BO1.1. History of Mathematics
Lecture XIII
Complex analysis

MT24 Week 7

Summary

- Complex numbers: validity and representation
- Substitution of complex values for real
- Cauchy's contributions
- Riemann
- What *is* an analytic function?

Early ideas about complex numbers

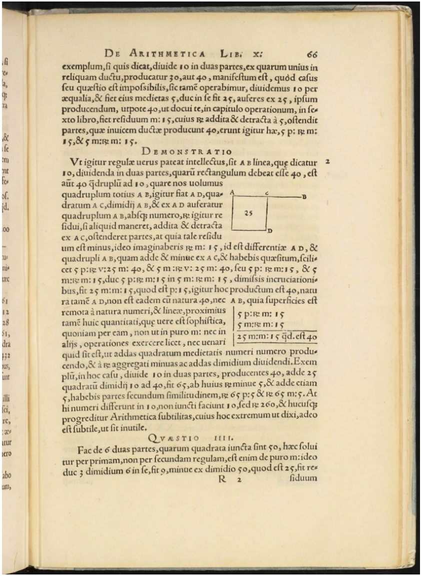
Before 1600, very faint beginnings:

- Cardano (1545) [from quadratics]
- Bombelli (1572) [from cubics]
- Harriot (c. 1600) [from quartics]

But:

> *For the most part such roots were ignored: negative roots were described merely as 'false', but complex roots as 'impossible'.*
>
> (*Mathematics emerging*, p. 459.)

Cardano and complex numbers



De Arithmetica Lib. x: 66

exemplum, si quis dicat, diuide 10 in duas partes, ex quarum unius in reliquam ductu, producatur 30, aut 40, manifestum est, quòd casus seu quæstio est impossibilis, sic tamẽ operabimur, diuidemus 10 per æqualia, & fiet eius medietas 5, duc in se fit 25, auferes ex 25, ipsum producendum, utpote 40, ut docui te, in capitulo operationum, in sexto libro, fiet residuum m: 15, cuius ℞ addita & detracta à 5, ostendit partes, quæ inuicem ductæ producunt 40, erunt igitur hæ, 5 p: ℞ m: 15, & 5 m: ℞ m: 15.

Demonstratio

Vt igitur regulæ uerus pateat intellectus, sit A B linea, quę dicatur 10, diuidenda in duas partes, quarũ rectangulum debeat esse 40, est aũt 40 q̃druplũ ad 10, quare nos uolumus quadruplum totius A B, igitur fiat A D, quadratum A C, dimidij A B, & ex A D auferatur quadruplum A B, absq; numero, ℞ igitur residui, si aliquid maneret, addita & detracta ex A C, ostenderet partes, at quia tale residuum est minus, ideo imaginaberis ℞ m: 15, id est differentiæ A D, & quadrupli A B, quam adde & minue ex A C, & habebis quæsitum, scilicet 5 p: ℞ v: 25 m: 40, & 5 m: ℞ v: 25 m: 40, seu 5 p: ℞ m: 15, & 5 m: ℞ m: 15, duc 5 p: ℞ m: 15 in 5 m: ℞ m: 15, dimissis incruciationibus, fit 25 m: m: 15, quod est p: 15, igitur hoc productum est 40, natura tamẽ A D, non est eadem cũ natura 40, nec A B, quia superficies est remota à natura numeri, & lineæ, proximius tamẽ huic quantitati, quę uere est sophistica, quoniam per eam, non ut in puro m: nec in aliꝭs, operationes exercere licet, nec uenari quid sit est, ut addas quadratum medietatis numeri numero producendo, & à ℞ aggregati minuas ac addas dimidium diuidendi. Exemplũ, in hoc casu, diuide 10 in duas partes, producentes 40, adde 25 quadratũ dimidij 10 ad 40, fit 65, ab huius ℞ minue 5, & adde etiam 5, habebis partes secundum similitudinem, ℞ 65 p: 5 & ℞ 65 m: 5. At hi numeri differunt in 10, non iuncti faciunt 10, sed ℞ 260, & hucusq; progreditur Arithmetica subtilitas, cuius hoc extremum ut dixi, adeo est subtile, ut sit inutile.

5 p: ℞ m: 15
5 m: ℞ m: 15
25 m: m: 15 q̃d. est 40

Qvæstio IIII.

Fac de 6 duas partes, quarum quadrata iuncta sint 50, hæc soluitur per primam, non per secundam regulam, est enim de puro m: ideo duc 3 dimidium 6 in se, fit 9, minue ex dimidio 50, quod est 25, fit residuum

R 2

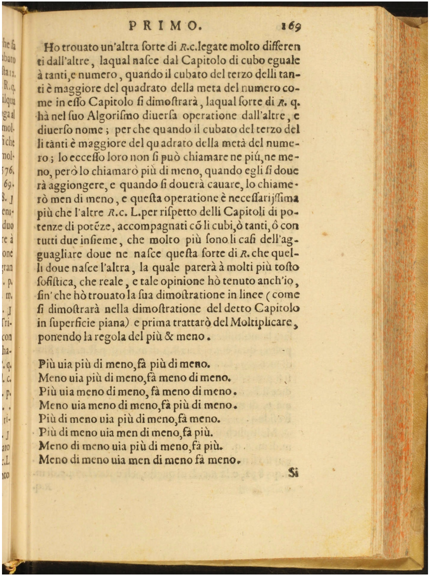
Problem: find two numbers that add to 10 and multiply to 40, i.e., solve an equation of the type ‘square plus number equals thing’

Cardano noted that $5+\sqrt{-15}$ and $5-\sqrt{-15}$ solve the problem, “dismissis incruciationibus”, meaning

“putting aside mental tortures”,
or
“the cross-multiples having canceled out”,
or
“the imaginary part being lost”

But regarded such ideas as absurd and useless

Bombelli and complex numbers



PRIMO. 169

Ho trouato un'altra ſorte di R.c.legate molto differenti dall'altre, laqual naſce dal Capitolo di cubo eguale à tanti, e numero, quando il cubato del terzo delli tanti è maggiore del quadrato della meta del numero come in eſſo Capitolo ſi dimoſtrarà, laqual ſorte di R. q. hà nel ſuo Algoriſmo diuerſa operatione dall'altre, e diuerſo nome; per che quando il cubato del terzo delli tanti è maggiore del quadrato della metà del numero; lo eccesſo loro non ſi può chiamare ne piú, ne meno, però lo chiamarò più di meno, quando egli ſi douerà aggiongere, e quando ſi douerà cauare, lo chiamerò men di meno, e queſta operatione è neceſſarijſſima più che l'altre R.c. L.per riſpetto delli Capitoli di potenze di potēze, accompagnati cō li cubi, ò tanti, ô con tutti due inſieme, che molto più ſono li caſi dell'agguagliare doue ne naſce queſta ſorte di R. che quelli doue naſce l'altra, la quale parerà à molti più toſto ſofiſtica, che reale, e tale opinione hò tenuto anch'io, ſin' che hò trouato la ſua dimoſtratione in linee (come ſi dimoſtrarà nella dimoſtratione del detto Capitolo in ſuperficie piana) e prima trattarò del Moltiplicare, ponendo la regola del più & meno.

Più uia più di meno, fà più di meno.
Meno uia più di meno, fà meno di meno.
Più uia meno di meno, fà meno di meno.
Meno uia meno di meno, fà più di meno.
Più di meno uia più di meno, fà meno.
Più di meno uia men di meno, fà più.
Meno di meno uia più di meno, fà più.
Meno di meno uia men di meno fà meno.

Si

"Another sort of cube root much different from the former ..."

Systematic rules:

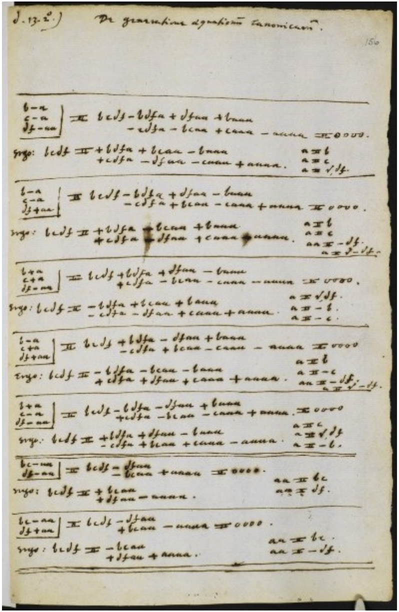
più di meno via più di meno, fà meno $(\sqrt{-1} \times \sqrt{-1} = -1)$

meno di meno via più di meno, fà più $(-\sqrt{-1} \times \sqrt{-1} = 1)$

But complex numbers were not admitted as solutions of equations — they could appear in calculations, provided they cancelled out by the end

Complex numbers justified through practical use?

Harriot and complex numbers



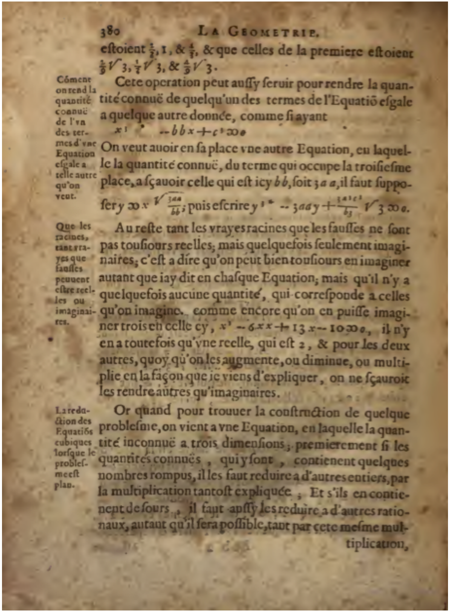
Add MS 6783 f. 156

Unpublished manuscripts contain systematic treatment of complex roots of equations — but these were removed by his editors

Cf. Harriot's *Artis analyticae praxis* (1931), pp. 14–15; see:

Muriel Seltman & Robert Goulding, *Thomas Harriot's Artis analyticae praxis: an English translation with commentary*, Springer, 2007

Descartes and ‘imaginaries’



estoient $\frac{2}{3}$, 1, & $\frac{4}{3}$, & que celles de la premiere estoient $\frac{2}{9}\sqrt{3}$, $\frac{1}{3}\sqrt{3}$, & $\frac{4}{9}\sqrt{3}$.

Cõment on rend la quantité connuë de l'vn des termes d'vne Equation esgale a telle autre qu'on veut.

Cete operation peut aussy seruir pour rendre la quantité connuë de quelqu'un des termes de l'Equatiõ esgale a quelque autre donnée, comme si ayant

$$x^3 * -bbx + c^3 \propto 0$$

On veut auoir en sa place vne autre Equation, en laquelle la quantité connuë, du terme qui occupe la troisiesme place, a sçauoir celle qui est icy bb, soit $3aa$, il faut supposer $y \propto x\sqrt{\frac{3aa}{bb}}$; puis escrire $y^3 * -3aay + \frac{3a^3c^3}{b^3}\sqrt{3} \propto 0$.

Que les racines, tant vrayes que fausses peuuent estre reelles ou imaginaires.

Au reste tant les vrayes racines que les fausses ne sont pas tousiours reelles; mais quelquefois seulement imaginaires; c'est a dire qu'on peut bien tousiours en imaginer autant que iay dit en chasque Equation; mais qu'il n'y a quelquefois aucune quantité, qui corresponde a celles qu'on imagine. comme encore qu'on en puisse imaginer trois en celle cy, $x^3 - 6xx + 13x - 10 \propto 0$, il n'y en a toutefois qu'vne reelle, qui est 2, & pour les deux autres, quoy qu'on les augmente, ou diminue, ou multiplie en la façon que ie viens d'expliquer, on ne sçauroit les rendre autres qu'imaginaires.

La reduction des Equatiõs cubiques lorsque le problesme est plan.

Or quand pour trouuer la construction de quelque problesme, on vient a vne Equation, en laquelle la quantité inconnuë a trois dimensions; premierement si les quantités connuës, qui y sont, contienent quelques nombres rompus, il les faut reduire a d'autres entiers, par la multiplication tantost expliquée; Et s'ils en contienent de sours, il faut aussy les reduire a d'autres rationaux, autant qu'il sera possible, tant par cete mesme multiplication,

La géométrie (1637):

introduced the term ‘imaginaire’ — meant to be derogatory?

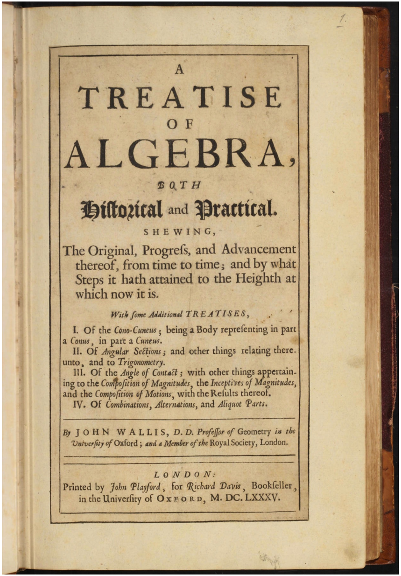
Didn't regard them as numbers

Ideas about complex numbers in the later 17th century

John Wallis, *A treatise of algebra* (1685): complex numbers based on insights derived from

- Euclidean geometry
- trigonometry
- properties of conics

(See: *Mathematics emerging*, §15.1.1.)



1.

A

TREATISE

OF

ALGEBRA,

BOTH

Historical and Practical.

SHEWING,

The Original, Progress, and Advancement thereof, from time to time; and by what Steps it hath attained to the Heighth at which now it is.

With some Additional TREATISES,

I. Of the *Cono-Cuneus*; being a Body representing in part a *Conus*, in part a *Cuneus*.
II. Of *Angular Sections*; and other things relating thereunto, and to *Trigonometry*.
III. Of the *Angle of Contact*; with other things appertaining to the *Composition of Magnitudes*, the *Inceptives of Magnitudes*, and the *Composition of Motions*, with the Results thereof.
IV. Of *Combinations*, *Alternations*, and *Aliquot Parts*.

By JOHN WALLIS, *D.D. Professor of* Geometry *in the University of* Oxford; *and a Member of the* Royal Society, London.

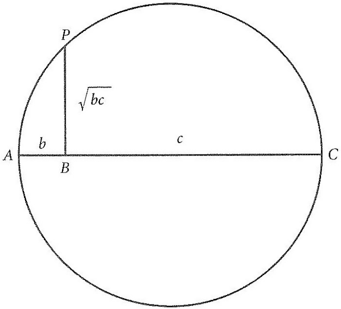
LONDON:
Printed by *John Playford*, for *Richard Davis*, Bookseller, in the University of OXFORD, M. DC. LXXXV.

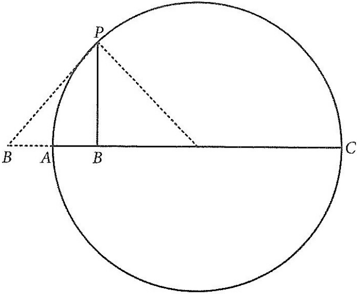
Wallis: justification of imaginary numbers

- A man starts at A and walks 5 yds to B, then retreats 2 yds to C: overall, he has covered 3 yds. If he instead retreats 8 yds to D, then we may say that he has covered -3 yds.
- Somewhere on the seashore, we gain 26 units of land from the sea, but lose 10 units. Thus, we have gained 16 units overall; if this is a perfect square, then it has side 4 units of length.
- If instead we lose 26 units of land, but gain 10, then we have lost 16 units overall, or gained -16. The area in question (assumed to be a square) might therefore be viewed as having side $\sqrt{-16}$.

(see: Leo Corry, *A brief history of numbers*, OUP, 2015, pp. 184–185)

Wallis: imaginary numbers as geometric means

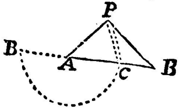




(see: Leo Corry, *A brief history of numbers*, OUP, 2015, pp. 185–186)

“A new Impossibility in Algebra”

John Wallis, *A treatise of algebra*, p. 267 ‘Of negative squares’:
... requires a new Impossibility in Algebra



Suppoſe again, $AP = 15$, $PC = 12$, (and therefore $AC = \sqrt{:225 - 144:} = \sqrt{81} = 9$,) $PB = 20$ (and therefore $BC = \sqrt{:400 - 144:} = \sqrt{256} = +16$, or -16:) Then is $AB = 9 + 16 = 25$, or $AB = 9 - 16 = -7$. The one Affirmative, the other Negative. (The ſame values would be, but with contrary Signs, if we take $AC = \sqrt{81} = -9$: That is, $AB = -9 + 16 = +7$, $AB = -9 - 16 = -25$.)

⋮

Which gives indeed (as before) a double value of AB, $\sqrt{175}, + \sqrt{-81}$, and $\sqrt{175}, - \sqrt{-81}$: But ſuch as requires a new Impoſſibility in Algebra, (which in Lateral Equations doth not happen;) not that of a Negative Root, or a Quantity leſs than nothing; (as before,) but the Root of a Negative Square. Which in ſtrictneſs of ſpeech, cannot be: ſince that no Real Root (Affirmative or Negative,) being Multiplied into itſelf, will make a Negative Square.

Complex numbers in the 18th century (1)

Nature remained unclear:

“that amphibian between being and not-being, which we call the imaginary root of negative unity” (Leibniz, 1702)

But complex numbers were increasingly being used . . .

Complex numbers in the 18th century (2)

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boles, dépend en partie de la quadrature du cercle, & en partie de la quadrature de l'hyperbole ou de la description de la Logarithmique.

Maniéres abrégées de transformer les différentielles composées en simples, & réciproquement; Et même les simples imaginaires en réelles composées.

PROBL. I. Transformer la différentielle $\frac{adz}{bb-zz}$ en une différentielle Logarithmique $\frac{adt}{2bt}$, & réciproquement.

Faites $z = \frac{t-1}{t+1} \times b$, & vous aurez $\frac{adz}{bb-zz} = \frac{adt}{2bt}$. Réciproquement prenez $t = \frac{+z+b}{-z+b}$, & vous aurez $\frac{adt}{2bt} = \frac{adz}{bb-zz}$.

Corol. On transformera de même la différentielle $\frac{adz}{bb+zz}$ en $\frac{-adt}{2bt\sqrt{-1}}$ différentielle de Logarithme imaginaire; & réciproquement.

PROBL. II. Transformer la différentielle $\frac{adz}{bb+zz}$ en différentielle de secteur ou d'arc circulaire $\frac{-adt}{2\sqrt{t-bbtt}}$; & réciproquement.

Faites $z = \sqrt{\frac{1}{t} - bb}$, & vous aurez $\frac{adz}{bb+zz} = \frac{-adt}{2\sqrt{t-bbtt}}$. Réciproquement prenez $t = \frac{1}{zz+bb}$, & vous aurez $\frac{-adt}{2\sqrt{t-bbtt}} = \frac{adz}{bb+zz}$.

PROBL. III. Transformer la différentielle $\frac{adz}{bb-zz}$ en différentielle de secteur hyperbolique $\frac{adt}{2\sqrt{t+bbtt}}$; & réciproquement.

Faites $z = \sqrt{\frac{1}{t} + bb}$, & ensuite $t = \frac{1}{bb-zz}$; & vous aurez ce qu'on demande.

PROBL.

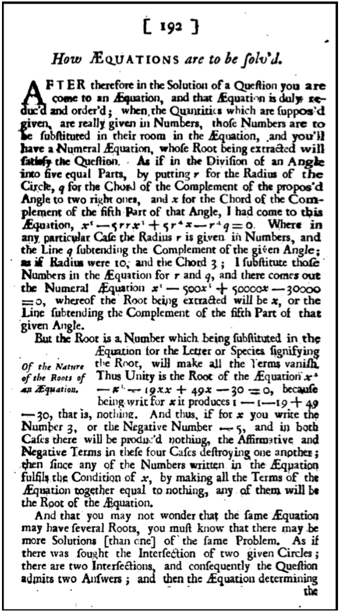
Johann Bernoulli, 'Solution d'un problème concernant le calcul intégrale, ...', *Mémoires de l'Académie royale des sciences*, 1702:

by making the substitution $z = \sqrt{\frac{1}{t} - bb}$, transform the differential $\frac{adz}{bb+zz}$ into $\frac{-adt}{2bt\sqrt{-1}}$

No worries about the validity of switching between real and complex integrals

(See *Mathematics emerging*, §15.2.1)

Complex numbers in the 18th century (3)



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How ÆQUATIONS *are to be ſolv'd.*

AFTER therefore in the Solution of a Queſtion you are come to an Æquation, and that Æquation is duly reduc'd and order'd; when the Quantities which are ſuppos'd given, are really given in Numbers, thoſe Numbers are to be ſubſtituted in their room in the Æquation, and you'll have a Numeral Æquation, whoſe Root being extracted will ſatisfy the Queſtion. As if in the Diviſion of an Angle into five equal Parts, by putting r for the Radius of the Circle, q for the Chord of the Complement of the propos'd Angle to two right ones, and x for the Chord of the Complement of the fifth Part of that Angle, I had come to this Æquation, $x^5 - 5rrx^3 + 5r^4x - r^4q = 0$. Where in any particular Caſe the Radius r is given in Numbers, and the Line q ſubtending the Complement of the given Angle; as if Radius were 10, and the Chord 3; I ſubſtitute thoſe Numbers in the Æquation for r and q, and there comes out the Numeral Æquation $x^5 - 500x^3 + 50000x - 30000 = 0$, whereof the Root being extracted will be x, or the Line ſubtending the Complement of the fifth Part of that given Angle.

Of the Nature of the Roots of an Æquation.

But the Root is a Number which being ſubſtituted in the Æquation for the Letter or Species ſignifying the Root, will make all the Terms vaniſh. Thus Unity is the Root of the Æquation $x^4 - x^3 - 19xx + 49x - 30 = 0$, becauſe being writ for x it produces $1 - 1 - 19 + 49 - 30$, that is, nothing. And thus, if for x you write the Number 3, or the Negative Number -5, and in both Caſes there will be produc'd nothing, the Affirmative and Negative Terms in theſe four Caſes deſtroying one another; then ſince any of the Numbers written in the Æquation fulfils the Condition of x, by making all the Terms of the Æquation together equal to nothing, any of them will be the Root of the Æquation.

And that you may not wonder that the ſame Æquation may have ſeveral Roots, you muſt know that there may be more Solutions [than one] of the ſame Problem. As if there was ſought the Interſection of two given Circles; there are two Interſections, and conſequently the Queſtion admits two Anſwers; and then the Æquation determining the

Isaac Newton, *Universal Arithmetick*, 1728:

p. 195: "it is just that the Roots of Equations should be often impossible, lest they should exhibit the cases of Problems that are impossible as if they are possible" — complex numbers as an indicator of real-world solvability of problems

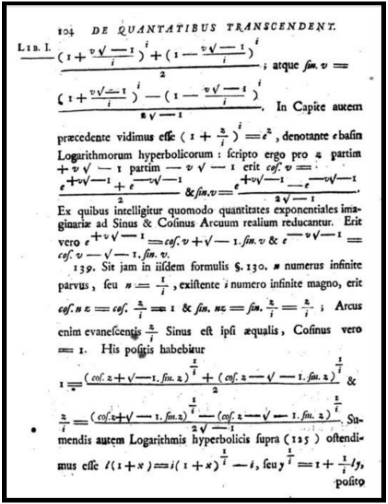
Complex numbers in the 18th century (4)

Leonhard Euler also used them freely: e.g., in *Introductio in analysin infinitorum*, 1748, §138:

$$e^{+v\sqrt{-1}} = \cos.v + \sqrt{-1}.\sin.v$$

$$e^{-v\sqrt{-1}} = \cos.v - \sqrt{-1}.\sin.v$$

(See *Mathematics emerging*, §9.2.3)



104 DE QUANTITATIBUS TRANSCENDENT.

LIB. I. $\frac{(1+\frac{v\sqrt{-1}}{i})^i+(1-\frac{v\sqrt{-1}}{i})^i}{2}$; atque *ſin.* $v =$ $\frac{(1+\frac{v\sqrt{-1}}{i})^i-(1-\frac{v\sqrt{-1}}{i})^i}{2\sqrt{-1}}$. In Capite autem præcedente vidimus eſſe $(1+\frac{z}{i})^i = e^z$, denotante e baſin Logarithmorum hyperbolicorum: ſcripto ergo pro z partim $+v\sqrt{-1}$ partim $-v\sqrt{-1}$ erit *coſ.* $v =$ $\frac{e^{+v\sqrt{-1}}+e^{-v\sqrt{-1}}}{2}$ & *ſin.* $v = \frac{e^{+v\sqrt{-1}}-e^{-v\sqrt{-1}}}{2\sqrt{-1}}$. Ex quibus intelligitur quomodo quantitates exponentiales imaginariæ ad Sinus & Coſinus Arcuum realium reducantur. Erit vero $e^{+v\sqrt{-1}} =$ *coſ.* $v + \sqrt{-1}$. *ſin.* v & $e^{-v\sqrt{-1}} =$ *coſ.* $v - \sqrt{-1}$. *ſin.* v.

139. Sit jam in iiſdem formulis §. 130. n numerus infinite parvus, ſeu $n = \frac{1}{i}$, exiſtente i numero infinite magno, erit *coſ.* $nz =$ *coſ.* $\frac{z}{i} = 1$ & *ſin.* $nz =$ *ſin.* $\frac{z}{i} = \frac{z}{i}$; Arcus enim evaneſcentis $\frac{z}{i}$ Sinus eſt ipſi æqualis, Coſinus vero $= 1$. His poſitis habebitur

$$1 = \frac{(\text{coſ.}\,z+\sqrt{-1}.\,\text{ſin.}\,z)^{\frac{1}{i}} + (\text{coſ.}\,z-\sqrt{-1}.\,\text{ſin.}\,z)^{\frac{1}{i}}}{2} \;\&$$

$$\frac{z}{i} = \frac{(\text{coſ.}\,z+\sqrt{-1}.\,\text{ſin.}\,z)^{\frac{1}{i}} - (\text{coſ.}\,z-\sqrt{-1}.\,\text{ſin.}\,z)^{\frac{1}{i}}}{2\sqrt{-1}}.$$

Sumendis autem Logarithmis hyperbolicis ſupra (125) oſtendimus eſſe $l(1+x) = i(1+x)^{\frac{1}{i}} - i$, ſeu $y^{\frac{1}{i}} = 1 + \frac{1}{i}ly$, poſito

The Fundamental Theorem of Algebra

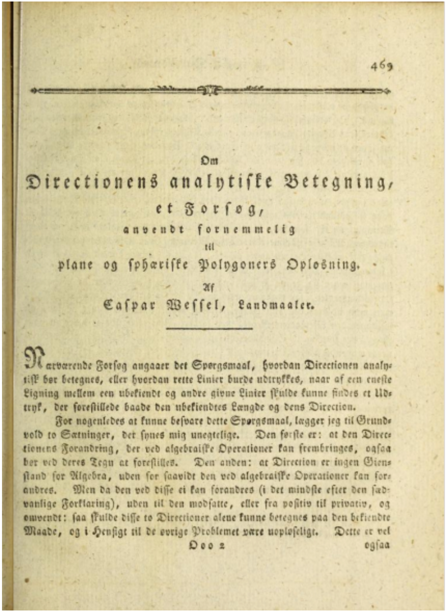
Every polynomial equation of degree n has exactly n roots.

- Early 17th century: known that an equation of degree n may have n roots
- During 17th century: complex numbers gradually admitted as roots
- 15 Sept 1759: Euler asserted theorem in a letter to Nicholas Bernoulli, but didn't prove it
- Mid/late 18th century: attempted proofs by Euler, d'Alembert, Lagrange, and others
- 1799: proof by Gauss in his doctoral dissertation, followed by several others
- 1806: new proof by Argand
- 1821: Argand's proof appears in Cauchy's *Cours d'analyse*

Gauss and complex numbers

"If this subject has hitherto been considered from the wrong viewpoint and thus enveloped in mystery and surrounded by darkness, it is largely an unsuitable terminology which should be blamed. Had $+1$, -1 and $\sqrt{-1}$, instead of being called positive, negative and imaginary (or worse still, impossible) unity, been given the names say, of direct, inverse and lateral unity, there would hardly have been any scope for such obscurity." (1831)

New ways of viewing complex numbers



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Om
Directionens analytiske Betegning,
et Forsøg,
anvendt fornemmelig
til
plane og sphæriske Polygoners Opløsning.
Af
Caspar Wessel, Landmaaler.

Nærværende Forsøg angaaer det Spørgsmaal, hvordan Directionen analytisk bør betegnes, eller hvordan rette Linier burde udtrykkes, naar af een eneste Ligning mellem een ubekiendt og andre givne Linier skulde kunne findes et Udtryk, der forestillede baade den ubekiendtes Længde og dens Direction.

For nogenledes at kunne besvare dette Spørgsmaal, lægger jeg til Grundvold to Sætninger, det synes mig unegtelige. Den første er: at den Directionens Forandring, der ved algebraiske Operationer kan frembringes, ogsaa bør ved deres Tegn at forestilles. Den anden: at Direction er ingen Gienstand for Algebra, uden for saavidt den ved algebraiske Operationer kan forandres. Men da den ved disse ei kan forandres (i det mindste efter den sædvanlige Forklaring), uden til den modsatte, eller fra positiv til privativ, og omvendt: saa skulde disse to Directioner alene kunne betegnes paa den bekiendte Maade, og i Henseende til de øvrige Problemet være uopløselige. Dette er vel

Ooo 2 ogsaa

Caspar Wessel, 'Om Directionens analytiske Betegning ...' ['On the analytic representation of direction ...'], *Nye Samling af det Kongelige Danske Videnskabers Selskabs Skrifter*, 1799

Published in Danish — not well known

French translation published in 1897

New ways of viewing complex numbers

ESSAI

SUR UNE MANIÈRE DE REPRÉSENTER

LES QUANTITÉS IMAGINAIRES

DANS

LES CONSTRUCTIONS GÉOMÉTRIQUES,

PAR R. ARGAND.

2e ÉDITION

PRÉCÉDÉE D'UNE PRÉFACE

PAR M. J. HOÜEL

ET SUIVIE D'UN APPENDICE

Contenant des Extraits des *Annales de Gergonne*, relatifs à la question des imaginaires.

PARIS,

GAUTHIER-VILLARS, IMPRIMEUR-LIBRAIRE

DU BUREAU DES LONGITUDES, DE L'ÉCOLE POLYTECHNIQUE,

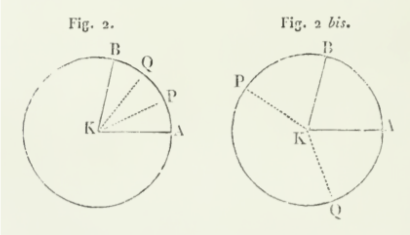
SUCCESSEUR DE MALLET-BACHELIER,

Quai des Augustins, 55.

1874

(Tous droits réservés.)

Robert Argand, *Essay on a method of representing imaginary quantities* ..., 1806



New ways of viewing complex numbers

Theory of Conjugate Functions, or Algebraic Couples; with a Preliminary and Elementary Essay on Algebra as the Science of Pure Time.

By WILLIAM ROWAN HAMILTON,

M. R. I. A., F. R. A. S., Hon. M. R. S. Ed. and Dub., Fellow of the American Academy of Arts and Sciences, and of the Royal Northern Antiquarian Society at Copenhagen, Andrews' Professor of Astronomy in the University of Dublin, and Royal Astronomer of Ireland.

Read November 4th, 1833, and June 1st, 1835.

General Introductory Remarks.

THE Study of Algebra may be pursued in three very different schools, the Practical, the Philological, or the Theoretical, according as Algebra itself is accounted an Instrument, or a Language, or a Contemplation; according as ease of operation, or symmetry of expression, or clearness of thought, (the *agere*, the *fari*, or the *sapere,)* is eminently prized and sought for. The Practical person seeks a Rule which he may apply, the Philological person seeks a Formula which he may write, the Theoretical person seeks a Theorem on which he may meditate. The felt imperfections of Algebra are of three answering kinds. The Practical Algebraist complains of imperfection when he finds his Instrument limited in power; when a rule, which he could happily apply to many cases, can be hardly or not at all applied by him to some new case; when it fails to enable him to do or to discover something else, in some other Art, or in some other Science, to which Algebra with him was but subordinate, and for the sake of which and not for its own sake, he studied Algebra. The Philological Algebraist complains of imperfection, when his Language presents him with an Anomaly; when he finds an Exception disturb the simplicity of his Notation, or the symmetrical structure of his Syntax; when a Formula must be written with precaution, and a Symbolism is not universal. The Theoretical Algebraist complains of imperfection, when the clearness of his Contemplation is obscured; when the Reasonings of his Science seem anywhere to oppose each other, or become in any part too complex or too little valid for his belief to rest firmly upon them; or when, though trial may have taught him that a rule is useful, or that a formula gives true results, he cannot prove that rule, nor understand that formula: when he cannot rise to intuition from induction, or cannot look beyond the signs to the things signified.

Transactions of the Royal Irish Academy, 1837

Complex numbers as ordered pairs subject to specified rules:

$$(a,b) \pm (c,d) = (a \pm c, b \pm d)$$
$$(a,b)(c,d) = (ac - bd, ad + bc)$$
$$\frac{(a,b)}{(c,d)} = \left(\frac{ac+bd}{c^2+d^2}, \frac{bc-ad}{c^2+d^2}\right)$$

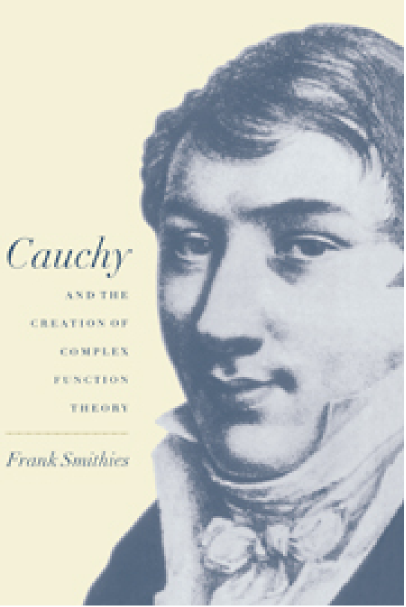
Led to the search for triples, and thence to quaternions

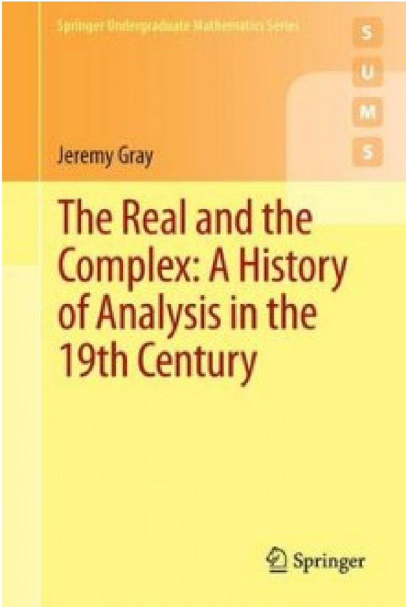
Complex analysis

The origins of complex analysis may be seen in early achievements by Johann Bernoulli, Euler, and others, using complex transformations to evaluate real integrals. But is substitution of complex variables for real variables permissible?

- Euler (posthumous, 1794): yes
- Laplace (1785, 1812): yes
- Poisson (1812): doubtful
- Cauchy (1814): inspired by Laplace, set to work on the problem

Sources for the origins of complex analysis





Cauchy as 'creator' of complex analysis

Some of Cauchy's contributions to complex analysis:

- integration along paths and contours (1814) [1827]
- calculus of residues (1826)
- integral formulae (1831)
- inferences about Taylor series expansions
- applications to evaluation of difficult definite integrals of real functions

Cauchy's changing views of complex numbers and variables

At different times, Cauchy regarded complex numbers in different ways:

- as formal (numerical) expressions $a + b\sqrt{-1}$;
- geometrically;
- by reducing $i = \sqrt{-1}$ to a "real but indeterminate quantity"

  This done, there is no need to torture the mind to discover what the symbolic sign $\sqrt{-1}$ could represent ...

  (in modern terms, Cauchy reduced complex arithmetic to calculations modulo $i^2 + 1$ in $\mathbb{R}[i]$)

Moreover, Cauchy's view of complex variables gradually shifted

- from quantities with two parts $x + y\sqrt{-1}$
- to single quantities z.

Cauchy's first 'Mémoire' (1814/1827)

MÉMOIRE

SUR

LES INTÉGRALES DÉFINIES[1].

INTRODUCTION.

La solution d'un grand nombre de problèmes se réduit, en dernière analyse, à l'évaluation des intégrales définies; aussi les géomètres se sont-ils beaucoup occupés de leur détermination. On trouve, à cet égard, une foule de théorèmes curieux et utiles dans les Mémoires et le *Calcul intégral* d'Euler, dans plusieurs Mémoires de M. Laplace, dans ses *Recherches sur les approximations de certaines formules*, et dans les *Exercices de Calcul intégral* de M. Legendre. Mais, parmi les diverses intégrales obtenues par les deux premiers géomètres que je viens de citer, plusieurs ont été découvertes pour la première fois à l'aide d'une espèce d'induction fondée sur le passage du réel à l'imaginaire. Les passages de cette nature conduisent souvent d'une manière très prompte à des résultats dignes de remarque. Toutefois cette portion de la théorie est, ainsi que l'a observé M. Laplace, sujette à plusieurs difficultés. Aussi, après avoir montré, dans le calcul des fonctions génératrices, les ressources que l'Analyse peut retirer de semblables considérations, l'auteur ajoute : « On peut donc considérer ces passages comme des moyens de découvertes semblables à l'induction dont les

[1] *Mémoires présentés par divers savants à l'Académie royale des Sciences de l'Institut de France et imprimés par son ordre. Sciences mathématiques et physiques.* Tome I. Imprimé, par autorisation du Roi, à l'Imprimerie royale; 1827.

Œuvres de C. — S. I, t. I. 42*

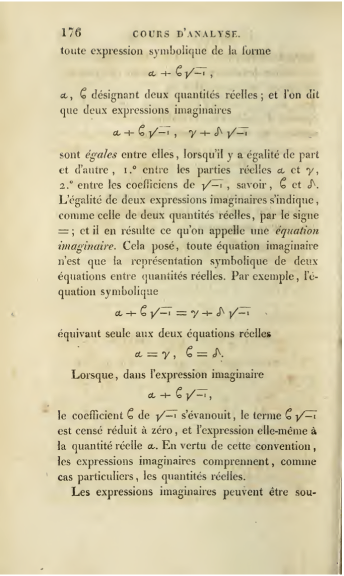
Cited Laplace's concerns about the solution of integrals by "the passage from the real to the imaginary"

First part: evaluation of improper integrals, such as

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}\, dx = \frac{\pi}{e}$$

Noted Cauchy–Riemann equations in passing (as had d'Alembert and Euler) as general useful property of analytic functions, rather than fundamental feature of the theory

Complex numbers in the *Cours d'analyse* (1821)



176 COURS D'ANALYSE.

toute expression symbolique de la forme

$$\alpha + \beta\sqrt{-1},$$

α, β désignant deux quantités réelles; et l'on dit que deux expressions imaginaires

$$\alpha + \beta\sqrt{-1}, \quad \gamma + \delta\sqrt{-1}$$

sont *égales* entre elles, lorsqu'il y a égalité de part et d'autre, 1.° entre les parties réelles α et γ, 2.° entre les coefficiens de $\sqrt{-1}$, savoir, β et δ. L'égalité de deux expressions imaginaires s'indique, comme celle de deux quantités réelles, par le signe =; et il en résulte ce qu'on appelle une *équation imaginaire*. Cela posé, toute équation imaginaire n'est que la représentation symbolique de deux équations entre quantités réelles. Par exemple, l'équation symbolique

$$\alpha + \beta\sqrt{-1} = \gamma + \delta\sqrt{-1}$$

équivaut seule aux deux équations réelles

$$\alpha = \gamma, \quad \beta = \delta.$$

Lorsque, dans l'expression imaginaire

$$\alpha + \beta\sqrt{-1},$$

le coefficient β de $\sqrt{-1}$ s'évanouit, le terme $\beta\sqrt{-1}$ est censé réduit à zéro, et l'expression elle-même à la quantité réelle α. En vertu de cette convention, les expressions imaginaires comprennent, comme cas particuliers, les quantités réelles.

Les expressions imaginaires peuvent être sou-

Defined as "symbolic expressions" $a + b\sqrt{-1}$

55-page development of formal definitions and properties

Consideration of multi-functions — which are the most natural branches to take?

Sought to extend ideas for real functions to the complex case, particularly those relating to power series and convergence

Cauchy's second 'Mémoire' (1825)

'Mémoire sur les intégrales définies, prises entre des limites imaginaires'

Direct adaptation of definition of real integral to the complex case:

$$\int_{x_0+y_0\sqrt{-1}}^{X+Y\sqrt{-1}} f(z)dz$$

is the limit (or one of the limits) of a sum of products of the form

$$\sum (x_{i-1}+y_{i-1}\sqrt{-1})f(x_{i-1}+y_{i-1}\sqrt{-1}).$$

NB. No explicit definition of a function of a complex variable; tacit assumption of differentiability, hence that the Cauchy–Riemann equations hold.

Contour integration

In any domain where the function does not become infinite, the value of a complex integral is independent of the path along which it is taken.

Cauchy: consider two different paths within the rectangle (x_0, y_0), (X, Y) such that the function $f(x + y\sqrt{-1})$ does not become infinite for values of x, y lying within the domain enclosed by the paths. Then the value of the integral $\int_{x_0+y_0\sqrt{-1}}^{X+Y\sqrt{-1}} f(z)dz$ is independent of the path taken.

Really a theorem about real functions in the plane?

(Gauss had discovered this in 1811, alongside a similar definition of a complex integral, but did not publish.)

Contour integration

For the case where $f(x + y\sqrt{-1})$ becomes infinite at the point $x = a, y = b$, Cauchy considered the limit

$$\mathrm{f} := \lim_{\substack{x \to a \\ y \to b}} \left(x - a + (y - b)\sqrt{-1}\right) f\left(x + y\sqrt{-1}\right),$$

and determined that the difference between the integrals of f along different paths that are infinitely close to each other as well as to (a, b) is $2\pi \mathrm{f}\sqrt{-1}$.

With a natural extension of this result for multiple and/or higher-order singularities, this became an ancestor of Cauchy's residue theorem — developed as part of Cauchy's calculus of residues in a paper of 1826 ('Sur un nouveau genre de calcul').

Taylor's Theorem for complex analytic functions

In *Cours d'analyse* (1821), Cauchy had considered the notion of radius of convergence for both real and imaginary power series.

1831: a complex function has a convergent power series if it is "finite and continuous"

Continued to refine the conditions for the theorem over many years.

Cauchy's language is not always satisfactory to modern eyes, but was considerably more rigorous than that of most of his contemporaries.

1841: extension to negative powers — Laurent's Theorem.

Cauchy's complex analysis

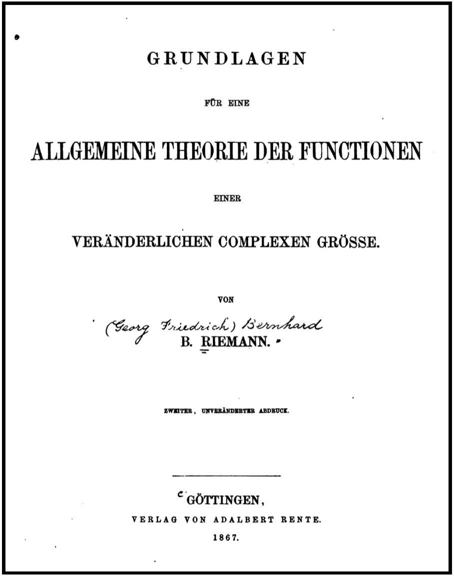
Cauchy's ideas concerning complex functions developed over many years. In the early stages

- did he appreciate the fundamental nature of the concepts and results that he was using and deriving?
- did he recognise the subtleties of working with complex numbers rather than simply with pairs of real numbers?

Have historians of mathematics read too much into the earlier work on the basis of what came later?

Point to note: Cauchy may be credited with many of the fundamental ideas of complex analysis, but this does not mean that they appeared fully-formed.

Riemann on complex analysis



GRUNDLAGEN

FÜR EINE

ALLGEMEINE THEORIE DER FUNCTIONEN

EINER

VERÄNDERLICHEN COMPLEXEN GRÖSSE.

VON

(Georg Friedrich) Bernhard
B. RIEMANN.

ZWEITER, UNVERÄNDERTER ABDRUCK.

GÖTTINGEN,
VERLAG VON ADALBERT RENTE.
1867.

Doctoral dissertation: *Foundations for a General Theory of Functions of a Variable Complex Quantity* (1851)

Started from the idea that a complex variable should be treated as a single quantity z

"The complex variable w is called a function of another complex variable z when its variation is such that the value of the derivative $\frac{dw}{dz}$ is independent of the value of dz"

That is: $\lim_{\delta \to 0} \frac{f(z+\delta)-f(z)}{\delta}$ exists

Riemann on complex analysis

so erhellt, dass er und zwar nur dann für je zwei Werthe von dx und dy denselben Werth haben wird, wenn

$$\frac{du}{dx} = \frac{dv}{dy} \quad \text{und} \quad \frac{dv}{dx} = -\frac{du}{dy}$$

ist. Diese Bedingungen sind also hinreichend und nothwendig, damit $w = u + vi$ eine Function von $z = x + yi$ sei. Für die einzelnen Glieder dieser Function fliessen aus ihnen die folgenden:

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = 0, \quad \frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} = 0,$$

welche für die Untersuchung der Eigenschaften, die Einem Gliede einer solchen Function einzeln betrachtet zukommen, die Grundlage bilden. Wir werden den Beweis für die wichtigsten dieser Eigenschaften einer eingehenderen Betrachtung der vollständigen Function voraufgehen lassen, zuvor aber noch einige Punkte, welche allgemeineren Gebieten angehören, erörtern und festlegen, um uns den Boden für jene Untersuchungen zu ebenen.

* * *

5.

Für die folgenden Betrachtungen beschränken wir die Veränderlichkeit der Grössen x, y auf ein endliches Gebiet, indem wir als Ort des Punktes O nicht mehr die Ebene A selbst, sondern eine über dieselbe ausgebreitete Fläche T betrachten. Wir wählen diese Einkleidung, bei der es unanstössig sein wird, von aufeinander liegenden Flächen zu reden, um die Möglichkeit offen zu lassen, dass der Ort des Punktes O über denselben Theil der Ebene sich mehrfach erstrecke; setzen jedoch für einen solchen Fall voraus, dass die auf einander liegenden Flächentheile nicht längs einer Linie zusammenhängen, so dass eine Umfaltung der Fläche, oder eine Spaltung in auf einander liegende Theile nicht vorkommt.

Die Anzahl der in jedem Theile der Ebene auf einander liegenden Flächentheile ist alsdann vollkommen bestimmt, wenn die Begrenzung der Lage und dem Sinne nach (d. h. ihre innere und äussere Seite) gegeben ist; ihr Verlauf kann sich jedoch noch verschieden gestalten.

In der That, ziehen wir durch den von der Fläche bedeckten Theil der Ebene eine beliebige Linie l, so ändert sich die Anzahl der über einander liegenden Flächentheile nur beim Ueberschreiten der Begrenzung, und zwar beim Uebertritt von Aussen nach Innen um $+1$, im entgegengesetzten Falle um -1, und ist also überall bestimmt. Längs des Ufers dieser Linie setzt sich nun jeder angrenzende Flächentheil auf ganz bestimmte Art fort, so lange die Linie die Begrenzung nicht trifft, da eine Unbestimmtheit jedenfalls nur in einem einzelnen Punkte und also entweder in einem Punkte der Linie selbst oder in einer endlichen Entfernung von derselben Statt hat; wir können daher, wenn wir unsere Betrachtung auf einen im Innern der Fläche verlaufenden Theil der Linie l und zu beiden Seiten auf einen hinreichend kleinen Flächenstreifen beschränken, von *bestimmten* angrenzenden Flächentheilen reden, deren Anzahl auf jeder Seite gleich ist, und die wir, indem wir der Linie eine bestimmte Richtung beilegen, auf der Linken mit $a_1, a_2, \dots a_n$, auf der Rechten mit $a'_1, a'_2, \dots a'_n$, bezeichnen. Jeder Flächentheil a wird sich dann in einen der Flächentheile a' fortsetzen; dieser wird zwar im Allgemeinen für den ganzen Lauf der Linie l derselbe sein, kann sich jedoch für besondere Lagen von l in einem ihrer Punkte ändern. Nehmen wir an, dass oberhalb eines solchen Punktes σ (d. h.

Cauchy–Riemann equations now taken as fundamental to the theory

Other key concepts appear explicitly:

- harmonic functions;
- conformality (a complex function preserves angles wherever its derivative does not vanish);
- ...

Early impact limited by abstraction and restricted publication

The word ‘analytic’

The words analysis, analytic have had many meanings:

Classical:	a method of investigating a problem, the opposite of synthesis
c. 1600:	algebra became known as the ‘analytic art’ or just ‘analysis’, using finite equations
1669:	Newton introduced ‘analysis with infinite equations’, that is, infinite series
1748:	Euler wrote on the analysis of infinitely large and infinitely small quantities
1790–1840:	in sections of journals, the Académie des Sciences, etc., Analyse could mean ‘pure mathematics’ though with a bias to algebra, calculus, etc.; compare Géométrie also meaning ‘pure mathematics’, but with (perhaps) spatial bias
1821:	Cauchy’s cours d’analyse shows similarities with our analysis courses today

What *is* an analytic function?

Lagrange, 1797: function is analytic if it has a power-series expansion

Cauchy's point of departure, 1814–1831: treated complex functions that are continuous and satisfy the Cauchy–Riemann equations (always true for analytic functions in the sense of Lagrange), but used no special terminology

Riemann, 1851: switched focus to complex functions for which $\lim_{h \to 0} \frac{f(z+h)-f(z)}{h}$ exists in the region of interest

Weierstrass, 1860s: applied Lagrange's term analytic to Riemann's conception of function

Oxford, 2024: we follow Riemann and Weierstrass, by using the words holomorphic, meromorphic, etc. as variants of analytic, with slightly different meanings