

BO1.1. History of Mathematics  
Lecture XIV  
Linear algebra

MT24 Week 7

# Summary

- ▶ Linear equations
- ▶ Determinants
- ▶ Eigenvalues
- ▶ Matrices
- ▶ Vector spaces

# Difficulties in the historical study of linear algebra

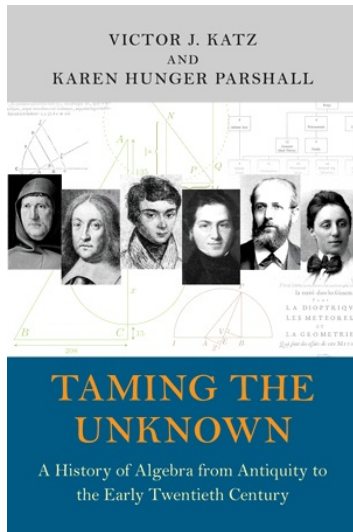
*Linear algebra may be mathematically simple but its history is more complicated than any other topic in this book. ... [Its development is] a very tangled tale.*

(*Mathematics Emerging*, p. 548)

- ▶ linear algebra is elementary but its manifestations are many and sophisticated
- ▶ there are hardly any obvious starting points
- ▶ theory often lagged behind practice
- ▶ practice sometimes lagged behind theory
- ▶ 19th-century reliance on theory of quadratic and bilinear forms (e.g.,  $ax^2 + 2bxy + cy^2$ ) — unfamiliar to students now

**Warning:** matrices (etc.) are primary in modern teaching, determinants secondary. For about 200 years until 1940 (or thereabouts) the reverse was the case: determinants came first.

# On the history of linear algebra



(Princeton University Press, 2014)



# Jiǔzhāng Suàنشù (China, c. 150 BC)



*Nine chapters of the mathematical art* 九章算術 (from a 16th-century edition, derived from a 3rd-century commentary by Liu Hui 劉徽)

Content: calculation of areas ( $\pi \approx 3.14159$ ), rates of exchange, computation with fractions, proportion, extraction of square and cube roots, calculation of volumes, systems of linear equations, Pythagoras' Theorem, ...

# Chinese calculation

|    ||    |||    ||||    |||||                 
 1   2   3   4   5     6   7   8   9

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$$\begin{array}{r}
 726 \\
 9 \overline{) 6538} \\
 \underline{63} \phantom{00} \\
 23 \phantom{00} \\
 \underline{18} \phantom{00} \\
 58 \phantom{00} \\
 \underline{54} \phantom{00} \\
 4
 \end{array}$$

Base 10 system of rods on counting board: red for positive, black for negative

# Early linear equations in China

Chapter 7: solution of pairs of equations in two unknowns by the method of false position

Chapter 8: solution of systems of  $n$  equations in  $n$  unknowns for  $n \leq 5$

*There are three types of grain*

*3 bundles of the first, 2 of the second, and 1 of the third  
contain 39 measures*

*2 of the first, 3 of the second, and 1 of the third contain 34*

*1 of the first, 2 of the second, and 3 of the third contain 26*

*How many measures in a bundle of each type?*

Solved on a counting board by **Gaussian elimination**, known here as 'fāngchéng' 方程

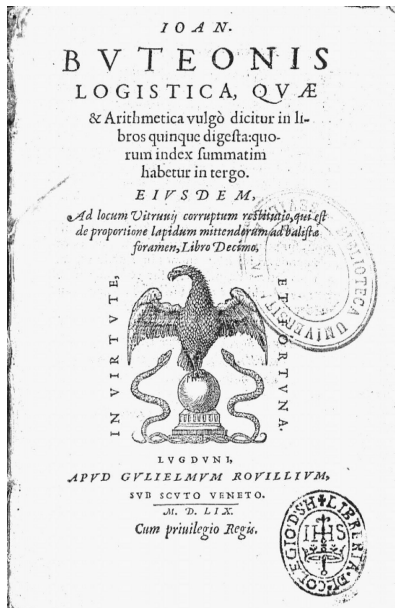
## Early linear equations in China

*There are five families which share a well. 2 of A's ropes are short of the well's depth by 1 of B's ropes. 3 of B's ropes are short of the depth by 1 of C's ropes. 4 of C's ropes are short by 1 of D's ropes. 5 of D's ropes are short by 1 of E's ropes. 6 of E's ropes are short by 1 of A's ropes. Find the depth of the well and the length of each rope.*

Five equations in six unknowns, so indeterminate

Liu Hui: we can only give a solution in terms of proportions of the lengths

# Early linear equations in Europe



Jean Borrel [Ioannes Buteus]  
*Logistica, quæ et Arithmetica  
vulgo dicitur in libros quinque  
digesta (Logistic, also known as  
Arithmetic, digested in five  
books), 1559*

# Linear equations in Borrel's *Logistica*

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*LIBER*

2 *A*, 1 *B* [60 singulatum in 3, fit 6 *A*, 3 *B*,  
[180. Ex his detrahe 1 *A*, 3 *B* [60, restat 5 *A*  
[120]. Partire in 5, prouenit 24, qui primus  
est numerus ex quesitis. Ex numero 30 aufer 24,  
residuum fit 6, quod est dimidium secundi, quare  
ipse est 12. Sunt igitur duo numeri 24, & 12,  
quos oportuit inuenire.

Tres numeros inuenire, quorum pri-  
mus cum triente reliquorum faciat 14. Se-  
cundus cum aliorum quadrante 8. Tertius  
item cum parte quinta reliquorum 8.

**P**One primum esse 1 *A*, secundum 1 *B*, tertium  
1 *C*. Erit igitur 1 *A*,  $\frac{1}{3}$  *B*,  $\frac{1}{5}$  *C* [14. Item  
1 *B*,  $\frac{1}{4}$  *A*,  $\frac{1}{5}$  *C* [8. Et etiam 1 *C*,  $\frac{1}{5}$  *A*,  $\frac{1}{3}$   
*B* [8. Ex his autem equationem secundam fa-  
ciendo, habebis pri-  
mam, secundam, et ter-  
tiam, quales hic ap-  
posui. Ex tribus istis  
equationibus aliæ, vel  
multiplicando, vel inuicem addendo sunt facien-  
dæ, quousque per detractionem minorum ex maio-  
ribus relinquatur sola quantitas vnius notæ, quod  
fiat hoc modo. Multiplica equationem secundam  
in 3, fit 3 *A*, 12 *B*, 3 *C* [96. Aufer primam, re-  
stat

To find three numbers, of which the  
first with a third of the rest makes  
14. The second with a quarter of the  
rest makes 8. Likewise the third with  
a fifth part of the rest makes 8.

*Put the first to be 1A, the second  
1B, the third 1C. ...*

[Derives a system of equations with  
'.' for addition and '[' for equality.]

Multiply by 3, by 4 and by 5  
respectively, etc.

(See *Mathematics emerging*,  
§17.1.1.)

## More unknowns

GVL. GOS. DE ARTE  
bunt 60 equalia 1 A, quare primus est  
60, iam vero 2 B 1 C equalia fuerunt  
100, tollamus 1 C hoc est 20, resta-  
bunt 80 equalia 2 B, & 1 B est 40,  
suntque tres numeri quæsi 60 40 20,  
quibus vestigatis opus fuit.

### Problema. v.

Inueniamus quatuor numeros quo-  
rum primus cum semisse reliquo-  
rum faciat 17, secundus cum aliorum  
triente 12, tertius cum aliorum qua-  
drante 13, quartus item cum aliorum  
sextante 13.

Sint illi quatuor A B C D, & sint 1 A  
 $\frac{1}{2}$  B  $\frac{1}{2}$  C  $\frac{1}{2}$  D equalia 17, 1 B  $\frac{1}{3}$  A  $\frac{1}{3}$  C  
 $\frac{1}{3}$  D equalia 12, 1 C  $\frac{1}{4}$  A  $\frac{1}{4}$  B  $\frac{1}{4}$  D æqua-  
lia 13, 1 D  $\frac{1}{6}$  A  $\frac{1}{6}$  B  $\frac{1}{6}$  C equalia 13, re-  
uocentur hæc ad integros numeros,  
existent 2 A 1 B 1 C 1 D æqualia 34, 1 A  
3 B 1 C 1 D æqualia 36, 1 A 1 B 4 C 1 D  
æqualia 52, 1 A 1 B 1 C 6 D æqualia 78,

Guillaume Gosselin, *De arte magna  
seu de occulta parte numerorum  
quæ et Algebra et Almucabala vulgo  
dicitur* (On the great art or the  
hidden part of numbers commonly  
called Algebra and Almucabala),  
1577

$$1A + \frac{1}{2}B + \frac{1}{2}C + \frac{1}{2}D = 17$$

$$1B + \frac{1}{3}A + \frac{1}{3}C + \frac{1}{3}D = 12$$

$$1C + \frac{1}{4}A + \frac{1}{4}B + \frac{1}{4}D = 13$$

$$1D + \frac{1}{6}A + \frac{1}{6}B + \frac{1}{6}C = 13$$

## A 17th-century example

After reading Gosselin ...

John Pell to Sir Charles Cavendish (1646):

Exemplum ... satis determinatis

$$3a - 4b + 5c = 2$$

$$5a + 3b - 2c = 58$$

$$7a - 5b + 4c = 14$$

(Solved via Pell's 'three-column method')

Exemplum ... non satis determinatis

$$5a + 3b - 2c = 24$$

$$-2a + 4b + 3c = 5$$

$(a, b, c > 0$ ; found bounds for the possible values: e.g.,  $a < 15\frac{9}{11}$ )



# Linear equations — systematic practical methods

Gaussian elimination:

- ▶ *The nine chapters of the mathematical art*, China (c. 150 BC)
- ▶ Colin Maclaurin, *A treatise of algebra* (1748), §§82–85

# Maclaurin on Gaussian elimination

## A TREATISE OF ALGEBRA, IN THREE PARTS.

- CONTAINING
- I. *The Fundamental Rules and Operations.*
  - II. *The Composition and Resolution of Equations of all Degrees; and the different Affections of their Roots.*
  - III. *The Application of Algebra and Geometry to each other.*

To which is added an

## APPENDIX, Concerning the general Properties of GEOMETRICAL LINES.

By COLIN MACLAURIN, M. A.  
Late PROFESSOR of MATHEMATICS in the University of Edinburgh, and Fellow of the Royal Society.

LONDON:

Printed for A. MILLAR, and J. NOURSSE,  
opposite to Catherine-Street, in the Strand.

M.DCC.XLVIII.

## Chap. II. ALGEBRA.

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$$\begin{cases} x:y::a:b \\ x^2-y^2=d \end{cases}$$

$$x = \frac{ay}{b} \text{ and } x^2 = \frac{a^2y^2}{b^2}$$

$$\text{but } x^2 = d + y^2$$

$$\text{whence } d + y^2 = \frac{a^2y^2}{b^2}$$

$$\text{and } a^2y^2 - b^2y^2 = db^2$$

$$y^2 = \frac{db^2}{a^2 - b^2}$$

$$y = \sqrt{\frac{db^2}{a^2 - b^2}}$$

$$\text{and } x = \sqrt{\frac{da^2}{a^2 - b^2}}$$

## DIRECTION V.

§ 82. "If there are three unknown Quantities, there must be three Equations in order to determine them, by comparing which you may, in all Cases, find two Equations involving only two unknown Quantities; and then, by Direct. 3d, from these two you may deduce an Equation involving only one unknown Quantity; which may be resolved by the Rules of the last Chapter."

From 3 Equations involving any three unknown Quantities,  $x$ ,  $y$ , and  $z$ , to deduce two Equations involving only two unknown Quantities, the following Rule will always serve.

RULE.

# Linear equations — systematic practical methods

Gaussian elimination:

- ▶ *The nine chapters of the mathematical art*, China (c. 150 BC)
- ▶ Colin Maclaurin, *A treatise of algebra* (1748), §§82–85
- ▶ C. F. Gauss: calculation of asteroid orbits (1810)
- ▶ from surveying, e.g., Wilhelm Jordan, *Handbuch der Vermessungskunde*, 3rd edition (1888)

# Maclaurin and linear equations

## Chap. 12. ALGEBRA.

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### EXAMPLE I.

$$\text{Supp. } \begin{cases} 5x+7y=100 \\ 3x+8y=80 \end{cases}$$

$$\text{then } y = \frac{5 \times 80 - 3 \times 100}{5 \times 8 - 3 \times 7} = \frac{100}{19} = 5 \frac{5}{19}$$

$$\text{and } x = \frac{240}{19} = 12 \frac{12}{19}$$

### EXAMPLE II.

$$\begin{cases} 4x+8y=90 \\ 3x-2y=160 \end{cases}$$

$$y = \frac{4 \times 160 - 3 \times 90}{4 \times -2 - 3 \times 8} = \frac{640 - 270}{-8 - 24} = \frac{370}{-32} = -11 \frac{9}{8}$$

### THEOREM II.

§ 87. Suppose now that there are three unknown Quantities and three Equations, then call the unknown Quantities  $x$ ,  $y$ , and  $z$ .

Thus,

$$\begin{cases} ax+by+cz=m \\ dx+ey+fz=n \\ gx+hy+kz=p \end{cases}$$

$$\text{Then shall } z = \frac{ap-abn+dbm-dbp+ebn-gem}{ack-abf+dbc-dbh+ebf-gdc}.$$

Where the Numerator consists of all the different Products that can be made of three opposite Coefficients taken from the Orders in which  $z$  is not found; and the Denominator consists of all the Products that can be made of the three opposite

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Colin Maclaurin, *A treatise of algebra*, 1748, p. 83

Three equations in three unknowns solved using a 'determinant-like' quantity

## Chap. 13. ALGEBRA.

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If four Equations are given, involving four unknown Quantities, their Values may be found much after the same Manner; by taking all the Products that can be made of four opposite Coefficients, and always prefixing contrary Signs to those that involve the Products of two opposite Coefficients.

Notational difficulties — we run out of letters!

# Determinants

Colin Maclaurin, *A treatise of algebra*, 1748, Ch. XII, pp. 81–85

# Determinants

Leibniz, unpublished works, 1680s/1690s.

Colin Maclaurin, *A treatise of algebra*, 1748, Ch. XII, pp. 81–85

## Leibniz and determinants

At least as early as June 1678, Leibniz devised a new notation for coefficients, writing

$$10 + 11x + 12y = 0,$$

$$20 + 21x + 22y = 0$$

for what we would write as

$$a_{10} + a_{11}x + a_{12}y = 0,$$

$$a_{20} + a_{21}x + a_{22}y = 0.$$

Leibniz used this notation to formulate general results on the solvability of systems of equations in terms of a determinant-like quantity (a sum of signed products of coefficients) — but these were not published during his lifetime

# Determinants

Leibniz, unpublished works, 1680s/1690s.

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# Determinants

Leibniz, unpublished works, 1680s/1690s.

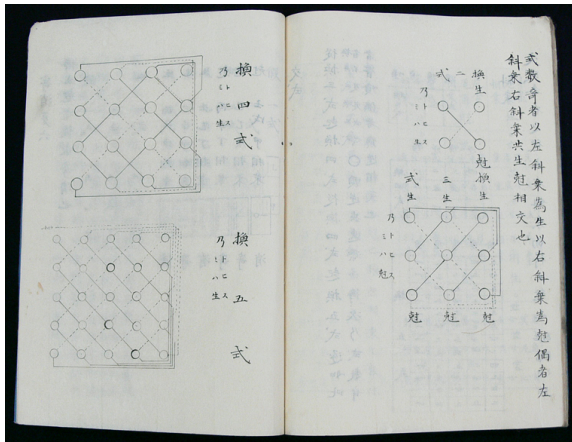
Seki Takakazu, *Kai-fukudai-no-hō* 解伏題之法, 1683

Colin Maclaurin, *A treatise of algebra*, 1748, Ch. XII, pp. 81–85

# Seki and determinants

Seki Takakazu, *Kai-fukudai-no-hō* 解伏題之法 (*Method for Solving Concealed Problems*), 1683

Arranged coefficients of systems of equations in a grid, and gave schematics for construction of determinants (dotted lines indicate positive products, and solid lines negative)



# Determinants

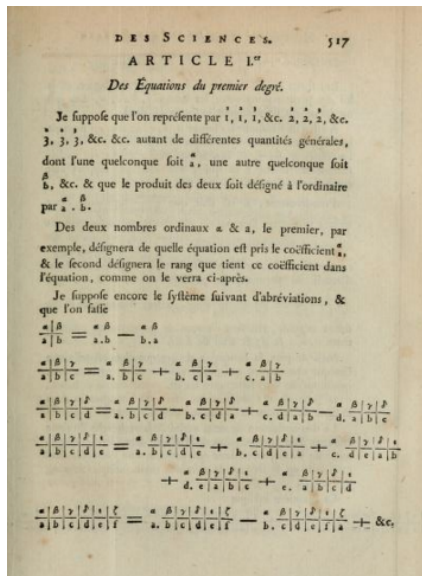
Leibniz, unpublished works, 1680s/1690s.

Seki Takakazu, *Kai-fukudai-no-hō* 解伏題之法, 1683

Colin Maclaurin, *A treatise of algebra*, 1748, Ch. XII, pp. 81–85

Vandermonde, 'Mémoire sur l'élimination', *Mémoires de l'Académie des sciences*, 1772: a recursive description of determinants of any size (but without a name and in an uncongenial notation — see *Mathematics emerging*, §17.1.3)

# Vandermonde on elimination



$\alpha$   
 $a$  denotes a single quantity, e.g.,  
a coefficient in a linear equation

Define:  $\frac{\alpha}{a} \bigg| \frac{\beta}{b} = \frac{\alpha}{a} \frac{\beta}{b} - \frac{\alpha}{b} \frac{\beta}{a}$

Anachronistically, this is the  
determinant of the matrix:

$$\begin{pmatrix} \alpha & \alpha \\ a & b \\ \beta & \beta \\ a & b \end{pmatrix}$$

Then continue recursively ...

# Determinants

Leibniz, unpublished works, 1680s/1690s.

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Vandermonde, 'Mémoire sur l'élimination', *Mémoires de l'Académie des sciences*, 1772: a recursive description of determinants of any size (but without a name and in an uncongenial notation — see *Mathematics emerging*, §17.1.3)

Gauss in *Disquisitiones arithmeticae* (1801) gave the name 'determinant' to what is now called the 'discriminant'  $B^2 - AC$  of the binary quadratic form  $Ax^2 + 2Bxy + Cy^2$ .

# Cauchy on determinants

QUI NE PEUVENT OBTENIR QUE DEUX VALEURS, ETC. 113

les propriétés générales des formes du second degré, c'est-à-dire des polynomes du second degré à deux ou à plusieurs variables, et il a désigné ces mêmes fonctions sous le nom de *déterminants*. Je conserverai cette dénomination qui fournit un moyen facile d'énoncer les résultats; j'observerai seulement qu'on donne aussi quelquefois aux fonctions dont il s'agit le nom de *résultantes* à deux ou à plusieurs lettres. Ainsi les deux expressions suivantes, *déterminant* et *résultante*, devront être regardées comme synonymes.

## DEUXIÈME PARTIE.

DES FONCTIONS SYMÉTRIQUES ALTERNÉES DESIGNÉES SOUS LE NOM  
DE DÉTERMINANTS.

### PREMIÈRE SECTION.

*Des déterminants en général et des systèmes symétriques.*

§ 1<sup>re</sup>. Soient  $a_1, a_2, \dots, a_n$  plusieurs quantités différentes en nombre égal à  $n$ . On a fait voir ci-dessus que, en multipliant le produit de ces quantités ou

$$a_1 a_2 a_3 \dots a_n$$

par le produit de leurs différences respectives, ou par

$$(a_2 - a_1)(a_3 - a_1) \dots (a_n - a_1)(a_3 - a_2) \dots (a_n - a_2) \dots (a_n - a_{n-1}),$$

on obtenait pour résultat la fonction symétrique alternée

$$S(\equiv a_1 a_2 a_3 \dots a_n)$$

qui, par conséquent, se trouve toujours égale au produit

$$a_1 a_2 a_3 \dots a_n (a_2 - a_1)(a_3 - a_1) \dots (a_n - a_1)(a_3 - a_2) \dots (a_n - a_2) \dots (a_n - a_{n-1}).$$

Supposons maintenant que l'on développe ce dernier produit et que, dans chaque terme du développement, on remplace l'exposant de

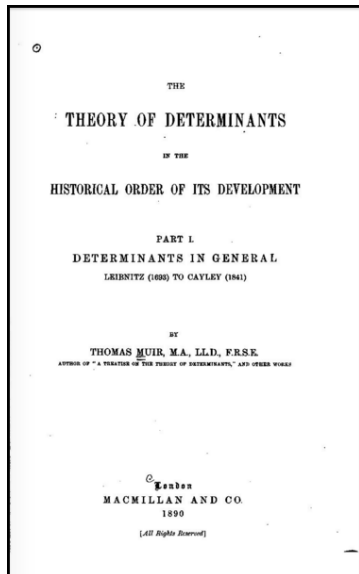
Cauchy, 'Mémoire sur les fonctions qui ne peuvent obtenir que deux valeurs égales et de signes contraires par suite des transpositions opérées entre les variables qu'elles renferment', *Journal de l'École polytechnique*, 1815

Referred to Laplace, Vandermonde, Gauss, and others

Introduced the term **determinant** for the function of  $n^2$  quantities (a sum of  $n!$  signed products) that we now know by that name.

(See *Mathematics emerging*, §17.1.4.)

# History of the theory of determinants



Determinants were studied extensively in the 19th century.

Sir Thomas Muir, *The theory of determinants in the historical order of development* (1890–1906)

- ▶ Part I: *Determinants in general: Leibnitz (1693) to Cayley (1841)*;
- ▶ Part II: *Special determinants up to 1841*

Second edition in 4 volumes, 1906–1923; supplement, 1930.

## 'Eigenvalue' problems

Euler (1748): change of coordinates to reduce equation of a quadric surface  $\alpha z^2 + \beta yz + \gamma xz + \delta y^2 + \epsilon xy + \zeta x^2 + \eta z + \theta y + \iota x + \chi = 0$  to its simplest form  $Ap^2 + Bq^2 + Cr^2 + K = 0$  (see: *Mathematics emerging*, §17.2.1.)

Laplace (1787): symmetry of coefficients in a set of linear differential equations leads to real 'eigenvalues' (see: *Mathematics emerging*, §17.2.2.)

Cauchy (1829): a symmetric matrix is diagonalisable by a real orthogonal change of variables (see: *Mathematics emerging*, §17.2.3.)



# Matrices and their determinants

Gauss, *Disquisitiones arithmeticae* (1801): transformation of quadratic forms  $ax^2 + 2bxy + cy^2$  by change of variables

$$x = \alpha x' + \beta y', \quad y = \gamma x' + \delta y'$$

followed by

$$x' = \alpha' x'' + \beta' y'', \quad y' = \gamma' x'' + \delta' y''$$

comes to the same as

$$x = (\alpha\alpha' + \beta\gamma')x'' + (\alpha\beta' + \beta\delta')y'', \quad y = (\gamma\alpha' + \delta\gamma')x'' + (\gamma\beta' + \delta\delta')y''$$

Moreover, the 'determinants' (our sense) multiply.

**NB.** All Gauss' coefficients were integers

(See *Mathematics emerging*, §17.3.1.)

## Early origins of matrices

The OED (3rd ed., March 2001) lists sense 2a of 'matrix' as

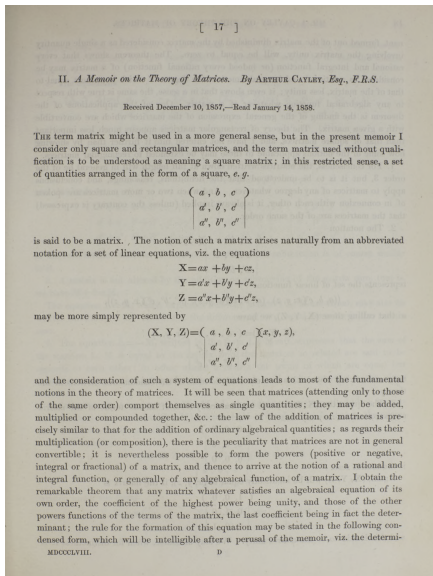
*A place or medium in which something is originated, produced, or developed . . .*

Thus, in 1850, J. J. Sylvester applied the word to the 'thing' from which determinants originate:

*For this purpose we must commence, not with a square, but with an oblong arrangement of terms consisting, suppose, of  $m$  lines and  $n$  columns. This will not in itself represent a determinant, but is, as it were, a Matrix out of which we may form various systems of determinants by fixing upon a number  $p$ , and selecting at will  $p$  lines and  $p$  columns, the squares corresponding of  $p$ th order.*

But he did not **operate** with matrices

# The definition of matrices



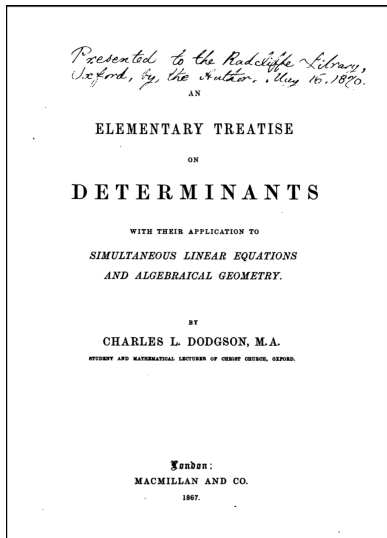
Arthur Cayley, 'A memoir on the theory of matrices', *Phil. Trans. Roy. Soc.*, 1858:

- ▶ defined matrices and their properties
- ▶ recognised connection to linear equations
- ▶ stated the Cayley–Hamilton Theorem
- ▶ investigated the matrices that commute with a given one

"It will be seen that matrices (attending only to those of the same order) comport themselves as single quantities..."

(See *Mathematics emerging*, §17.3.2.)

# Determinants persist



*"I am aware that the word 'Matrix' is already in use to express the very meaning for which I use the word 'Block'; but surely the former word means rather the mould, or form, into which algebraical quantities may be introduced, than an actual assemblage of such quantities ..."*

Criticised notation ' $a_{ij}$ ':

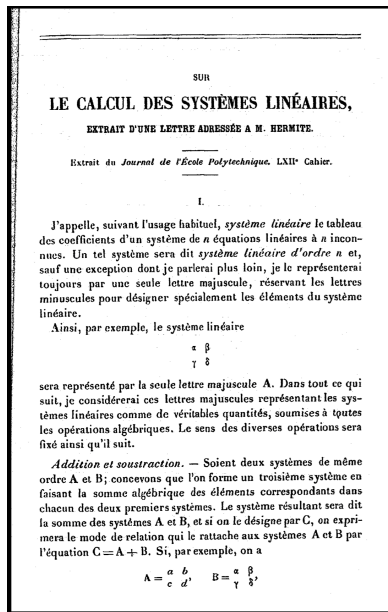
*"it seems a fatal objection to this system that most of the space is occupied by a number of a's, which are wholly superfluous, while the only important part of the notation is reduced to minute subscripts, alike difficult to the writer and the reader."*

Proposed  $i \setminus j$  instead

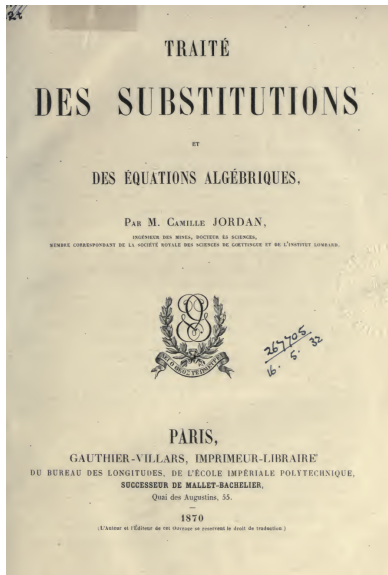
# Matrices elsewhere

Matrix algebra appears in Hamilton's *Lectures on Quaternions* (1853) as 'linear and vector functions' (including his version of the Cayley–Hamilton Theorem, stated and proved in terms of quaternions)

Matrices were also devised by Laguerre in his paper 'Sur le calcul des systèmes linéaires' (*J. École polytechnique*, 1867)



# Jordan and linear substitutions



Camille Jordan, *Traité des substitutions*, 1870:

- ▶ studied matrices over integers modulo  $n$  as part of an extensive study of linear substitutions (in connection with Galois theory); developed 'canonical forms' to study conjugacy classes in these groups
- ▶ developed his ideas to 'Jordan canonical form' for complex matrices in his studies 1872–4 of linear differential equations

# German contributions

2

Frobenius, über lineare Substitutionen und bilineare Formen.

führt. Diese Erwägungen leiteten mich darauf, statt der Transformation der bilinearen Formen die Zusammensetzung der linearen Substitutionen zu behandeln.

## § 1. Multiplication.

1. Sind  $A$  und  $B$  zwei bilineare Formen der Variablen  $x_1, \dots, x_n$ ;  $y_1, \dots, y_n$ , so ist auch

$$P = \sum_{y,y'} \frac{\partial A}{\partial y} \frac{\partial B}{\partial y'}$$

eine bilineare Form derselben Variablen. Dieselbe nenne ich aus den Formen  $A$  und  $B$  (in dieser Reihenfolge) *zusammengesetzt*\*). Es werden in Folgenden nur solche Operationen mit bilinearen Formen vorgenommen, bei welchen sie bilineare Formen bleiben\*\*). Ich werde z. B. eine Form mit einer Constanten (von  $x_1, y_1; \dots, x_n, y_n$  unabhängigen Grösse) multiplizieren, zwei Formen addiren, eine Form, deren Coefficienten von einem Parameter abhängen, nach demselben differenziren. Ich werde aber nicht zwei Formen mit einander multiplizieren. Aus diesem Grunde kann kein Missverständniß entstehen, wenn ich die aus  $A$  und  $B$  zusammengesetzte Form  $P$  mit

$$AB = \sum_{y,y'} \frac{\partial A}{\partial y} \frac{\partial B}{\partial y'}$$

bezeichne, und sie das *Product* der Formen  $A$  und  $B$ , diese die *Factoren* von  $P$  nenne. Für diese Bildung gilt

a) das *distributive* Gesetz:

$$A(B+C) = AB+AC, \quad (A+B)C = AC+BC,$$

$$(A+B)(C+D) = AC+BC+AD+BD.$$

\*) Borchardt, Neue Eigenschaft der Gleichung, mit deren Hilfe man die säcclären Störungen der Planeten bestimmt. Dieses Journal Bd. 30, S. 38.

Cayley, Remarques sur la notation des fonctions algebriques. Dieses Journal Bd. 50, S. 282.

Hesse, Neue Eigenschaften der linearen Substitutionen, welche gegebene homogene Functionen des zweiten Grades in andere transformiren, die nur die Quadrate der Variablen enthalten. Dieses Journal Bd. 57, S. 175.

Christoffel, Theorie der bilinearen Formen. Dieses Journal Bd. 68, S. 253.

Rosanes, Ueber die Transformation einer quadratischen Form in sich selbst. Dieses Journal Bd. 80, S. 52.

\*\*) Unter dem Bilde einer bilinearen Form fasse ich ein System von  $n^2$  Grössen zusammen, die nach  $n$  Zeilen und  $n$  Columnen geordnet sind. Eine Gleichung zwischen zwei bilinearen Formen repräsentirt daher einen Complex von  $n^2$  Gleichungen. Ich werde bisweilen von dem Bilde der Form absehen und unter dem Zeichen  $A$  das System der  $n^2$  Grössen  $a_{\alpha\beta}$ , unter der Gleichung  $A = B$  das System der  $n^2$  Gleichungen  $a_{\alpha\beta} = b_{\alpha\beta}$  verstehen.

Georg Frobenius, in 1878, working with bilinear forms, produced more canonical forms, and gave a satisfactory proof of the Cayley–Hamilton Theorem

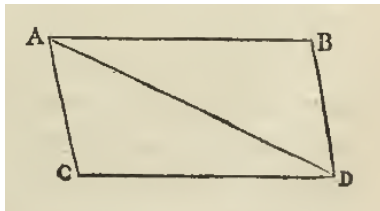
(See *Mathematics emerging*, §17.3.3.)

Other mathematicians in Germany (e.g., Kronecker, Hurwitz) contributed similarly

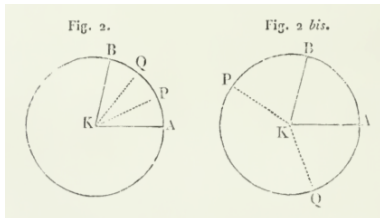
A recommended secondary source: Thomas Hawkins, 'Another look at Cayley and the theory of matrices', *Archives internationales d'histoire des sciences* **26** (1977), 82–112

# Vectors

Newton (1687): parallelogram of forces



Argand (1806): complex numbers as directed quantities in the plane





## 14 — APPLICATIONS DU CALCUL INFINITESIMAL.

ments directs de rotation autour de ces demi-axes auront lieu de droite à gauche, et les mouvements rétrogrades de gauche à droite.

Nous appliquerons les mêmes dénominations aux deux espèces de mouvements que peut prendre un rayon vecteur mobile en tournant autour d'un point de manière à parcourir successivement les trois faces d'un angle solide quelconque; et quand le mouvement de rotation du rayon vecteur sur chaque face aura lieu de droite à gauche autour de l'arête située hors de cette face, ce mouvement sera nommé *direct* ou *rétrograde*, suivant que les mouvements de rotation des plans coordonnés, tournant de droite à gauche autour des demi-axes  $\overline{OX}$ ,  $\overline{OY}$ ,  $\overline{OZ}$ , seront eux-mêmes directs ou rétrogrades.

Une droite  $\overline{AB}$ , menée d'un point A supposé fixe à un point B supposé mobile, sera généralement désignée sous le nom de *rayon vecteur*. Nommons R ce rayon vecteur,

$$x_0, y_0, z_0$$

les coordonnées du point A :

$$x, y, z$$

celles du point B; et

$$a, b, c$$

les angles formés par la direction  $\overline{AB}$  avec les demi-axes des coordonnées positives;

$$\alpha, \beta, \gamma$$

seront les angles formés par le même rayon vecteur avec les demi-axes des coordonnées négatives. De plus, la *projection orthogonale* du rayon vecteur sur l'axe des  $x$  sera égale, d'après un théorème connu de Trigonométrie, au produit de ce rayon vecteur par le cosinus de l'angle aigu qu'il forme avec l'axe des  $x$  prolongé dans un certain sens. Cette projection se trouvera donc représentée : si l'angle  $\alpha$  est aigu, par le produit

$$R \cos \alpha,$$

et si l'angle  $\alpha$  est obtus, par le produit

$$R \cos(\pi - \alpha) = -R \cos \alpha,$$

Word applied mostly to **radius vectors**

e.g., as **rayon vecteur** in Laplace's *Mécanique Céleste* (1799–1825)

Also in Cauchy's *Leçons sur les Applications du Calcul Infinitésimal à la Géométrie* (1826), p. 14:

A line  $\overline{AB}$ , taken from a point A, supposed to be fixed, to a moving point B, will in general be referred to as a **radius vector**.

# Hamilton and vectors

Sir William Rowan Hamilton drew a distinction between a ‘vector’ and a ‘radius vector’:

Between 1843–1866, developed quaternions — 4-dimensional quantities  $a + bi + cj + dk$ , where  $i^2 = j^2 = k^2 = ijk = -1$ , designed for use in mechanics (and geometry of 3 dimensions)

*“A VECTOR is thus . . . a sort of NATURAL TRIPLET (suggested by Geometry): and accordingly we shall find that QUATERNIONS offer an easy mode of symbolically representing every vector by a TRINOMIAL FORM ( $ix + jy + kz$ ); which form brings the conception and expression of such a vector into the closest possible connexions with Cartesian and rectangular co-ordinates.”*

So a quaternion is a scalar + a vector

# Vector spaces appear



Die *Ausdehnungslehre*  
**Ausdehnungslehre.**

—•••••—

Vollständig und in strenger Form

bearbeitet

VON

**Hermann Grassmann,**

Professor am Gymnasium zu Stettin.

\*\*\*\*\*

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**BERLIN, 1862.**

VERLAG VON TH. CHR. FR. ENSLIN.  
(ADOLPH ENSLIN.)

# Grassmann's 'doctrine of extension'

Die Wissenschaft  
der  
**extensiven Grösse**

oder  
**die Ausdehnungslehre,**  
eine neue mathematische Disciplin  
dargestellt und durch Anwendungen erläutert

von  
**Hermann Grassmann**  
Lehrer an der Friedrich-Wilhelms-Schule zu Stettin.

---

*Erster Theil,*  
die **lineale Ausdehnungslehre** enthaltend.

---

Leipzig, 1844.  
Verlag von Otto Wigand.

*Die Ausdehnungslehre* [*Doctrine of extension*] (1862) is a heavily reworked version of an earlier (1844) work:

*The Science of Extensive Quantities, or the Doctrine of Extension, a New Mathematical Discipline, Presented and Explained through Examples*

Introduced idea of objects generated by motion — a single element generates an object of order 1, an object of order 1 generates an object of order 2, etc.

Objects of the same order can be added together or scaled by real numbers

Little impact at the time

# Grassmann's 'extensive quantities'

4

(9

$$\begin{aligned} 3) \quad a + b - b &= \sum \overline{ae} + \sum \overline{\beta e} - \sum \overline{\beta e} \\ &= \sum (\overline{a + \beta})e - \sum \overline{\beta e} & [6]. \\ &= \sum (\overline{a + \beta - \beta})e & [7]. \\ &= \sum \overline{ae} = a \end{aligned}$$

$$\begin{aligned} 4) \quad a - b + b &= \sum \overline{ae} - \sum \overline{\beta e} + \sum \overline{\beta e} \\ &= \sum (\overline{a - \beta})e + \sum \overline{\beta e} & [7]. \\ &= \sum (\overline{a - \beta + \beta})e & [6]. \\ &= \sum \overline{ae} = a \end{aligned}$$

**9.** Für extensive Größen gelten die sämtlichen Gesetze algebraischer Addition und Subtraktion.

Beweis. Denn diese Gesetze können, wie bekannt, aus den 4 Fundamentalförmeln in No. 8 abgeleitet werden.

**10.** Erklärung. Eine extensive GröÖe mit einer Zahl multipliciren heisst ihre sämtlichen Ableitungszahlen mit dieser Zahl multipliciren, d. h.

$$\sum \overline{ae} \cdot \beta = \beta \cdot \sum \overline{ae} = \sum (\overline{a\beta}) \cdot e$$

**11.** Erklärung. Eine extensive GröÖe durch eine Zahl, die nicht gleich null ist, dividiren, heisst ihre sämtlichen Ableitungszahlen durch diese Zahl dividiren, d. h.

$$\sum \overline{ae} : \beta = \sum \frac{\overline{a}}{\beta} e$$

**12.** Für die Multiplikation und Division extensiver GröÖen (a, b) durch Zahlen ( $\beta$ ,  $\gamma$ ) gelten die Fundamentalförmeln:

- 1)  $a\beta = \beta a$ ,
- 2)  $a\beta\gamma = a(\beta\gamma)$ ,
- 3)  $(a + b)\gamma = a\gamma + b\gamma$ ,
- 4)  $a(\beta + \gamma) = a\beta + a\gamma$ ,
- 5)  $a \cdot 1 = a$ ,
- 6)  $a\beta = 0$  dann und nur dann, wenn entweder  $a = 0$ , oder  $\beta = 0$ ,
- 7)  $a : \beta = a \frac{1}{\beta}$ , wenn  $\beta \geq 0$  ist \*).

Beweis. Es sei  $a = \sum \overline{ae}$ ,  $b = \sum \overline{\beta e}$ , wo die Summe sich auf das System der Einheiten  $e_1 \dots e_n$  bezieht, so ist

\*) Das Zeichen  $\geq$  zusammengesetzt aus  $\supset$  und  $\angle$  soll ungleich bedeuten.

The 1862 text contains a theory of **extensive** quantities

$$a_1 e_1 + a_2 e_2 + \dots,$$

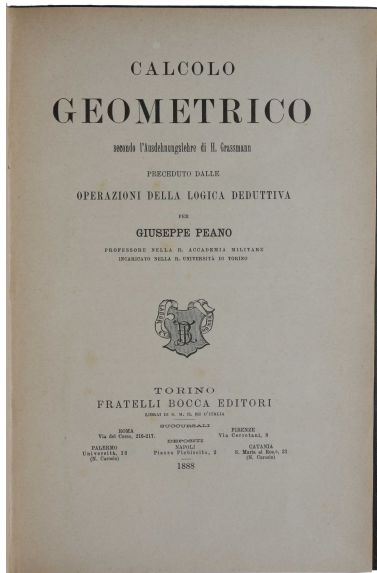
where the  $e_i$  are 'units' and the  $a_i$  are real numbers, including

- rules for the arithmetic of such quantities
- a notion of linear independence
- dimension
- ...

But still had little impact

(See *Mathematics emerging*, §17.4.1.)

# Vector spaces defined



On the way towards developing a 'geometric calculus', Giuseppe Peano axiomatised Grassmann's collections of extensive quantities as **linear systems** (**sistemi lineari**), and moved to a fully abstract setting

Clarified connection between dimension and linear independence — noted existence of linear systems with infinite dimension

Also no immediate impact!

# Vector spaces develop

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## Algebraische Theorie der Körper.

Von Herrn Ernst Steinitz in Berlin.

In dem vorliegenden Aufsatz ist der Begriff „Körper“ in derselben abstrakten und allgemeinen Weise gefaßt wie in H. Webers Untersuchungen über die allgemeinen Grundlagen der Galoisschen Gleichungstheorie\*, nämlich als ein System von Elementen mit zwei Operationen: Addition und Multiplikation, welche dem assoziativen und kommutativen Gesetz unterworfen, durch das distributive Gesetz verbunden sind und unbeschränkte und eindeutige Umkehrungen zulassen\*\*. Während aber bei Weber das Ziel eine allgemeine, von der Zahlenbedeutung der Elemente unabhängige Behandlung der Galoisschen Theorie ist, steht für uns der Körperbegriff selbst im Mittelpunkt des Interesses. Eine Übersicht über alle möglichen Körpertypen zu gewinnen und ihre Beziehungen untereinander in ihren Grundzügen festzustellen, kann als Programm dieser Arbeit gelten\*\*\*. Da hierbei die der Arithmetik im engeren Sinn angehörigen Unterscheidungen zwischen ganzen und gebrochenen Größen nicht weiter zu verfolgen waren, wurde der Titel *Algebraische Theorie der Körper* gewählt.

Durch die hier gekennzeichnete Tendenz ist auch der Weg, den wir einzuschlagen haben, vorgezeichnet. Wir werden von der Bildung der einfachsten Körper ausgehen und sodann die Methoden betrachten, durch

\* Math. Ann. 43. S. 521. — \*\* Nur die Division durch Null ist auszuschließen.

\*\*\* Zu diesen allgemeinen Untersuchungen wurde ich besonders durch Hensels Theorie der algebraischen Zahlen (Leipzig, 1908) angeregt, in welcher der Körper der  $p$ -adischen Zahlen den Ausgangspunkt bildet, ein Körper, der weder den Funktionen- noch den Zahlkörpern im gewöhnlichen Sinne des Wortes beizuzählen ist.

Journal für Mathematik. Bd. 137. Heft 3.

Dedekind (1879): fields and ‘modules’ needed for algebraic number theory in famous appendices to his third edition of Dirichlet, *Vorlesungen über Zahlentheorie* [*Lectures on number theory*]; published also separately in France, 1876–77

Ernst Steinitz (1910), ‘Algebraische Theorie der Körper’ [‘Algebraic theory of fields’] — contains a beautifully crystallised theory of linear dependence and independence, bases, dimension, etc., in the form it is now taught

# Vector spaces develop

## FINITE DIMENSIONAL VECTOR SPACES

BY  
PAUL R. HALMOS

PRINCETON  
PRINCETON UNIVERSITY PRESS  
LONDON HUMPHREY MILFORD  
OXFORD UNIVERSITY PRESS  
**1948**

B. L. van der Waerden (1930–31),  
*Moderne Algebra*, incorporating  
material from lectures by Emil Artin  
and Emmy Noether (1926–1928)

Paul Halmos (1942),  
*Finite-dimensional vector spaces* —  
made the subject accessible to 1st  
and 2nd year undergraduates