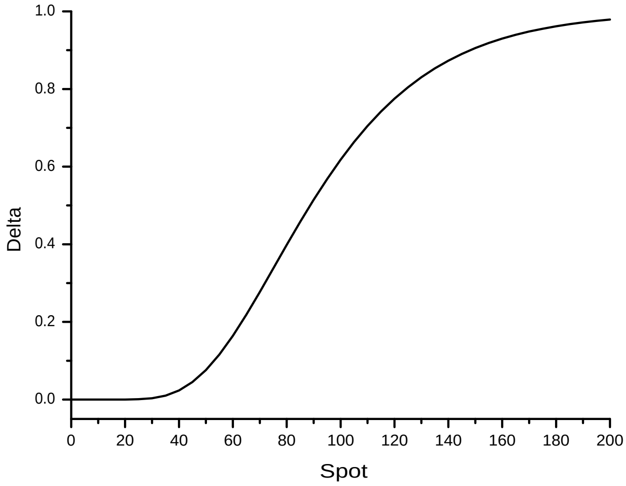
Solution



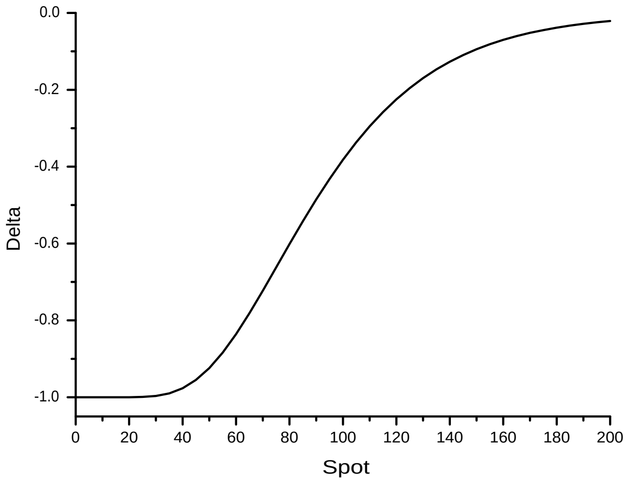
Variation of a Call Option that pays no dividends with respect to the spot price. Parameters: $K = 100$, $r = 0.04$, $\sigma = 0.4$ and $T - t = 1$.



Variation of a Put Option that pays no dividends with respect to the spot price. Parameters: $K = 100$, $r = 0.04$, $\sigma = 0.4$ and $T - t = 1$.



Variation of the Delta of the Call Option with respect to the spot price.



Variation of the Delta of the Put Option with respect to the spot price.

4. Find the most general solution of the Black–Scholes PDE

$$V_t + \frac{1}{2}\,\sigma^2\,S^2\,V_{SS} + r\,S\,V_S - r\,V = 0 \tag{41}$$

that has the special form
(a) $V = V(S)$,
(b) $V = A(t)\,B(S)$, where V, A, B are 'nicely-behaved' functions.

Solution

(a) We seek a solution to the Black–Scholes equation independent of time, this is $V = V(S)$ such that it satisfies an Euler differential equation

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r\,S\,\frac{\partial V}{\partial S} - r\,V = 0\,. \tag{42}$$

Let us solve by proposing a solution of the form $V = S^m$; and consequently: $V_S = m\,S^{m-1}$, $V_{SS} = m\,(m-1)\,S^{m-2}$. Upon replacing these results into (42) we obtain

$$(m-1)\left(\frac{\sigma^2}{2}m + r\right) = 0, \tag{43}$$

which yields two solutions: $m_1 = 1$ and $m_2 = -\frac{2r}{\sigma^2}$.

Finally, replacing the two roots of (43) into $V = S^m$ we may write the most general time-independent solution as

$$V(S) = AS + BS^{-\frac{2r}{\sigma^2}}. \tag{44}$$

(b) Now, we propose a solution of the form $V(S,t) = A(t)B(S)$ and solve for Black–Scholes time-dependent equation, this is

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0\,. \tag{45}$$

Replacing the partial derivatives: $V_t = A_t B$, $V_S = AB_S$ and $V_{SS} = AB_{SS}$ into (45) we obtain

$$A_t B + \frac{1}{2}\sigma^2 S^2 A B_{SS} + rSAB_S - rAB = 0, \tag{46}$$

and by dividing this equation by $V = AB$ we arrive to

$$\left(\frac{A_t}{A} - r\right) + \frac{1}{B}\left(\frac{\sigma^2 S^2 B_{SS}}{2} + rSB_S\right) = 0. \tag{47}$$

Now, since A depends exclusively on time, and B on S, both terms in parenthesis must be equal to a constant such that

$$\left(\frac{A_t}{A} - r\right) = \xi; \tag{48}$$

$$\frac{1}{B}\left(\frac{\sigma^2 S^2 B_{SS}}{2} + rSB_S\right) = -\xi. \tag{49}$$

Solving for (48) we readily obtain

$$A(T) = A(t)e^{(r+\xi)(T-t)}; \tag{50}$$

and solving for (49) we get[1]

$$B(S) = CS^{n_+} + DS^{n_-}, \tag{51}$$

where C, D are constants and n_+ and n_- are such that

$$n_\pm = \frac{\frac{\sigma^2}{2} - r \pm \sqrt{\left(r - \frac{\sigma^2}{2}\right)^2 - 2\sigma^2\xi}}{\sigma^2} \tag{52}$$

Finally, since we had proposed a general solution of the form $V(S,t) = A(t)B(S)$ we obtain

$$V(S,t) = \left(A(t)e^{(r+\xi)(T-t)}\right)\left(CS^{n_+} + DS^{n_-}\right). \tag{53}$$

Part C

Let the price of a stock follow the dynamics $dS_t = \mu\, S_t\, dt + \sigma\, S_t\, dW_t$ and let

$$\mathcal{H}(x) = \begin{cases} 0 & \text{for} \quad x < 0\,, \\ 1 & \text{for} \quad x \geq 0\,, \end{cases} \tag{54}$$

i.e., the Heaviside function

1. (a) What is the value of a European option struck at K and expiring at T with payoff $\mathcal{H}(S-K)$?

   (b) What is the value of a European option struck at K and expiring at T with payoff $\frac{1}{d}\left(\mathcal{H}(S-K) - \mathcal{H}(S-K-d)\right)$?

   Solution

   The Heaviside function is related to the delta function by

$$\int_{-\infty}^{x} \delta(s)ds = \mathcal{H}(x), \tag{55}$$

[1] Equation (49) is again an Euler differential equation, thus we solve it as in (a) straightforwardly.

where $\mathcal{H}(x)$ is the *Heaviside function*, defined by

$$\mathcal{H}(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ 1 & \text{for} \quad x \geq 0. \end{cases} \tag{56}$$

(a) Hence, if the payoff is denoted by $\mathcal{H}(S-K)$, then by (56)

$$\mathcal{H}(S-K) = \begin{cases} 0 & \text{for} \quad S-K < 0\,, \\ 1 & \text{for} \quad S-K \geq 0\,, \end{cases} \tag{57}$$

or in the more familiar notation, the payoff of the option is

$$V(S,T) = \begin{cases} 0 & \text{for} \quad S < K\,, \\ 1 & \text{for} \quad S \geq K\,. \end{cases} \tag{58}$$

It can be shown (slides derive general PDE for European-style options; students solve the PDE or use Feyman-Kac). Either way, the price of the option at time t is given by

$$V(S,t;K) = \mathbb{E}_t^Q \left[e^{-r(T-t)}\, V(S_T, K) \right], \tag{59}$$

where $\mathcal{Q}$ denotes we are in the risk-neutral measure, or equivalently for the Black–Scholes model we are pricing for the SDE

$$dS = r\, S\, dt + \sigma\, S\, dW^Q. \tag{60}$$

Recall that

$$S_T = S_t e^{\left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma Z_{t,T}}; \tag{61}$$

and we write the Wiener process $Z_{t,T}$ as

$$Z_{t,T} \overset{d}{=} \sqrt{T-t}\, \phi, \qquad \phi \sim N(0,1)\,, \tag{62}$$

where $\overset{d}{=}$ is equality in distribution. Thus, solving (59) implies evaluating

$$V(S,t;K) = e^{-r(T-t)} \int_{-\infty}^{\infty} V(S_T, K) d\mathcal{P}\,, \tag{63}$$

where $d\mathcal{P}$ is the density function of a normal distribution with zero mean and unit variance, hence

$$V(S,t;K) = e^{-r(T-t)} \frac{1}{\sqrt{2\,\pi}} \int_{-\infty}^{\infty} V(S_T, K)\, e^{\frac{-\phi^2}{2}}\, d\phi\,. \tag{64}$$

Now, by (58) this integral will only have value different from zero when $S_T \geq K$; and from (61)and (62) this implies

$$\phi \geq \frac{\log\left(\frac{K}{S_t}\right) - \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \doteq x, \tag{65}$$

where for the mean time we are calling this limit x.

Hence, (64) becomes

$$\begin{aligned} V(S,t;K) &= e^{-r(T-t)}\frac{1}{\sqrt{2\,\pi}}\int_x^{\infty} e^{\frac{-\phi^2}{2}}d\phi \\ &= e^{-r(T-t)}\frac{1}{\sqrt{2\,\pi}}\int_{-\infty}^{-x} e^{\frac{-\phi^2}{2}}d\phi, \end{aligned} \tag{66}$$

where we inverted the limits of integration because the normal distribution is symmetric around the mean.

Finally, noting that

$$-x = \frac{\log\left(\frac{S_t}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} = d_2 \tag{67}$$

then we may at last write the price of the option at time t as

$$V(S,t) = e^{-r(T-t)}N(d_2); \tag{68}$$

$$N(d_2) = \frac{1}{\sqrt{2\,\pi}}\int_{-\infty}^{d_2} e^{\frac{-\phi^2}{2}}d\phi; \tag{69}$$

$$d_2 = \frac{\log(S_t/K) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}. \tag{70}$$

(b) The payoff function is now given by

$$\frac{1}{d}\left(\mathcal{H}(S-K) - \mathcal{H}(S-K-d)\right), \tag{71}$$

where

$$\mathcal{H}(S-K) = \begin{cases} 0 & \text{for } \; S < K\,, \\ 1 & \text{for } \; S \geq K\,, \end{cases} \tag{72}$$

and

$$\mathcal{H}(S-K-d) = \begin{cases} 0 & \text{for } \; S < K+d\,, \\ 1 & \text{for } \; S \geq K+d\,. \end{cases} \tag{73}$$

Hence, from (71), (72) and (73) we have

$$V(S,T) = \begin{cases} 0 & \text{for} \quad S < K \\ \frac{1}{d} & \text{for} \quad K \leq S < K+d \\ 0 & \text{for} \quad S \geq K+d; \end{cases} \tag{74}$$

(notice that when $d \to 0$ the payoff becomes a delta function).

As before, we calculate the option price at time t ,expiring at T, through (64), where the payoff function is now that in (74). Clearly, the integral will only exist for $K \leq S < K+d$; by (61) and (62) this implies

$$x \doteq \frac{\log\left(\frac{K}{S_t}\right) - \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \leq \phi < \frac{\log\left(\frac{K+d}{S_t}\right) - \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \doteq y. \tag{75}$$

Hence (64) becomes

$$V(S,t;K) = e^{-r(T-t)}\frac{1}{\sqrt{2\pi}}\int_x^y \frac{1}{d}e^{\frac{-\phi^2}{2}}d\phi, \tag{76}$$

which we can rewrite as

$$V(S,t;K) = \frac{1}{d}e^{-r(T-t)}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-x} e^{\frac{-\phi^2}{2}}d\phi - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-y} e^{\frac{-\phi^2}{2}}d\phi\right), \tag{77}$$

where we have used that for the normal distribution $\int_x^y(\cdot) = \int_{-y}^{-x}(\cdot) = \int_{-\infty}^{-x}(\cdot) - \int_{-\infty}^{-y}(\cdot)$.

Finally write $-x$ as d_2 (as in (67)) and let $-y = \tilde{d}_2$ such that

$$\tilde{d}_2 = \frac{\log\left(\frac{S_t}{K+d}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \tag{78}$$

the price of the option at time t as

$$V(S,t) = \frac{1}{d}e^{-r(T-t)}(N(d_2) - N(\tilde{d}_2)). \tag{79}$$

Exercise Sheet 3

Part A

1. The European asset-or-nothing call pays S if $S > K$ at expiry, and nothing if $S \leq K$. What is its value?

Solution The European asset-or-nothing call pays S if $S > K$ at expiry, and nothing if $S \leq K$, hence the payoff function is given by

$$V(S,T) = \begin{cases} 0 & \text{for} \quad S \leq K \\ S & \text{for} \quad S > K. \end{cases} \tag{80}$$

We have to evaluate

$$\begin{aligned} V(S,t;K) &= e^{-r\,(T-t)} \frac{1}{\sqrt{2\,\pi}} \int_{-\infty}^{d_2} V(S_T, K)\, e^{\frac{-x^2}{2}}\, dx \\ &= e^{-r\,(T-t)} \frac{1}{\sqrt{2\,\pi}} \int_{-\infty}^{d_2} S_T\, e^{\frac{-x^2}{2}}\, dx\,, \end{aligned} \tag{81}$$

and as in the Black–Scholes model we write

$$S_T = S_t e^{\left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma\sqrt{T-t}\,\phi} \qquad \phi \sim N(0,1)\,; \tag{82}$$

thus, (81) becomes

$$\begin{aligned} V(S,t;K) &= e^{-r(T-t)} \frac{1}{\sqrt{2\,\pi}} \int_{-\infty}^{d_2} S_t e^{\left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma\sqrt{T-t}\,x} e^{\frac{-x^2}{2}} dx \\ &= S_t e^{-\frac{\sigma^2}{2}(T-t)} \frac{1}{\sqrt{2\,\pi}} \int_{-\infty}^{d_2} e^{\sigma\sqrt{T-t}\,x - \frac{-x^2}{2}} dx \\ &= S_t\, e^{-\frac{\sigma^2}{2}(T-t)} \frac{1}{\sqrt{2\,\pi}} \int_{-\infty}^{d_2} e^{\Psi} dx, \end{aligned} \tag{83}$$

where $\Psi = \sigma\sqrt{T-t}\,x - \frac{-x^2}{2}$.

To evaluate the integral in (83) we complete the square in Ψ:

$$\begin{aligned} -\frac{\left(x - \sigma\sqrt{T-t}\right)^2}{2} &= \frac{-\phi^2 + 2\,x\sigma\sqrt{T-t} - \sigma^2(T-t)}{2} \\ &= \left(\frac{-x^2}{2} + \sigma\sqrt{T-t}\,x\right) - \frac{\sigma^2(T-t)}{2} \\ &= \Psi - \frac{\sigma^2(T-t)}{2}\,, \end{aligned} \tag{84}$$

then

$$\Psi = \frac{\sigma^2 (T-t)}{2} - \frac{\left(x - \sigma\sqrt{T-t}\right)^2}{2}; \tag{85}$$

and substituting back into (83) we obtain

$$\begin{aligned} V(S,t;K) &= S_t e^{-\frac{\sigma^2}{2}(T-t)} \frac{1}{\sqrt{2\,\pi}} \int_{-\infty}^{d_2} e^{\frac{\sigma^2(T-t)}{2} - \frac{(x-\sigma\sqrt{T-t})^2}{2}} dx \\ &= S_t \frac{1}{\sqrt{2\,\pi}} \int_{-\infty}^{d_2} e^{-\frac{(x-\sigma\,\sqrt{T-t})^2}{2}}\, dx \\ &= S_t\, N(d_2)\,, \end{aligned} \tag{86}$$

where $N \sim N(\sigma\sqrt{T-t}\,, 1)$.

2. What is the probability that a European call will expire in the money?

Solution

The probability that at expiry $S > K$ is given by

$$\text{Prob}(S > K) = \mathbb{E}[\mathcal{H}(S-K)]; \tag{87}$$

hence, we need to calculate

$$\text{Prob}(S > K) = \int_{-\infty}^{\infty} \mathcal{H}(S-K) d\mathcal{P}\,. \tag{88}$$

We use the SDE

$$dS = \mu\, S\, dt + \sigma\, S\, dW\,, \tag{89}$$

so

$$S_T = S_t\, e^{\left(\mu - \frac{\sigma^2}{2}\right)(T-t) + \sigma\,\sqrt{T-t}\,\phi}\,. \tag{90}$$

Now we may continue to calculate (87); from the previous exercises we know that this integral will only have value different from zero when $S > K$, which from (90) implies

$$\phi \geq \frac{\log\left(\frac{K}{S_t}\right) - \left(\mu - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \doteq -d_2^{\mu}; \tag{91}$$

thus, (88) becomes

$$\begin{aligned} \text{Prob}(S > K) &= \int_{-d_2^{\mu}}^{\infty} e^{-\frac{\phi^2}{2}} d\phi \\ &= \int_{-\infty}^{d_2^{\mu}} e^{-\frac{\phi^2}{2}} d\phi \\ &= N(d_2^{\mu}). \end{aligned} \tag{92}$$

Finally, note that d_2^μ is not the same as d_2 from the previous exercises – they are the same if $r = \mu$.

Part B

1. In the Black–Scholes model, show that the value of European call option on an asset that pays a constant continuous dividend yield (i.e., $S\,D\,\mathrm{d}t$ as in the lectures) lies below the payoff for large enough values of S. Show also that the call on an asset with dividends is less valuable than the call on an asset without dividends.

   Solution

   The price of a Call option on an asset that pays a constant continuous dividend yield is given by
   $$C(S,t) = e^{-D\,(T-t)}\,S\,N(\hat{d}_+) - K\,e^{-r\,(T-t)}N(\hat{d}_-)\,, \tag{93}$$
   with
   $$\hat{d}_\pm = \frac{\log\left(\frac{S}{K}\right) + \left(r - D \pm \frac{\sigma^2}{2}\right)(T-t)}{\sigma\,\sqrt{T-t}}\,. \tag{94}$$

   Clearly, in the limit $S \to \infty$, $\hat{d}_\pm \to \infty$, thus, $N(\hat{d}_\pm) \to 1$. Therefore, for arbitrarily large values of S we have
   $$C(S,t) \sim S\,e^{-D\,(T-t)} - K\,e^{-r\,(T-t)}\,. \tag{95}$$

   Clearly, for any $D > 0$ and $r > 0$ the value of the call in this limit is such that
   $$S\,e^{-D\,(T-t)} - K\,e^{-r\,(T-t)} < S - K\,; \tag{96}$$

   so the value of the option lies below its payoff.

   Finally, by simple inspection of (93) we notice that the call with dividends is less valuable than the call which pays no dividends.

2. What is the random walk followed by the the forward price $F(S,t) = S\,e^{r\,(T-t)}$ in the Black–Scholes model?

   Solution We want to write the SDE followed by the forward price $F(S,t) = S\,e^{r\,(T-t)}$

   in the Black–Scholes model.

Let us write Itô's Lemma for the futures price F, then

$$dF = \left(F_t + \frac{1}{2}\,\sigma^2\, S^2\, F_{SS} + r\, S\, F_S \right) dt + \sigma\, S\, F_S\, dW\,. \tag{97}$$

Finally, evaluating the partial derivatives: $F_t = -r\, S\, e^{r\,(T-t)}$, $F_S = e^{r\,(T-t)}$, $F_{SS} = 0$ and replacing these in (97) we obtain

$$dF = \sigma\, F\, dW\,. \tag{98}$$

3. Suppose that $V(S,t)$ satisfies the Black–Scholes problem

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2\, S^2\, \frac{\partial^2 V}{\partial S^2} + (r-q)\, S\, \frac{\partial V}{\partial S} - r\, V = 0, \quad S > 0,\ t < T,$$
$$V(S,T) = P_{\rm o}(S), \quad S > 0.$$

Use the chain rule to show that if $F = S\, e^{(r-q)(T-t)}$ (the forward price of S over the time interval $[t,T]$), $t' = t$ and $\hat{V}(F,t') = V(S,t)$ then

$$\frac{\partial \hat{V}}{\partial t'} + \tfrac{1}{2}\sigma^2\, F^2\, \frac{\partial^2 \hat{V}}{\partial F^2} - r\, \hat{V} = 0, \quad F > 0,\ t' < T,$$
$$\hat{V}(F,T) = P_{\rm o}(F), \quad F > 0.$$

Put $F = S\, e^{(r-q)(T-t)}$, $t' = t$ and $\hat{V}(F,t') = V(S,t)$. We have

$$\frac{\partial V}{\partial S} = \frac{\partial F}{\partial S}\,\frac{\partial \hat{V}}{\partial F} = e^{(r-q)(T-t)}\,\frac{\partial \hat{V}}{\partial F}$$

and hence

$$S\,\frac{\partial V}{\partial S} = S\, e^{(r-q)(T-t)}\,\frac{\partial \hat{V}}{\partial F} = F\,\frac{\partial \hat{V}}{\partial F}. \tag{99}$$

Similarly, we find that

$$S^2\,\frac{\partial^2 V}{\partial S^2} = F^2\,\frac{\partial^2 \hat{V}}{\partial F^2}. \tag{100}$$

We also have

$$\frac{\partial V}{\partial t} = \frac{\partial t'}{\partial t}\,\frac{\partial \hat{V}}{\partial t'} + \frac{\partial F}{\partial t}\,\frac{\partial \hat{V}}{\partial F} = \frac{\partial \hat{V}}{\partial t'} - (r-q)\, F\,\frac{\partial \hat{V}}{\partial F} \tag{101}$$

Substituting (99)–(101) into the Black–Scholes equation gives

$$\frac{\partial \hat{V}}{\partial t'} - (r-q)\, F\,\frac{\partial \hat{V}}{\partial F} + \tfrac{1}{2}\sigma^2\, F^2\,\frac{\partial^2 \hat{V}}{\partial F^2} + (r-q)\, F\,\frac{\partial \hat{V}}{\partial F} - r\, \hat{V} = 0,$$

which clearly simplifies to

$$\frac{\partial \hat{V}}{\partial t'} + \tfrac{1}{2}\sigma^2 F^2 \frac{\partial^2 \hat{V}}{\partial F^2} - r\,\hat{V} = 0,$$

and holds for $F > 0$ and $t' < T$. When $t = T$ we have $t' = T$ and $F = S > 0$ and so the terminal condition becomes

$$\hat{V}(F, T) = P_o(F), \quad F > 0.$$

Part C

1. Consider the following perpetual American option problem. The option's payoff is

$$P_o(S) = \begin{cases} K - S/3 & \text{if } 0 < S \le K, \\ 0 & \text{if } S > K. \end{cases}$$

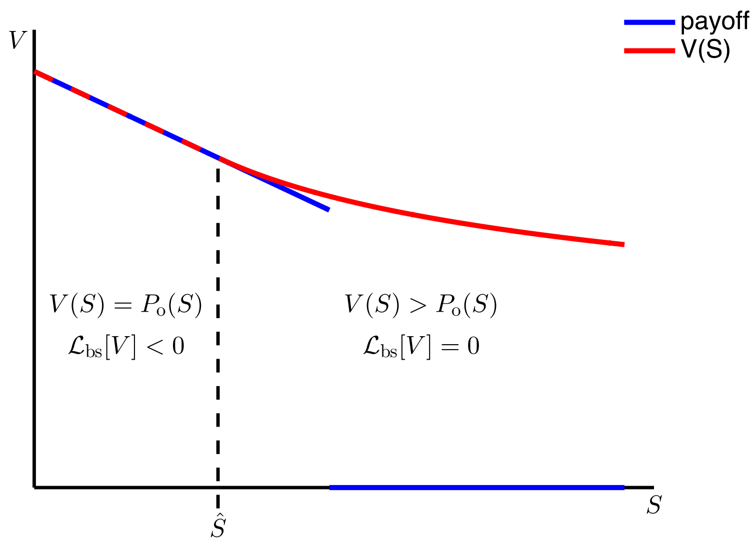
Assume that the option value satisfies the steady-state Black–Scholes equation

$$\mathcal{L}_{\text{ssbs}}[V] = \tfrac{1}{2}\sigma^2 S^2 V''(S) + (r - q)\,S\,V'(S) - r\,V = 0, \quad \hat{S} < S,$$

where $0 < \hat{S} \le K$ is the optimal exercise boundary and where $\sigma > 0$, $r > 0$ and $q > 0$ are constants. The option satisfies the boundary conditions

$$V(\hat{S}) = K - \hat{S}/3, \qquad \lim_{S\to\infty} V(S) \to 0.$$

(a) Give a *sketch* of the payoff and option price as functions of S and indicate where $\mathcal{L}_{\text{ssbs}}[V] = 0$, where $\mathcal{L}_{\text{ssbs}}[V] < 0$, where $V(S) > P_o(S)$ and where $V(S) = P_o(S)$.



(b) Show that, under the assumptions given above, the quadratic

$$p(m) = \tfrac{1}{2}\,\sigma^2\, m\,(m-1) + (r-q)\,m - r$$

has two distinct real roots and only one of these is strictly negative. It is clear that

$$\lim_{m\to-\infty} p(m) \to \infty, \quad \lim_{m\to\infty} p(m) \to \infty$$

and that

$$p(0) = -r < 0\,, \quad p(1) = -q < 0\,.$$

Given that a quadratic can has only one turning point.

So one root is negative and the other is greater than one.

(c) Assume that we have smooth pasting at $\hat{S}$, i.e., $V'(\hat{S}) = -1/3$. Show that this implies that

$$\hat{S} = \frac{3m^-}{m^- - 1}\,K,$$

where $m^- < 0$ is the negative root of the quadratic $p(m)$.

First note that if we assume that $V(S) = S^m$ then m satisfies the quadratic equation

$$p(m) = \tfrac{1}{2}\sigma^2\, m(m-1) + (r-q)\,m - r = 0$$

so there are two real roots, $m^- < 0$ and $m^+ > 1$. Thus the general solution of the ODE for $V(S)$ is

$$V(S) = A\,S^{m^-} + B\,S^{m^+}.$$

Second note that $V(\hat{S}) = K - \hat{S}/3$ and $\lim_{S\to\infty} V(S) \to 0$ imply that

$$V(S) = (K - \hat{S}/3)\left(\frac{S}{\hat{S}}\right)^{m^-}$$

for $S \geq \hat{S}$.

If the smooth pasting condition $V'(\hat{S}) = -1/3$ applies then we have

$$V'(S) = m^-\left(\frac{K - \hat{S}/3}{S}\right)\left(\frac{S}{\hat{S}}\right)^{m^-}$$

and so

$$V'(\hat{S}) = m^-\left(\frac{K - \hat{S}/3}{\hat{S}}\right) = -\frac{1}{3}.$$

When solved for $\hat{S}$ this gives

$$\hat{S} = \frac{3m^- K}{m^- - 1}.$$

(d) Show that smooth pasting only makes sense if $-\frac{1}{2} < m^- < 0$.

Clearly we need $\hat{S} > 0$ and $\hat{S} \leq K$. In the first case, the asset price can never reach $S = 0$ or $S < 0$ and so the option would never be exercised. In the latter case if $\hat{S} > K$ then we are exercising the option when its payoff is zero, i.e., for nothing, and this is clearly not optimal.

Thus we need

$$0 < \frac{3m^- K}{m^- - 1} \leq K$$

and since $K > 0$ this translates to

$$0 < \frac{3m^-}{m^- - 1} \leq 1$$

Given that we know $m^- < 0$ from Part (b) we automatically have

$$0 < \frac{3m^-}{m^- - 1}.$$

The other inequality, together with $m^- < 0$, gives

$$3m^- \geq m^- - 1,$$

which is equivalent to

$$m^- \geq -\tfrac{1}{2}.$$

(e) What is the optimal exercise boundary if $m^- < -\frac{1}{2}$? Justify your answer.

If $m^- < -\frac{1}{2}$ then we must have $\hat{S} = K$, i.e., the optimal exercise boundary is at the strike.

It is clear that we can't have $\hat{S} > K$ as this implies we exercise the option when the payoff is zero, which is clearly not optimal.

Suppose that we have $0 < \hat{S} < K$. Then, as above, we find that

$$V'(\hat{S}) = m^- \left(\frac{K - \hat{S}/3}{\hat{S}} \right) = m^- \left(\frac{K}{\hat{S}} - \tfrac{1}{3} \right) \leq \tfrac{2}{3}\, m^- < -\tfrac{1}{3}.$$

(Note that as $K/\hat{S} > 1$, it follows that $K/\hat{S} - \frac{1}{3} > \frac{2}{3}$ and hence that $m^-\big(K/\hat{S} - \frac{1}{3}\big) \leq \frac{2}{3}\, m^-$, because $m^- < 0$.) This means that the option's value falls below the payoff for S greater than but close to $\hat{S}$, which is an arbitrage for an American option.

Since both $\hat{S} < K$ and $\hat{S} > K$ are both impossible, the only option is $\hat{S} = K$. (When $\hat{S} = K$ we have, for $S > \hat{S}$, a non-zero value for the option which is always above the (zero) payoff.)

(f) Suppose that $-\frac{1}{2} < m^- < 0$, so that smooth pasting does give the correct optimal exercise boundary. Suppose also that the holder of the option decides that they are going to ignore the optimal exercise boundary $\hat{S}$ and simply exercise the option as soon as $S \le \bar{S}$ where $0 < \bar{S} < K$ is chosen by the holder. In this case the value of the option, $\bar{V}(S,t)$, satisfies the problem

$$\mathcal{L}_{\mathrm{ssbs}}[\,\bar{V}\,] = 0, \quad S > \bar{S},$$
$$\bar{V}(\bar{S}) = K - \bar{S}/3, \quad \lim_{S\to\infty} \bar{V}(S) \to 0.$$

Find $\bar{V}(S)$ and show that

i. if $\bar{S} > \hat{S}$ then one could increase the value of the option by decreasing $\bar{S}$ (hint; differentiate with respect to $\bar{S}$);

ii. if $\bar{S} < \hat{S}$ then there is a potential arbitrage in the price $\bar{V}(S)$ (hint; differentiate with respect to S).

As above the general solution of the ODE is

$$\bar{V}(S;\bar{S}) = A\,S^{m^-} + B\,S^{m^+},$$

where $m^- < 0$, $m^+ \ge 1$. The condition that $\lim_{S\to\infty} \bar{V}(S) = 0$ shows that $B = 0$, so

$$\bar{V}(S;\bar{S}) = A\,S^{m^-}.$$

The condition $\bar{V}(\bar{S};\bar{S}) = K - \bar{S}/3$ gives

$$A\,\bar{S}^{m^-} = K - \bar{S}/3 \quad \Longrightarrow \quad A = (K - \bar{S}/3)/\bar{S}^{m^-}$$

and hence

$$\bar{V}(S;\bar{S}) = (K - \bar{S}/3)\left(\frac{S}{\bar{S}}\right)^{m^-}.$$

It follows that

$$\begin{aligned}
\frac{\partial \bar{V}}{\partial S}(S;\bar{S}) &= m^- \frac{(K-\bar{S}/3)}{S}\left(\frac{S}{\bar{S}}\right)^{m^-} \\
\frac{\partial \bar{V}}{\partial \bar{S}}(S;\bar{S}), &= -\left(\frac{1}{3} + m^-\left(\frac{K}{\bar{S}} - \frac{1}{3}\right)\right)\left(\frac{S}{\bar{S}}\right)^{m^-}.
\end{aligned}$$

On the one hand, if $\bar{S} > \hat{S}$ then it follows that $K/\bar{S} < K/\hat{S}$. Noting that $m^- < 0$ and that $\hat{S}$ satisfies the equation

$$\frac{1}{3} + m^-\left(\frac{K - \hat{S}/3}{\hat{S}}\right) = \frac{1}{3} + m^-\left(\frac{K}{\hat{S}} - \frac{1}{3}\right) = 0,$$

it follows that

$$\frac{\partial \bar{V}}{\partial \bar{S}}(S;\bar{S}) < 0.$$

This implies that by *decreasing* the value of $\bar{S}$ we can *increase* the value of $\bar{V}$. On the other hand, if $\hat{S} > \bar{S}$ then $K/\hat{S} < K/\bar{S}$. Then we see that

$$\begin{aligned}\frac{\partial \bar{V}}{\partial S}(S;\bar{S}) &= m^{-}\left(\frac{K}{\bar{S}} - \frac{1}{3}\right) \\ &< m^{-}\left(\frac{K}{\hat{S}} - \frac{1}{3}\right) \qquad (recall m^{-} < 0) \\ &= -\frac{1}{3}.\end{aligned}$$

This implies that the option's price falls *strictly below* the payoff (to the right of $\bar{S}$, near to $\bar{S}$), which is an arbitrage.

Exercise Sheet 4

Part A

1. The instalment option has the same payoff as that of a vanilla call or a put option; it may be European or American. Its unusual feature is that, as well as paying the initial premium, the holder must pay 'instalments' during the life of the option. The instalments may be paid either continuously or discretely. The holder can choose at any time to stop paying the instalments, at which point the contract is cancelled and the option ceases to exist.

   Assume instalments are paid continuously at a rate $L(t)$ per unit time. Derive the differential equation satisfied by the option price. What new constraint must it satisfy?

   Solution

   Since instalments are paid continuously at a rate $L(t)$, when we build our Black–Scholes portfolio we will have to subtract Ldt to account for the continuously paid instalments.

   Hence, we build our portfolio as usual,

$$\Pi = V - \Delta S, \tag{102}$$

   and therefore

$$d\Pi = dV - \Delta dS - Ldt; \tag{103}$$

where at this stage we subtract the contribution of the instalments.

Now we invoke Itô's Lemma and write

$$\begin{aligned} d\Pi &= \left(V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + \mu S V_S\right) dt + \sigma S V_S dZ - \Delta\mu S dt - \Delta\sigma S dZ - L dt \\ &= \left(V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + \mu S V_S - \Delta\mu S - L\right) dt + \sigma S\left(V_S - \Delta\right) dZ. \end{aligned} \tag{104}$$

In order to eliminate the randomness from this equation we clearly must set $\Delta = V_S$, as customary, therefore (104) becomes

$$d\Pi = \left(V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} - L\right) dt; \tag{105}$$

and in order to eliminate any possibilities of arbitrage our portfolio must grow at a risk-less rate, hence

$$\begin{aligned} d\Pi &= r\Pi dt \\ &= r(V - \Delta S) \\ &= r(V - S V_S). \end{aligned} \tag{106}$$

Finally, replacing (106) back into (105) and arranging terms we obtain the differential equation satisfied by this option price,

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + r S V_S - r V - L = 0. \tag{107}$$

Part B

1. Assume that the USD/GBP exchange rate, X_t, evolves according to the SDE

$$\frac{dX_t}{X_t} = \mu\, dt + \sigma\, dW_t.$$

   (a) Today's exchange rate is X_0, find the expected USD/GBP exchange rate $\mathbb{E}[X_T]$ at time $T > 0$.

   Write $dX_t = \mu\, X_t\, dt + \sigma\, X_t\, dW_t$ and integrate

$$X_t - X_0 = \mu \int_0^t X_u\, du + \sigma \int_0^T X_u\, dW_u$$

then take expectations to get

$$\begin{aligned} \mathbb{E}\big[X_t\big] - X_0 &= \mu\, \mathbb{E}\Big[\int_0^t X_u\, du\Big] + \sigma\, \mathbb{E}\Big[\int_0^t X_u\, dW_u\Big] \\ &= \mu \int_0^t \mathbb{E}[X_u]\, du \end{aligned}$$

and then differentiate with respect to t,

$$\frac{d\mathbb{E}\big[X_t\big]}{dt} = \mu\, \mathbb{E}\big[X_t\big].$$

Solve this for $\mathbb{E}\big[X_t\big]$ to find

$$\mathbb{E}\big[X_t\big] = X_0\, e^{\mu t},$$

which gives $\mathbb{E}\big[X_T\big] = X_0\, e^{\mu T}$.

(b) Find the SDE followed by the GBP/USD exchange rate, i.e., find the dynamics for $Y_t = 1/X_t$. Applying Itô's lemma to $Y_t = f(X_t)$ where $f(X) = 1/X$ gives

$$\begin{aligned} dY_t &= -\frac{dX_t}{X_t^2} + \frac{d[X]_t}{X_t^3} \\ &= -\frac{\mu}{X_t}\, dt - \frac{\sigma}{X_t}\, dW_t + + \frac{\sigma^2}{X_t}\, dt \\ &= (\sigma^2 - \mu)\, Y_t\, dt - \sigma\, Y_t\, dW_t, \end{aligned}$$

that is,

$$\frac{dY_t}{Y_t} = (\sigma^2 - \mu)\, dt - \sigma\, dW_t.$$

(c) Given that $Y_0 = 1/X_0$ today, find the expected GBP/USD exchange rate $\mathbb{E}[Y_T]$ at time $T > 0$.

The SDE satisfied by Y_t has the same form as that for X_t but with μ replaced by $\sigma^2 - \mu$ and σ replaced by $-\sigma$. Therefore

$$\mathbb{E}[Y_T] = Y_0\, e^{(\sigma^2 - \mu)T} = \frac{1}{X_0}\, e^{(\sigma^2-\mu)T}.$$

(d) Show that, although $X_T\, Y_T = 1$ for any $T > 0$,

$$\mathbb{E}[X_T]\, \mathbb{E}[Y_T] = e^{\sigma^2 T}.$$

By definition, $Y_T = 1/X_T$ so $X_T\, Y_T = 1$. From the calculations above,

$$\mathbb{E}[X_T]\, \mathbb{E}[Y_T] = e^{\sigma^2 T},$$

which is strictly greater than one unless $T = 0$ or $\sigma = 0$.

2. A European log-put option has the payoff

$$V_T = \big(-\log(S_T/K)\big)^+$$

(a) Show that if S_u evolves as

$$\frac{dS_u}{S_u} = r\,du + \sigma\,dW_u,\ t < u \le T, \quad S_t = S,$$

then

$$\text{prob}(S_T < K) = \text{N}(-d_-), \quad d_- = \frac{\log(S/K) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sqrt{\sigma^2\,(T-t)}}.$$

As $S_t = S$, we can write

$$S_T = S\,e^{(r-\sigma^2/2)\tau + \sigma\,W_\tau}, \quad \tau = T - t,$$

from which we can see that

$$\begin{aligned}
\text{prob}(S_T < K) &= \text{prob}\big(\log(S_T) < \log(K)\big) \\
&= \text{prob}\big(\log(S) + (r - \tfrac{1}{2}\sigma^2)\tau + \sigma\,W_\tau < \log(K)\big) \\
&= \text{prob}\big(\sigma\,W_\tau < -\log(S/K) - (r - \tfrac{1}{2}\sigma^2)\tau\big) \\
&= \text{prob}\Big(\big(W_\tau/\sqrt{\tau}\big) < -\frac{\log(S/K) + (r - \frac{1}{2}\sigma^2)\tau}{\sqrt{\sigma^2\,\tau}}\Big) \\
&= \text{N}(-d_-),
\end{aligned}$$

because $(W_\tau/\sqrt{\tau}) \sim N(0,1)$.

(b) Assuming the underlying share pays no dividends, show that the Black–Scholes formula for the log-put is

$$V(S,t) = e^{-r(T-t)}\sqrt{\sigma^2\,(T-t)}\,\Big(d_-\,\text{N}(-d_-) - e^{-\frac{1}{2}d_-^2}/\sqrt{2\,\pi}\Big).$$

The simplest way to do this is to use the formula

$$V(S,t) = e^{-r(T-t)}\,\mathbb{E}_t^{\mathbb{Q}}\Big[\,\big(-\log(S_T/K)\big)^+\,\big|\,S_t = S\,\Big],$$

where for $t < u$

$$\frac{dS_u}{S_u} = r\,du + \sigma\,dW_u, \quad S_t = S.$$

With $\tau = T - t$, we have

$$\begin{aligned}
S_T &= S\,\exp\big((r - \tfrac{1}{2}\sigma^2)(T-t) + \sigma\,W_\tau\big) \\
&= S\,\exp\big((r - \tfrac{1}{2}\sigma^2)(T-t) + \sqrt{\sigma^2\,\tau}\,Z\big),
\end{aligned}$$

where $Z = W_\tau/\sqrt{\tau} \sim N(0,1)$, and so

$$\log(S_T/K) = \log(S/K) + (r - \tfrac{1}{2}\sigma^2)\tau + \sqrt{\sigma^2\,\tau}\,Z. \tag{108}$$

Following the same idea as in (a), $-\log(S_T/K) > 0$ iff and only if

$$\begin{aligned} & 0 < -\log(S/K) + (r - \tfrac{1}{2})\tau - \sqrt{\sigma^2\,\tau}\,Z \\ \iff \quad & \sqrt{\sigma^2\,\tau}\,Z < -\log(S/K) + (r - \tfrac{1}{2})\tau \\ \iff \quad & Z < -\frac{\log(S/K) + (r - \frac{1}{2})\tau}{\sqrt{\sigma^2\,\tau}} \\ \iff \quad & Z < -d_-. \end{aligned}$$

Regarding $\log(S_T/K)$ as a function of the random variable $Z \sim \mathcal{N}(0,1)$, as in (108), we see that

$$\begin{aligned} & \mathbb{E}_t^{\mathbb{Q}}\big[\,\big(-\log(S_T/K)\big)^+ \,\big|\, S_t = S\big] \\ & = \frac{1}{\sqrt{2\,\pi}} \int_{-\infty}^{\infty} \big(-\log(S_T/K)\big)^+ e^{-z^2/2}\,dz \\ & = -\frac{1}{\sqrt{2\,\pi}} \int_{-\infty}^{-d_-} \log(S_T/K)\,e^{-z^2/2}\,dz. \end{aligned}$$

Using (108) to express $\log(S_T/K)$ in terms of Z, we get

$$\begin{aligned} & \mathbb{E}_t^{\mathbb{Q}}\Big[\,\big(-\log(S_T/K)\big)^+ \,\Big|\, S_t = S\,\Big] \\ & = -\frac{1}{\sqrt{2\,\pi}} \int_{-\infty}^{-d_-} \Big(\log(S/K) + (r - \tfrac{1}{2}\sigma^2)\tau + \sqrt{\sigma^2\,\tau}\,z\Big)\,e^{-\frac{1}{2}z^2}\,dz \\ & = -\frac{1}{\sqrt{2\,\pi}} \int_{-\infty}^{-d_-} \sqrt{\sigma^2\tau}\,\big(d_- + z\big)\,e^{-\frac{1}{2}z^2}\,dz \\ & = -\sqrt{\sigma^2\tau}\,\Big(d_-\,\mathrm{N}(-d_-) - \frac{1}{\sqrt{2\,\pi}}\,e^{-\frac{1}{2}z^2}\Big|_{-\infty}^{-d_-}\Big) \\ & = \sqrt{\sigma^2(T-t)}\,\Big(\frac{e^{-\frac{1}{2}d_-^2}}{\sqrt{2\,\pi}} - d_-\,\mathrm{N}(-d_-)\Big). \end{aligned}$$

Multiplying this by $e^{-r(T-t)}$ gives the result.

3. An investor has the choice of investing their wealth of 1 unit of currency in either a risky asset whose price evolves as

$$\frac{dS_t}{S_t} = \mu\,dt + \sigma\,dW_t,\ \ t > 0, \quad S_0 = 1,$$

where $\sigma > 0$, or in a risk-free bond whose price evolves as

$$\frac{dB_t}{B_t} = r\,dt,\ t > 0, \quad B_0 = 1,$$

where $0 < r < \mu - \frac{1}{2}\sigma^2$. The investment horizon is $[0, T]$. The investor decides to invest their funds in the risky asset, but is worried that when they withdraw the funds, at time T, the risk-free bonds may have outperformed the risky assets. So they consider the possibility of purchasing a put option with maturity T to protect themselves against this possibility. (They borrow money to buy the put.)

(a) What is the probability of the risky asset underperforming the risk-free one, i.e., what is the probability that $S_T < e^{rT}$?
As we start with $S_0 = 1$ we have

$$S_T = e^{(\mu - \frac{1}{2}\sigma^2)T + \sigma W_T}$$

and we want to know

$$\begin{aligned} \text{prob}\big(\, S_T < e^{rT} \,\big) &= \text{prob}\big(\, (\mu - \tfrac{1}{2}\sigma^2)T + \sigma W_T < rT \,\big) \\ &= \text{prob}\big(\, \sigma W_T < (r - \mu + \tfrac{1}{2}\sigma^2)T \,\big) \\ &= \text{prob}\Big(\, \frac{W_T}{\sqrt{T}} < \frac{(r - \mu + \frac{1}{2}\sigma^2)T}{\sqrt{\sigma^2\, T}} \,\Big) \\ &= \text{N}(\, x \,), \end{aligned}$$

because $W_T/\sqrt{T} \sim \mathcal{N}(0, 1)$ and where

$$x = \frac{(r - \mu + \frac{1}{2}\sigma^2)\sqrt{T}}{\sigma}.$$

(b) What happens to this probability as $T \to \infty$?
We have $r + \frac{1}{2}\sigma^2 < \mu$ and $\sigma > 0$, so

$$\frac{(r - \mu + \frac{1}{2}\sigma^2)}{\sigma} < 0$$

and so $x \to -\infty$ as $T \to \infty$. Thus, the probability of underperformance goes to zero as $T \to \infty$.[2]

[2]It goes to zero extremely rapidly. To see this consider

$$\int_{-\infty}^{-z} e^{-y^2/2}\,dy = \int_z^{\infty} e^{-y^2/2}\,dy = \int_z^{\infty} y\,e^{-y^2/2}\,\frac{dy}{y} = \frac{e^{-z^2/2}}{z} - \int_z^{\infty} e^{-y^2/2}\,\frac{dy}{y^2}$$

which shows that as $z \to \infty$, $\text{N}(-z) \sim \dfrac{e^{-z^2/2}}{\sqrt{2\,\pi}\,z}$.

(c) What should the strike of the put be so that the investor is fully insured against the possibility of underperformance?

$K = e^{rT}$. Then at T if $S_T < e^{rT}$ the investor can sell the risky asset for e^{rT} by exercising the put and if $S_T \geq e^{rT}$ they can let the put expire worthless. Thus they are guaranteed a final value of $\max(e^{rT}, S_T)$.

(d) What happens to the price of the insurance as $T \to \infty$?

As $S_0 = 1$ and $K = e^{rT}$, the price of the put at $t = 0$ is

$$\begin{aligned} P(1, 0; T) &= e^{rT} e^{-rT} \mathrm{N}(-d_-) - (- d_+) \\ &= \mathrm{N}(-d_-) - \mathrm{N}(-d_+) \end{aligned}$$

with

$$d_\pm = \frac{\log(1/e^{rT}) + (r \pm \frac{1}{2}\sigma^2)T}{\sqrt{\sigma^2 T}} = \frac{\pm\frac{1}{2}\sigma^2 T}{\sqrt{\sigma^2 T}} = \pm\tfrac{1}{2}\sqrt{\sigma^2 T}.$$

This shows that

$$\begin{aligned} P(1, 0; T) &= \mathrm{N}\big(\sqrt{\sigma^2 T}\,\big) - \mathrm{N}\big(-\sqrt{\sigma^2 T}\,\big) \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\sqrt{\sigma^2 T}}^{\sqrt{\sigma^2 T}} e^{-p^2/2}\, dp \leq 1, \end{aligned} \tag{109}$$

and it follows that

$$\lim_{T \to \infty} P(1, 0; T) = 1,$$

i.e., the cost of insurance against underperformance tends to the total value available for investment, even though the probability of underperformance tends to zero.

More generally, it is clear that the risk of underperformance is monotonically decreasing in T but the cost of insurance against underperformance, i.e., the put, is an increasing function of T. If the investor borrows the money to buy the put at $t = 0$, they will owe $e^{rT} P(1, 0; T)$ at time T and they are guaranteed to have $\max(e^{rT}, S_T)$ and so their overall position at time T is

$$\max\Big(\big(1 - P(1, 0; T)\big)\, e^{rT}, S_T - e^{rT} P(1, 0; T)\Big) \geq 0.$$

4. Let T_1 and T_2 be given times with $0 < T_1 < T_2$ and let $\alpha > 0$ be a given constant. A forward-start put is a European put option written on an asset whose price is S_t, but where the strike is not given at time zero, rather it is set equal to $\alpha\, S_{T_1}$, where S_{T_1} is the share price at time T_1. Find the option price and Δ for $T_1 < t < T_2$ and then for $0 \leq t \leq T_1$.

For time $T_1 < t < T_2$ we know $K = \alpha\, S_{T_1}$ and so we have a regular put option with price

$$P(S,t) = K\, e^{-r(T_2-t)}\,\mathrm{N}(-d_-) - S\, e^{-q(T_2-t)}\,\mathrm{N}(-d_+)\,,$$

where

$$d_\pm = \frac{\log(S/K) + (r - q \pm \frac{1}{2}\sigma^2)(T_2-t)}{\sqrt{\sigma^2(T_2-t)}}, \qquad K = \alpha\, S_{T_1}\,,$$

and the delta of the option is $-e^{-q(T_2-t)}N(-d_+)$.

At time $t = T_1$ we have $S = S_{T_1}$ and $K = \alpha\, S = \alpha\, S_{T_1}$ by definition. Thus we have

$$P(S,T_1) = \alpha\, S\, e^{-r(T_2-T_1)}\,\mathrm{N}(-\hat{d}_-) - S\, e^{-q(T_2-T_1)}\,\mathrm{N}(-\hat{d}_+)$$

where

$$\hat{d}_\pm = \frac{\log(\alpha) + (r - q \pm \frac{1}{2}\sigma^2)(T_2-T_1)}{\sqrt{\sigma^2(T_2-T_1)}}.$$

Thus we can write

$$P(S,T_1) = S\, A(T_2, T_1, r, y, \sigma, \alpha)$$

where

$$A(T_2, T_1, r, q, \sigma, \alpha) = \alpha\, e^{-r(T_2-T_1)}\,\mathrm{N}(-\hat{d}_-) - e^{-q(T_2-T_1)}\,\mathrm{N}(-\hat{d}_+)$$

is independent of both S and t.

Solving that Black–Scholes equation backwards from T_1 we see that for $t \le T_1$ we have

$$P(S,t) = S\, e^{-q(T_1-t)}\, A(T_2, T_1, r, q, \sigma, \alpha).$$
$$\Delta(S,t) = e^{-q(T_1-t)}\, A(T_2, T_1, r, q, \sigma, \alpha).$$

Part C

1. Assume the stock price S follows the usual Black–Scholes dynamics and that there are no dividend payments. Let $0 < T_1 < T_2$ and $K > 0$. A derivative security with the following properties is written on a share (which does not pay any dividends between time $t = 0$ and $t = T_2$). If at time T_1 the share price is greater than or equal to K, $S_{T_1} \ge K$, then the derivative security becomes a European call option with strike S_{T_1} and expiry date T_2. If $S_{T_1} < K$, it becomes a European put option with strike S_{T_1} and expiry date T_2. What is the price of this derivative for $T_1 < t < T_2$ and for $0 \le t \le T_1$.

   For $T_1 \le t \le T_2$ we know the value of S_{T_1}. If $S_{T_1} \ge K$ then we have a call option with strike S_{T_1}, so

$$V(S,t) = C_{\mathrm{bs}}(S, t; \text{strike} = S_{T_1})\,,$$

while if $S_{T_1} < K$, we have a put option with strike S_{T_1}, so

$$V(S,t) = P_{\text{bs}}(S,t;\text{strike} = S_{T_1})\,.$$

Thus, for $T_1 \le t \le T_2$ we have

$$V(S,t) = \begin{cases} S_{T_1}\, e^{-r(T_2-t)}\,\mathrm{N}(-d_-) - S\,\mathrm{N}(-d_+) & \text{if } S_{T_1} < K, \\ S\,\mathrm{N}(d_+) - S_{T_1}\, e^{-r(T_2-t)}\,\mathrm{N}(d_-) & \text{if } S_{T_1} \ge K\,, \end{cases}$$

where

$$d_\pm = \frac{\log(S/S_{T_1}) + (r \pm \frac{1}{2}\sigma^2)(T_2 - t)}{\sqrt{\sigma^2(T_2-t)}}.$$

At time T_1, by definition $S = S_{T_1}$ so

$$V(S,T_1) = \begin{cases} A(T_1,T_2)\,S & \text{if } S < K\,, \\ B(T_1,T_2)\,S & \text{if } S \ge K\,, \end{cases}$$

where

$$\begin{aligned} A(T_1,T_2) &= e^{-r(T_2-T_1)}\,\mathrm{N}(-\hat{d}_-) - \mathrm{N}(-\hat{d}_+) \\ B(T_1,T_2) &= \mathrm{N}(\hat{d}_+) - e^{-r(T_2-T_1)}\,\mathrm{N}(\hat{d}_-) \end{aligned}$$

and

$$\hat{d}_\pm = \frac{(r \pm \frac{1}{2}\sigma^2)(T_2 - T_1)}{\sqrt{\sigma^2(T_2-T_1)}}\,.$$

Use $\mathrm{N}(x) + \mathrm{N}(-x) = 1$ to write

$$\begin{aligned} &B(T_1,T_2) - A(T_1,T_2) \\ &\qquad = \mathrm{N}(\hat{d}_+) + \mathrm{N}(-\hat{d}_+) - e^{-r(T_2-T_1)}\big(\mathrm{N}(\hat{d}_-) + \mathrm{N}(-\hat{d}_-)\big) \\ &\qquad = 1 - e^{-r(T_2-T_1)}\,, \end{aligned}$$

so

$$B(T_1,T_2) = A(T_1,T_2) + C(T_1,T_2)\,\mathbf{1}_{\{S\ge K\}},$$

where $C(T_1,T_2) = 1 - e^{-r(T_2-T_1)}$. Thus,

$$V(S,T_1) = A(T_1,T_2)\,S + C(T_1,T_2)\,S\,\mathbf{1}_{\{S<K\}}.$$

As the share pays no dividends, any multiple of S is a solution of the Black–Scholes equation and so the component $A(T_1,T_2)\,S$ of the payoff leads to a price that is always $A(T_1,T_2)\,S$. The component of the payoff $C(T_1,T_2)\,S\,\mathbf{1}_{\{S\ge K\}}$ is $C(T_1,T_2)$ gap-calls — $S\,\mathbf{1}_{\{S\ge K\}}$ is zero if $S < K$ and S if $S \ge K$. Thus if we denote the price function for a gap-call with strike K by $C_g(S,t;K,T_1)$ we find that for $t < T_1$ we have

$$V(S,t) = A(T_1,T_2)\,S + C(T_1,T_2)\,C_g(S,t;K,T_1).$$

Although you were not asked to find a formula for $C_g(S,t)$, it is a relatively simple thing to do. Recall that

- if $U(S,t)$ is a solution of the Black–Scholes equation then so is $S\,(\partial U/\partial S)$;
- the delta of a call option is given by $\Delta(S,t) = (\partial C/\partial S)(S,t) = \mathrm{N}(d_+)$; and
- $\Delta_c(S,T) = \mathbf{1}_{\{S\geq K\}}$.

It follows that $S\,\Delta_c(S,t) = S\,\mathrm{N}(d_+)$ is a solution of the Black–Scholes equation with the property that $S\,\Delta_c(S,T) = S\,\mathbf{1}_{\{S\geq K\}}$, so we must have

$$C_g(S,t;K,T) = S\,\mathrm{N}(d_+),$$

where, as usual,

$$d_+ = \frac{\log(S/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sqrt{\sigma^2(T-t)}}.$$