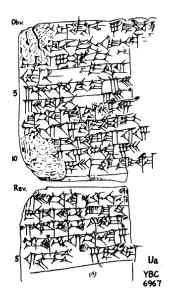
BO1.1. History of Mathematics Lecture IX Classical algebra: equation solving 1800 BC – AD 1800

MT24 Week 5

Summary

- Early quadratic equations
- Cubic and quartic equations
- ► Further 16th-century developments
- ▶ 17th century ideas
- ▶ 18th century ideas
- ► Looking back

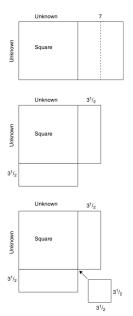
Completing the square, c. 1800 BC



A Babylonian scribe, clay tablet BM 13901, c. 1800 BC:

A reciprocal exceeds its reciprocal by 7. What are the reciprocal and its reciprocal? Break in half the 7 by which the reciprocal exceeds its reciprocal, and $3\frac{1}{2}$ will come up. Multiply $3\frac{1}{2}$ by $3\frac{1}{2}$ and $12\frac{1}{4}$ will come up. Append 60, the area, to the $12\frac{1}{4}$ which came up for you and $72\frac{1}{4}$ will come up. What is the square-side of $72\frac{1}{4}$? $8\frac{1}{2}$. Put down $8\frac{1}{2}$ and $8\frac{1}{2}$ and subtract $3\frac{1}{2}$ from one of them; append $3\frac{1}{2}$ to one of them. One is 12, the other is 5. The reciprocal is 12, its reciprocal 5.

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Points to note

- ► We have used the word 'equation' without writing down anything in symbols
- Solution recipe derived from geometrical insight
- Not (explicitly) a general solution but reader ought to be able to adapt the method
- Is this algebra? Geometrical algebra?

Diophantus of Alexandria (3rd century AD)

Arithmeticorum Liber I

... Abiiciatur denominator partium, Erit itaque primusto. fe-H4. & fatisfaciunt quæstioni.

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IN QUAESTIONEM XXVI.

E A D Z M ratio off huius quaritionis , qua & pracedentis. Quaritio infinitas reci-pit folutiones , & fi determinanda fit ad vnicam , praferibendus off numerus in quo fieri debetæqualitas, tunoque operabimur vt in præcedente traditum est. Quod autem denominatores abiici iubet Diophantus, vt solutio in integris habestur, id sit quia fi inuenti femel numeri quæltioni fatisfacientes , per eundem multiplicentur vel dissidantur, produkta itidem & quotientes quartionem folsent, cuius rei zatio est quam attingit Xilander, quia feibert questin numeri, partes proportionales vicillim dant & accipiun, que autem pariom cognomisme achem tocrum inter fe, a c vicillim estratio. Vnde etiam colligi potest alius modus folsendi huiusmodi questioper, cum numerus przeferibitur in quo fise zqualitas. N. m.fi quæftio prius foluatur per operationem Diophanti, & numerus in quo fize qualitas diuidatur per eum qui przeferibitur, & per quotientem diuidantur teem inuenti numeri per operationem Diophanti, habebuntur quæfiti numeri. Verbi gratia, fi quærantur quatuor numeridantes & accipientes ealdem partes quas requirit Diophantus, ita ve facta contributione quilibet reperiator 59 ; folues prius que flionem cum Diophanto, & inue-ntes numeros 150. 9. 120. 114. Et numerus in quo fix equalitas erit 119. Hunc. ergo i dividas per numerum præfcitprum 59. l erit quotieros 2. per quem fi diuidas fi-gillatim inuentos numeros, fieta 75. 46. 65. 77. qua fii numeri. Polifecteam ram guardinamente de la control de dando fui quadrantem,& accipiendo trientem primi fiat 7. Tertius dando fui quintandem, & accipiendo quadrantem fecundi fiat 14. Quartus dando fui fextantem , & recipiendo quintanté tertij, fiat 15. Et tunc imitabimur artificium operationis quæ ad przeodente tradita eft, hoc modo. Ponatur primus 3 N. cum ergo multatur fuo trien-te & auclus fextante quarti faciat 6. erit 6 ~ 2 N. fextans quarti, & ipfe quartus 36 – 12 N. vnde ablato fextante, manent 30. – 10 N. quæ cum quintante tertij debent facere 23. Igitur quintans terrijest 10 N. - 7. Ideoque ipse terrius est 50 N. - 35. qui multatus quintante maner 40 N - 18, debetque tunc cum quadrante fecundi facere 14. Quare 42 - 40 N. eft quadrans fecundi, & ipfe fecundus 168 - 150 N. vnde ablato quadrante manent 126 - 110 N. quæ cum triente primi debene facere 7. fed facient 116 - 119 N. hoc ergo acquatur 7. & fit 1 N. 1. Ad politiones primuseft 3. focundus 8. tertius 15. quartus 24.

OVÆSTIO XXVIL

ΤΡΕΙΝ Είνε Σειθμοις όπως INVENTRE tres numeros ve quilibet à reliquis duobus έκαςος παρά τ λοιπών δύο ώς consunctis partem imperatam coos 20 69 mipo po Antardos. accipiat, & fiant æquales. Staputum fit primum à reliquis

Problem 1.27: Find two numbers such that their sum and product are given numbers

Muḥammad ibn Mūsā al-Khwārizmī (c. 780-c. 850)



Noted six cases of equations:

- 1. Squares are equal to roots $(ax^2 = bx)$
- 2. Squares are equal to numbers $(ax^2 = c)$
- 3. Roots are equal to numbers (bx = c)
- 4. Squares and roots are equal to numbers $(ax^2 + bx = c)$
- 5. Squares and numbers are equal to roots $(ax^2 + c = bx)$
- 6. Roots and numbers are equal to squares $(bx + c = ax^2)$

Muḥammad ibn Mūsā al-Khwārizmī (c. 780-c. 850)

العظم وهوسطح دي وقلطمناان دلك وإنما نصفنا العشرة الإحلاد وصوساها ذوبتلها وردنا متر رضوت تصفه في مثله فاستعنا الضوب وهداصورته

> وله اليفاصورة احرى تودى للهذا وهو اتب وهوالمال فاددنا ان تريد عليه

An algorithm for case (4) on the previous slide

Leonardo of Pisa (Fibonacci) (c. 1175-c. 1240/50)

Liber abaci (or Liber abbaci), Pisa, 1202:

- included al-Khwārizmi's recipes
- geometrical demonstrations and lots of examples
- didn't go very far beyond predecessors, but began transmission of Islamic ideas to Europe



Cubic equations (1)

Italy, early 16th century:

solutions to cubics of the form $x^3 + px = q$

- ▶ found by Scipione del Ferreo (or Ferro) (c. 1520)
- taught to Antonio Maria Fiore (pupil)
- and Annibale della Nave (son-in-law)
- rediscovered by Niccolò Tartaglia (1535)
- passed in rhyme to Girolamo Cardano (1539)

Cubic equations (2)

$$x^3 + px = q$$

When the cube with the things next after
Together equal some number apart
Find two others that by this differ
And this you will keep as a rule
That their product will always be equal
To a third cubed of the number of things
The difference then in general between
The sides of the cubes subtracted well
Will be your principal thing.

(Tartaglia, 1546; see: Mathematics emerging, §12.1.1)

Cubic equations (3)

$$x^3 + px = q$$

When the cube with the things next after Together equal some number apart Find two others that by this differ And this you will keep as a rule That their product will always be equal To a third cubed of the number of things The difference then in general between The sides of the cubes subtracted well Will be your principal thing.

Interpretation of Tartaglia's rhyme:

Find u, v such that

$$u-v=q, \quad uv=\left(\frac{p}{3}\right)^3.$$

Then

$$x = \sqrt[3]{u} - \sqrt[3]{v}$$

NB: In an equation $y^3 + ay^2 + by + c = 0$ we can put $y = x - \frac{a}{3}$ to remove the square term, so this solution is general.

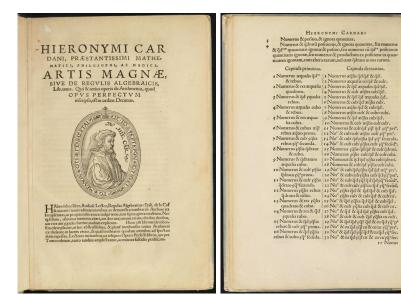
Cubic equations (4)

In modern terms, one of the solutions of the equation $ax^3 + bx^2 + cx + d = 0$ has the form

$$x = \sqrt[3]{\left(-\frac{b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) + \sqrt{\left(-\frac{b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}} + \sqrt[3]{\left(-\frac{b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) - \sqrt{\left(-\frac{b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}} - \frac{b}{3a}}$$

with similar expressions (in radicals) for the remaining two roots

Cardano's Ars magna, sive de regulis algebraicis (1545)



Cardano on the cubic



- Geometrical justification remains
- General solution (to particular case), rather than example to be followed
- Make substitution $x = y \frac{a}{3}$ in $y^3 + ax^2 + bx + c = d$ to suppress square term and obtain equation of the form $x^3 + px = q$ — manipulation of equations prior to solution

Quartic equations (1)

General solution discovered (again on a case-by-case basis) by Lodovico Ferrari (c. 1540) and published by Cardano, in the form of worked examples, alongside solution of cubic

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Angunt future audier quantum

1. Aufgrunt future; audier

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1. positioner pris, fit totum 2 positiones numeri quadratorum, 2 positioner pris, fit totum 2 positiones numeri quadratorum, 2 positioner pris, fit totum 2 positiones numeri quadratorum, 2 positiones quantum 2 positiones numeri quadratorum, 2 positiones numeri q

Quartic equations (2)

In modern terms, suppose that

$$x^4 = px^2 + qx + r.$$

Add $2yx^2 + y^2$ to each side to give

$$(x^2 + y)^2 = (p + 2y)x^2 + qx + (r + y^2).$$

Now we seek y such that the right hand side is a perfect square:

$$8y^3 + 4py^2 + 8ry + (4pr - q^2) = 0.$$

So the problem is reduced to solving a cubic equation and then a quadratic.

NB: In an equation $y^4 + ay^3 + by^2 + cy + d = 0$ we can put $y = x - \frac{a}{4}$ to remove the cube term, so this solution is general.

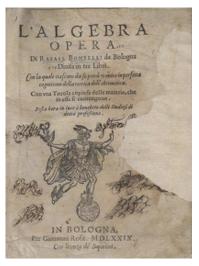
Quartic equations (3)

Formulae for the solutions of the general quartic equation, in all their unedifying glory, may be found at:

http://planetmath.org/QuarticFormula

Cardano's Ars Magna may also be found online here

Further 16th-century developments



Rafael Bombelli, L'algebra (1572):

- heavily influenced by Cardano
- equation solving, new notation
- exploration of complex numbers[to be dealt with in a later lecture]

Further 16th-century developments

L'ARITHMETIQUE DE SIMON STEVIN DE BRUGES:

Contenant les computations des nombres Arithmetiques ou vulgaires :

Außi l'Algebre, auec les equations de cinc quamitez. Ensemble les quatre premiers liures d'Algebre de Diophante d'Alexandrie, maintenant premierement traduicts en François.

Encore vn liure particulier de la Pratique d'Arithmetique, contenant entre autres, Les Tables d'Interes?, La Difine; Et vn traillé des Incommensurables grandeurs: Aucc l'Explication du Dixtesme Liure d'Euclide.



A LEYDE,
De l'Imprimerie de Christophle Plantin.
c15. 15. LXXXV.

Simon Stevin, L'arithmetique ... aussi l'algebre (1585):

- heavily influenced by Cardano through Bombelli
- appended his treatise on decimal notation

Further 16th-century developments

François Viète (1590s):

- links between algebra and geometry
- (algebra as 'analysis' or 'analytic art')
- notation [recall Lecture III]
- numerical methods for solving equations



Thomas Harriot (c. 1600)

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Note:

- notation [see lecture III];
- appearance of polynomials as products of linear factors.

Thomas Harriot (1631)

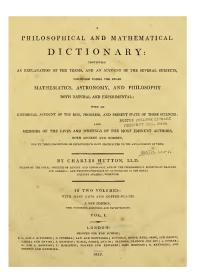
ARTIS ANALYTICAE PRAXIS. Ad æquationes Algebraïcas nouâ, expeditâ, & generali methodo, refoluendas: TRACTATVS E posthumis THOME HARRIOTI Philosophi ac Mathematici celeberrimi schediasmatis summa fide & diligentia descriptus: FLLVSTRISSIMO DOMINO DOM. HENRICO PERCIO, NORTHYMBRIE COMITI Oui bac primo, sub Patronatus & Munificentia sua auspicijs ad proprios vius elucubrata, in communem Mathematicorum vtilitatem, denuò reuisenda, describenda, & publicanda mandauit, meritiflimi Honoris ergò Nuncupatus. LONDINI Apud ROBERTYM BARKER, Typographum Regium: Et Hæred. I o. BILLIL

Some of Harriot's ideas found their way into his *Artis* analyticae praxis (The practice of the analytic art), published posthumously in 1631

But editors did not permit negative or imaginary roots [to be discussed further in a later lecture]

See *Mathematics emerging*, §12.2.1.

Commentary on Harriot



Charles Hutton, *A mathematical and philosophical dictionary*, London, 1795, vol. 1, p. 91 (p. 96 of revised edition, 1815):

He shewed the universal generation of all the compound or affected equations, by the continual multiplication of so many simple ones; thereby plainly exhibiting to the eye the whole circumstances of the nature, mystery and number of the roots of equations; with the composition and relations of the coefficients of the terms: . . .

Algebra in the 17th century

From 1600 onwards, 'algebra' as a set of recipes and techniques began to diverge in two (linked) directions:

- 'algebra' as a tool or a language (a.k.a. 'analysis' or the 'analytic art')
- 'algebra' as an object of study in its own right (the 'theory of equations')

Descartes on algebra

Polynomials feature in Descartes' La géométrie (1637), e.g.:

- one example to show that polynomials can be constructed from their roots (influenced by Harriot?);
- 'rule of signs': the number of positive ('true') roots of a polynomial is at most the number of times that the sign changes as we read term-by-term; the number of negative ('false') roots is at most the number of successions of the same sign; for example,

$$x^4 - 4x^3 - 19xx + 106x - 120 = 0$$

has at most 3 positive roots and at most one negative;

can always make a transformation to remove the second-highest term.

Descartes on cubics

LIVRE TROISISSME.

2.

tiplication, que par dissers autres moyens, qui fonc affar facilers a trouver. Puis examinant par ordre outres les quantités, qui peunent dissifer fans fixédion le dernier cerme, il faut voir, fi quelqu'une d'éliels, sointe auce la quantité discommé parle figne + ou -, peu compofer vo binome, qui dusife toute la fomme, & fi cela et le Problefine et Pala , c'êt à direil peu et fire confluir auce la reigle & de compas ; Car obbien la quantité comno é de ce binofine et la racine cherchée ; oubien l'Esquation estant distifée par luy ; se reduit a decas dimensions, en forte qu'on en peut trouver aprés la racine, par ce qui a els dies au premier liure.

Par exemple fi on a

le deraier terme, qui est 64, pour estre dinisé sans frachion par 1, 3, 4, 8, 16; 33, & 64, C'est pourquoy al laut examiner par ordre si exte Equation ne peut point estre d'unsée par quelqu'un des binomes , y - 2 ou y y + 1, y y - 204y y + 2, y - 4 & c. & con tronné qu'elle peut l'estre par y - 1 de ne ces forte.

y + 8 yy + 4 200.

te commence par le dornier terme, & diufie - 64 par da l'about - 16, co qui fait + 4, que l'eferis dans le quotient , puis voitsusie multiplie + 4 par + 17 ye, ce qui fait + 4 79 5; chi pour - bialoquoy l'eferis - 47 ye et a fomme, qu'il faut diufier, cert ly me qu'
B b b 1 faut constit fa

Search for roots of a cubic by examining the factors of the constant term:

if α is such a factor, test whether $x-\alpha$ divides the polynomial.

Examines the example

$$y^6 - 8y^4 - 124y^2 - 64 = 0$$

Descartes on quartics

LIVAE TROISIESME.

Befoinde paffer outre; card fuit de la infulliblement, que le problème et foide. Mais fi en la troune, en peu dionier par fou moyen la precedente Equation en deux autres, en chafcone desquelles la quantié inconnét d'aira que deux dimensions, & données racines fenon les mesmes que les fienes. A s'ganoir, au lieu de traisité par en present les memes que les fienes. A s'ganoir, au lieu de traisité par en present les memes que les fienes.

il faut escrite ces deux autres

+ xx-yx+ ½yy .½p .½, ∞o,&

+xx+yx+1xy, 1p. 1 200.

Et pour les fignes + & - que lay omis, s'il y a + β en l'Equation precedente, il faut mettre $+\frac{1}{2}\beta$ en chaffune de celles cy, & $-\frac{1}{2}\beta$, s'il y a en l'autre $-\beta$. A las il faut mettre $+\frac{2}{4}\beta$, en celle où il y a $+\gamma s_1$ lor fqu'il y a $+\gamma$ que la premiere. Et au contraire s'il y a $-\gamma$, il faut mettre $-\frac{2}{4}\beta$, en celle où il y a trais c'il y a $-\gamma$, il faut mettre $-\frac{2}{4}\beta$, en celle où il y a

-yx; & $+\frac{1}{y}$; en celle outly y = +yx. En fuite dequoy if eft sylé de connoitrertoutes les racines de l'Equation propolée, & par confequent de confirmure le probletime, dont elle confient la foliation , fans y employer que des vercles, & des lignes droites.

Par exemple à cause que faisant

y -347° +313yy - 400 me, ponr "
x "-17xx-20x-6 me, on troute que yy est 16, ondoifau lien de cete Equation

+ xh' - 17 dx - 20 n - 20 n - 6 20 e, escrire ces denx

To solve $+x^4 \star .pxx.qx.r = 0$ (Descartes' notation), that is,

$$x^4 \pm pxx \pm qx \pm r = 0,$$

he sought to write the quartic as a product of two quadratics. This led him to

$$y^6 \pm 2py^4 + (pp \pm 4r)yy - qq = 0$$

As in Ferrari's/Cardano's method: a quartic is reduced to a cubic

Summary and a glance ahead

By 1600, general solutions were available for quadratic, cubic and quartic equations — specifically, general solutions in radicals, i.e., solutions constructed from the coefficients of a given polynomial equation via +, -, \times , \div , \sqrt , $\sqrt[3]{}$, $\sqrt[4]{}$, ...

NB: A solution in radicals may be constructed by ruler and compass.

Spoiler: the general quintic equation is not solvable in radicals.

By the 1770s, mathematicians (notably Lagrange) had come to suspect this, but it was not proved until the 1820s.

So did anything interesting happen in algebra during the 17th and 18th centuries?

A typical 20th-century view

Luboš Nový, Origins of modern algebra (1973), p. 23:

From the propagation of Descartes' algebraic knowledge up to the publication of the important works of Lagrange [and others] in the years 1770–1, the evolution of algebra was, at first glance, hardly dramatic and one would seek in vain for great and significant works of science and substantial changes.

Fair point? Or not?

Some 17th-century developments: Hudde's rule (1657)

434 IOHANNIS HUDDENII EPIST. L. quæro, per Methodum superiùs explicatam, maximum earum communem diviforem; arque hujus ope zquationem Propositam toties divido, quoties id fieri po-

Exempli gratia, proponatur hac aquatio x1-4xx+ (x-1mo. in qua duz funt zquales radices. Multiplico ergo ipfam per Arithmeticam Progressionem qualemcunque, hoc est, cujus incrementum vel decrementum fit vel 1 , vel 2 , vel 4 , vel alius quilibet numerus; & cujus primus terminus sit vel o, vel +, vel quam o : Ita ut semper ejus ope talis terminus æquationis tolli possit, qualem quis voluerit, collocando tantum sub eo o. Ut fi , exempli causa, ultimum ejus terminum auferre velim.

multiplicatio fieri potelt iplius x1-4xx+ (x-2 20 2 per hanc progressionem 3. 2. 1. 0
fietque 1x1-8xx+(x * 200.

tionis est x - 1 20 0, per quam Proposita bis dividi potest; ita ut eiusdem radices sint 1 . 1 . & 2 . Sic fi cupiam 1 mum æquationis terminum auferre, multiplica-

tio inflitui poteft ipfius x1 - 4xx + 1 x - 2 mo per hanc progressionem o. 1. 2. 3.

& fit * -4xx+10x-6 200. Cujus quidem ac Propolitæ æquationis maximus communis divisor, ut antea, est x - 1 200.

Similiter si 2dam terminum tollere lubeat, multiplicatio sieri potest, hoc pacto: x1-4xx+5x-1x00

$$+1.$$
 0. $-1.-1$
& prodibit x^1 * $-5x+4\infty0$.

Cujus item & Propofitæ maximus communis divifor est

Ubi notandum, non neccliarium effe, semper uti Progressione cujus excessus sit 1, quanquam ea communiter sit optima.

Published 1659 as an addendum to van Schooten's Latin translation of Descartes' La géométrie:

take
$$x^3 - 4xx + 5x - 2 = 0$$

multiply the terms of the equation by numbers in arithmetic progression:

$$3x^3 - 8xx + 5x = 0$$

$$\gcd(x^3-4xx+5x-2,3x^3-8xx+5x) = x-1$$

so the original polynomial can be divided by x-1 twice — hence it has a double root at x = 1

See Mathematics emerging, §12.2.2.

Some 17th-century developments: Tschirnhaus transformations (1683)

ACTA ERUDITORUM
METHODUS AUFERENDI OMNES TER.
minos intermedios ex data aquatione,
per D. T.

EN Geometria Do. Des Cartes notum eft, qua tationa femper fem, des retrains ex dra ac quation polificatieri; quond plures terme nos intermedies auferendos, hadenus nibil inventum vidi in Artenia, priteria, inmo no pauco offendi, qui credidentun; la in ulla arte pede polife. Quapropere hie quardam circa hoc negotium aperite conflieta, verum filiem pro itis, qui Artis Analytica apprime garat; cum alta tam brevi explicatione via fatisficir polific: reliqua, qua hie defidenti polifica, talif tempori refervans.

—pyy −2p y −pa I is nihlo (quia hunc auferre offir a; tento) prinque 3 yy −p yy ∞. Inde −re = fento) prinque 3 yy −p yy ∞. Inde −re = ± i quod indicat, ad auferendam fecundum terminum in ≈quaitone Cubica, fipponendum effe less xy y −4 (prout modo fecimus) x=y +ξ. Hac jam vulgata admodum funt, nec hie referentum taliam ob exulum, quam quia fequenții

dum tunt, nee ne reteruntu anam ob causam, quam quia tequenta admodum illuftrant, dum hifce bene intellectis, eo facilius, qua modo proponam, capientur.

 For an equation $x^3 - px^2 + qx - r = 0$

- ▶ to remove one term put x = y + a (where a = p/3)
- can we remove both the middle terms?
- to remove two terms put $x^2 = bx + y + a$

See Mathematics emerging, §12.2.3.

An 18th-century development: Newton's *Arithmetica* universalis (1707)

newten, bir sease
Universal Arithmetick:

OR, A

TREATISE

O F

ARITHMETICAL

Composition and Resolution.

To which is added,

Dr. HALLEY'S Method of finding the Roots of Æquations Arithmetically.

Translated from the LATIN by the late Mr. RAPHSON, and revised and corrected by Mr. CUNN.



Printed for J. Senex at the Globs in Salisbury-Carr; W. Taylor at the Ship, T. Warner at the Blat-Boy, in Pater-nofer Rem, and J. Osborn at the Original-Arm in Lomberd Servet. 1720. Rules for sums of powers of roots of

$$x^{n}-px^{n-1}-qx^{n-2}-rx^{n-3}-sx^{n-4}-\cdots=0$$

sum of roots
$$=$$
 a $=$ p
sum of roots² $=$ b $=$ $pa+2q$
sum of roots³ $=$ c $=$ $pb+qa+3r$
sum of roots⁴ $=$ d $=$ $pc+qb+ra$
 $+4s$

Extension of Descartes's Rule of Signs to cover numbers of complex roots, and other techniques

Developments of the 17th and 18th centuries

- Symbolic notation
- Understanding of the structure of polynomials
- ... of the number and nature of their roots
- ... of the relationship between roots and coefficients
- ... of how to manipulate them
- ... of how to solve them numerically
- ▶ The leaving behind of geometric intuition?

Some 18th-century theory of equations

Recall:

- cubic equations can be solved by means of quadratics
- quartic equations can be solved by means of cubics

Some 18th-century theory of equations

Recall:

- quadratic equations can be solved by means of linear equations
- cubic equations can be solved by means of quadratics
- quartic equations can be solved by means of cubics

The 'reduced' or 'resolvent' equation:

- for cubics, the reduced equation is of degree 2
- for quartics, the reduced equation is of degree 3
- for quintics, the reduced equation is of degree ?

Some 18th-century hypotheses

Euler's hypothesis (1733):

• for an equation of degree n the degree of the reduced equation will be n-1

Bézout's hypothesis (1764):

- ▶ for an equation of degree *n* the degree of the reduced equation will in general be *n*!
- ▶ though always reducible to (n-1)!
- ▶ possibly further reducible to (n-2)!

Lagrange's 'Réflexions' 1770/71

J.-L. Lagrange, 'Réflexions sur la résolution algébrique des équations', Berlin (1770/1):

Examined all known methods of solving

- quadratics: the well-known solution
- cubics: methods of Cardano, Tschirnhaus, Euler, Bézout
- quartics: methods of Cardano, Descartes, Tschirnhaus, Euler, Bézout

seeking to identify a uniform method that could be extended to higher degree

A typical 20th-century view revisited

Luboš Nový, Origins of modern algebra (1973), p. 23:

From the propagation of Descartes' algebraic knowledge up to the publication of the important works of Lagrange [and others] in the years 1770–1, the evolution of algebra was, at first glance, hardly dramatic and one would seek in vain for great and significant works of science and substantial changes.

Filling a gap in the history of algebra (2011)

Heritage of European Mathematics Jacqueline Stedall From Cardano's great art to Lagrange's reflections: filling a gap in the history of algebra European Mathematical Society

The hitherto untold story of the slow and halting journey from Cardano's solution recipes to Lagrange's sophisticated considerations of permutations and functions of the roots of equations . . . [Preface]

From Stedall's preface:

This assertion . . . from Nový quoted above, betrays yet another fundamental shortcoming of several earlier accounts, a view that mathematics somehow progresses only by means of 'great and significant works' and 'substantial changes'. Fortunately, the truth is far more subtle and far more interesting: mathematics is the result of a cumulative endeavour to which many people have contributed, and not only through their successes but through half-formed thoughts, tentative proposals, partially worked solutions, and even outright failure. No part of mathematics came to birth in the form that it now appears in a modern textbook: mathematical creativity can be slow, sometimes messy, often frustrating.