BO1.1. History of Mathematics
Lecture XV
Geometry and number theory

MT24 Week 8

Summary

- Euclid's *Elements* revisited
- The parallel postulate
- Non-Euclidean geometry
- Number theory down the centuries

Euclid's *Elements*

Euclid's Elements, in 13 books, compiled c. 250 BC.

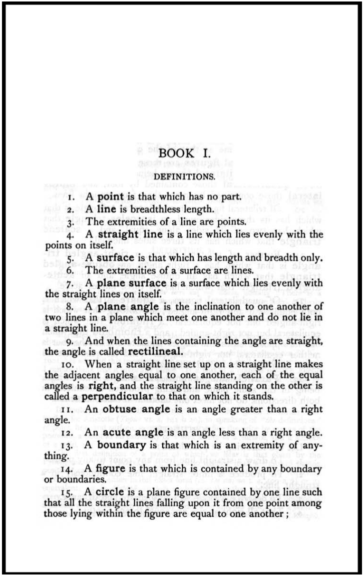
Books I–V:	definitions, postulates, plane geometry of lines and circles
Book VI:	similarity, proportion
Books VII–IX:	number theory
Book X:	commensurability, irrational numbers, surds
Books XI–XIII:	solid geometry ending with the classification of the regular polyhedra

Euclid's *Elements*

Euclid's Elements, in 13 books, compiled c. 250 BC.

Books I–V:	definitions, postulates, plane geometry of lines and circles
Book VI:	similarity, proportion
Books VII–IX:	number theory
Book X:	commensurability, irrational numbers, surds
Books XI–XIII:	solid geometry ending with the classification of the regular polyhedra

Euclid in English



BOOK I.

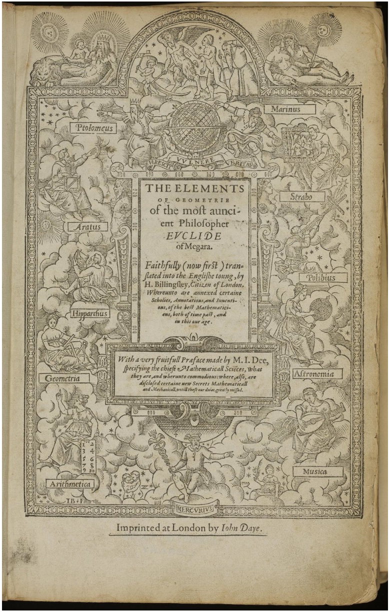
DEFINITIONS.

1. A **point** is that which has no part.
2. A **line** is breadthless length.
3. The extremities of a line are points.
4. A **straight line** is a line which lies evenly with the points on itself.
5. A **surface** is that which has length and breadth only.
6. The extremities of a surface are lines.
7. A **plane surface** is a surface which lies evenly with the straight lines on itself.
8. A **plane angle** is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.
9. And when the lines containing the angle are straight, the angle is called **rectilineal.**
10. When a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is **right,** and the straight line standing on the other is called a **perpendicular** to that on which it stands.
11. An **obtuse angle** is an angle greater than a right angle.
12. An **acute angle** is an angle less than a right angle.
13. A **boundary** is that which is an extremity of anything.
14. A **figure** is that which is contained by any boundary or boundaries.
15. A **circle** is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another;

Canonical English edition by Sir Thomas L. Heath, 1908

See also the Reading Euclid Project

Billingsley's Euclid, 1570



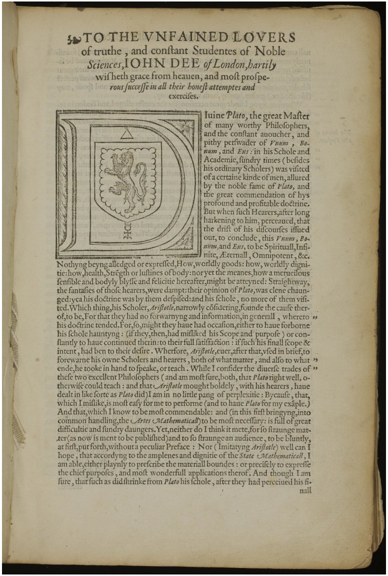
The Elements of Geometrie:

"Faithfully (now first) translated into the Englishe toung" by H. Billingsley, London, 1570

Available online

Preface by John Dee

Dee's Preface

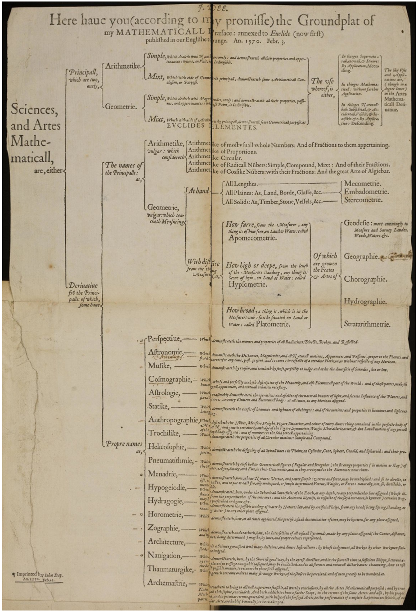


TO THE VNFAINED LOVERS of truthe, and conſtant Studentes of Noble *Sciences,* IOHN DEE *of London, hartily* wiſheth grace from heauen, and moſt proſperous *ſucceſſe in all their honeſt attemptes and* exerciſes.

Diuine *Plato,* the great Maſter of many worthy Philoſophers, and the conſtant auoucher, and pithy perſwader of *Vnum, Bonum,* and *Ens:* in his Schole and Academie, ſundry times (beſides his ordinary Scholers) was viſited of a certaine kinde of men, allured by the noble fame of *Plato,* and the great commendation of hys profound and profitable doctrine. But when ſuch Hearers, after long harkening to him, perceaued, that the drift of his diſcourſes iſſued out, to conclude, this *Vnum, Bonum,* and *Ens,* to be Spirituall, Infinite, Æternall, Omnipotent, &c. Nothyng beyng alledged or expreſſed, How, worldly goods: how, worldly dignitie: how, health, Strẽgth or luſtines of body: nor yet the meanes, how a merueilous ſenſible and bodyly blyſſe and felicitie hereafter, might be atteyned: Straightway, the fantaſies of thoſe hearers, were dampt: their opinion of *Plato,* was clene chaunged: yea his doctrine was by them deſpiſed: and his ſchole, no more of them viſited. Which thing, his Scholer, *Ariſtotle,* narrowly cõſidering, founde the cauſe therof, to be, For that they had no forwarnyng and information, in generall, whereto ” his doctrine tended. For, ſo, might they haue had occaſion, either to haue forborne his ſchole hauntyng: (if they, then, had miſliked his Scope and purpoſe) or conſtantly to haue continued therin: to their full ſatiſfaction: if ſuch his finall ſcope & intent, had ben to their deſire. Wherfore, *Ariſtotle,* euer, after that, vſed in brief, to forewarne his owne Scholers and hearers, both of what matter, and alſo to what ” ende, he tooke in hand to ſpeake, or teach. While I conſider the diuerſe trades of ” theſe two excellent Philoſophers (and am moſt ſure, both, that *Plato* right well, otherwiſe could teach: and that *Ariſtotle* mought boldely, with his hearers, haue dealt in like ſorte as *Plato* did) I am in no little pang of perplexitie: Bycauſe, that, which I miſlike, is moſt eaſy for me to performe (and to haue *Plato* for my exãple.) And that, which I know to be moſt commendable: and (in this firſt bringyng, into common handling, the *Artes Mathematicall*) to be moſt neceſſary: is full of great difficultie and ſundry daungers. Yet, neither do I think it mete, for ſo ſtraunge matter (as now is ment to be publiſhed) and to ſo ſtraunge an audience, to be bluntly, at firſt, put forth, without a peculiar Preface: Nor (Imitatyng *Ariſtotle*) well can I hope, that accordyng to the amplenes and dignitie of the *State Mathematicall,* I am able, either playnly to preſcribe the materiall boundes: or preciſely to expreſſe the chief purpoſes, and moſt wonderfull applications therof. And though I am ſure, that ſuch as did ſhrinke from *Plato* his ſchole, after they had perceiued his fi-

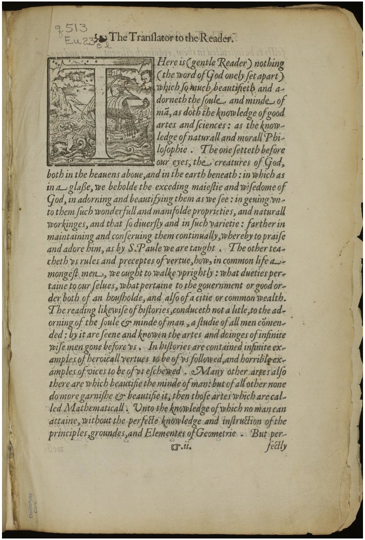
nall

Dee's 'Groundplat'



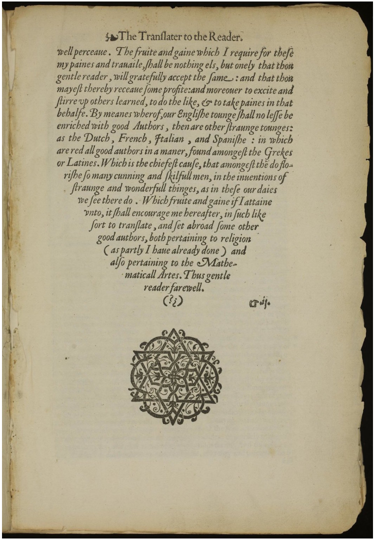
See: Jennifer M. Rampling, 'The Elizabethan mathematics of everything: John Dee's 'Mathematicall praeface' to Euclid's *Elements*', *BSHM Bulletin: Journal of the British Society for the History of Mathematics* **26**(3) (2011) 135–146

Billingsley's Preface, pp. 1, 3



There is (gentle Reader) nothing (the word of God onely ſet apart) which ſo much beautifieth and adorneth the ſoule and minde of mã, as doth the knowledge of good artes and ſciences: as the knowledge of naturall and morall Philoſophie. The one ſetteth before our eyes, the creatures of God, both in the heauens aboue, and in the earth beneath: in which as in a glaſſe, we beholde the exceding maieſtie and wiſedome of God, in adorning and beautifying them as we ſee: in geuing vnto them ſuch wonderfull and manifolde proprieties, and naturall workinges, and that ſo diuerſly and in ſuch varietie: farther in maintaining and conſeruing them continually, whereby to praiſe and adore him, as by S. Paule we are taught. The other teacheth vs rules and preceptes of vertue, how, in common life amongeſt men, we ought to walke vprightly: what dueties pertaine to our ſelues, what pertaine to the gouernment or good order both of an houſholde, and alſo of a citie or common wealth. The reading likewiſe of hiſtories, conduceth not a litle, to the adorning of the ſoule & minde of man, a ſtudie of all men cõmended: by it are ſeene and knowen the artes and doinges of infinite wiſe men gone before vs. In hiſtories are contained infinite examples of heroicall vertues to be of vs followed, and horrible examples of vices to be of vs eſchewed. Many other artes alſo there are which beautifie the minde of man: but of all other none do more garniſhe & beautifie it, then thoſe artes which are called Mathematicall. Unto the knowledge of which no man can attaine, without the perfecte knowledge and inſtruction of the principles, groundes, and Elementes of Geometrie. But per-

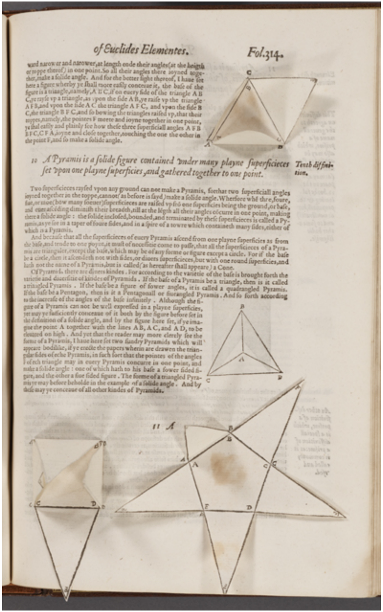
. . .



well perceaue. The fruite and gaine which I require for theſe my paines and trauaile, ſhall be nothing els, but onely that thou gentle reader, will gratefully accept the ſame: and that thou mayeſt thereby receaue ſome profite: and moreouer to excite and ſtirre vp others learned, to do the like, & to take paines in that behalfe. By meanes wherof, our Engliſhe tounge ſhall no leſſe be enriched with good Authors, then are other ſtraunge tounges: as the Dutch, French, Italian, and Spaniſhe: in which are red all good authors in a maner, found amongeſt the Grekes or Latines. Which is the chiefeſt cauſe, that amongeſt thẽ do floriſhe ſo many cunning and ſkilfull men, in the inuentions of ſtraunge and wonderfull thinges, as in theſe our daies we ſee there do. Which fruite and gaine if I attaine vnto, it ſhall encourage me hereafter, in ſuch like ſort to tranſlate, and ſet abroad ſome other good authors, both pertaining to religion (as partly I haue already done) and alſo pertaining to the Mathematicall Artes. Thus gentle reader farewell.

(?¿)

Pop-up Euclid



Book I: definitions

¶ The first booke of Euclides Elementes.

The argument of the first Booke.

In this first booke is intreated of the most simple, easie, and first matters and groundes of Geometry, as, namely, of Lynes, Angles, Triangles, Parallels, Squares, and Parallelogrammes. First of theyr definitions, shewyng what they are. After that it teacheth how to draw Parallel lynes, and how to forme diuersly figures of three sides, & foure sides, according to the varietie of their sides, and Angles : & cōpareth them all with Triangles, & also together the one with the other. In it also is taught how a figure of any forme may be chaunged into a Figure of an other forme. And for that it entreateth of these most common and generall thynges, thys booke is more vniuersall then is the seconde, third, or any other, and therefore iustly occupieth the first place in order : as that without which, the other bookes of *Euclide* which follow, and also the workes of others which haue written in Geometry, cannot be perceaued nor vnderstanded. And forasmuch as all the demonstrations and proofes of all the propositions in this whole booke, depende of these groundes and principles following, which by reason of their playnnes neede no greate declaration, yet to remoue all (be it neuer so litle) obscuritie, there are here set certayne shorte and manifest expositions of them.

Definitions.

Definition of a poynt.

1. *A signe or point is that, which hath no part.*

The better to vnderstand what maner of thing a signe or point is, ye must note that the nature and propertie of quantitie (wherof Geometry entreateth) is to be deuided, so that whatsoeuer may be deuided into sundry partes, is called quantitie. But a point, although it pertayne to quantitie, and hath his beyng in quantitie, yet is it no quantitie, for that it cannot be deuided. Because (as the definition saith) it hath no partes into which it should be deuided. So that a pointe is the least thing that by minde and vnderstanding can be imagined and conceyued : then which, there can be nothing lesse, as the point *A* in the margent.

A.

Definition of a poynt after Pithagoras.

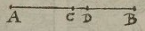
A signe or point is of *Pithagoras* Scholers after this manner defined: *A poynt is an vnitie which hath position*. Nūbers are conceaued in mynde without any forme & figure, and therfore without matter wheron to receaue figure, & consequently without place and position. Wherfore vnitie beyng a part of number, hath no position, or determinate place. Wherby it is manifest, that number is more simple and pure then is magnitude, and also immateriall: and so vnity which is the beginning of number, is lesse materiall then a signe or poynt, which is the beginnyng of magnitude. For a poynt is materiall, and requireth position and place, and therby differeth from vnitie.

Definition of a line.

2. *A line is length without breadth.*

There pertaine to quantitie three dimensions, length, bredth, & thicknes, or depth: and by these thre are all quātities measured & made known. There are also, according

to these three dimensions, three kyndes of continuall quantities : a lyne, a superficies, or plaine, and a body. The first kynde, namely, a line is here defined in these wordes, *A lyne is length without breadth*. A point, for that it is no quantitie nor hath any partes into which it may be deuided, but remaineth indiuisible, hath not, nor can haue any of these three dimensions. It neither hath length, breadth, nor thickenes. But to a line, which is the first kynde of quantitie, is attributed the first dimension, namely, length, and onely that, for it hath neither breadth nor thicknes, but is conceaued to be drawne in length onely, and by it, it may be deuided into partes as many as ye list, equall, or vnequall. But as touching breadth it remaineth indiuisible. As the lyne A B, which is onely drawen in length, may be deuided in the pointe C equally, or in the point D vnequally, and so into as many partes as ye list. There are also of diuers other geuen other definitions of a lyne: as these which follow.



An other definition of a line.

A lyne is the mouyng of a poynte, as the motion or draught of a pinne or a penne to your sence maketh a lyne.

An other.

Agayne, *A lyne is a magnitude hauing one onely space or dimension*, namely, *length wantyng breadth and thicknes*.

The endes of a line.

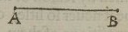
3 *The endes or limites of a lyne, are pointes.*

For a line hath his beginning from a point, and likewise endeth in a point: so that by this also it is manifest, that pointes, for their simplicitie and lacke of composition, are neither quantitie, nor partes of quantitie, but only the termes and endes of quantitie. As the pointes *A*, *B*, are onely the endes of the line *A B*, and no partes thereof. And herein differeth a poynte in quantitie, from vnitie in number: for that although vnitie be the beginning of nombers, and no number (as a point is the beginning of quantitie, and no quantitie) yet is vnitie a part of number. For number is nothyng els but a collection of vnities, and therfore may be deuided into them, as into his partes. But a point is no part of quantitie, or of a lyne: neither is a lyne composed of pointes, as number is of vnities. For things indiuisible being neuer so many added together, can neuer make a thing diuisible, as an instant in time, is neither tyme, nor part of tyme, but only the beginning and end of time, and coupleth & ioyneth partes of tyme together.

Difference of a point frō vnity.

Vnitie is a part of number.

A poynt is no part of quantitie.



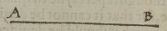
Definition of a right line.

4 *A right lyne is that which lieth equally betwene his pointes.*

As the whole line *A B* lyeth straight and equally betwene the poyntes *A B* without any going vp or comming downe on eyther side.

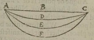
Definition of a right line after Campanus.

Campanus and certain others, define a right line thus: *A right line is the shortest extension or draught, that is or may be from one poynt to an other.* *Archimedes* defineth it thus.



Definitiō therof after Archimedes.

A right line is the shortest of all lines, which haue one and the self same limites or endes: which is in maner al one with the definitiō of *Campanus*. As of all these lines *A B C*, *A D C*, *A E C*, *A F C*, which are all drawen from the point *A*, to the poynte *B*, as *Campanus* speaketh, or which haue the selfsame limites or endes, as *Archimedes* speaketh, the lyne *A B C*, beyng a right line, is the shortest.



Definitiō therof after Plato.

Plato defineth a right line after this maner: *A right line is that whose middle part shadoweth the extremes.* As if you put any thyng in the middle of a right lyne, you shall not see from the one ende to the other, which thyng happeneth not in a crooked lyne. The Ecclipse of the Sunne (say Astronomers) then happeneth, when the Sunne, the Moone, & our eye are in one right line. For the Moone then being in the midst betwene vs and the Sunne, causeth it to be darkened. Diuers other define a right line diuersly, as followeth.

An other definition.

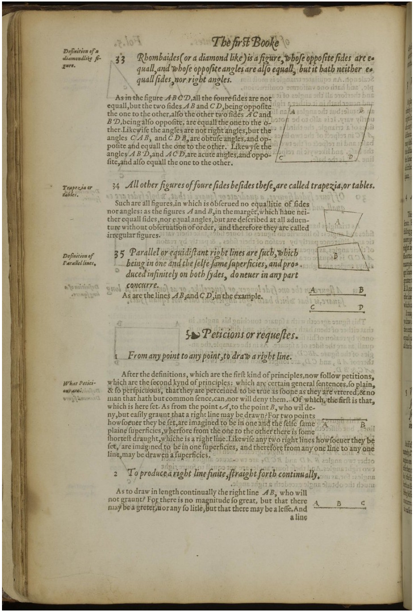
A right lyne is that which standeth firme betwene his extremes.

An other.

Agayne, *A right line is that which with an other line of lyke forme cannot make a figure.*

Agayne,

Book I: postulates



Definition of a diamondlike figure.

33 *Rhombaides (or a diamond like) is a figure, whoſe oppoſite ſides are equall, and whoſe oppoſite angles are alſo equall, but it hath neither equall ſides, nor right angles.*

As in the figure *ABCD*, all the foure ſides are not equall, but the two ſides *AB* and *CD*, being oppoſite the one to the other, alſo the other two ſides *AC* and *BD*, being alſo oppoſite, are equall the one to the other. Likewiſe the angles are not right angles, but the angles *CAB*, and *CDB*, are obtuſe angles, and oppoſite and equall the one to the other. Likewyſe the angles *ABD*, and *ACD*, are acute angles, and oppoſite, and alſo equall the one to the other.

Trapezia or tables.

34 *All other figures of foure ſides beſides theſe, are called trapezia, or tables.*

Such are all figures, in which is obſerued no equallitie of ſides nor angles: as the figures *A* and *B*, in the margēt, which haue neither equall ſides, nor equal angles, but are deſcribed at all aduenture without obſeruation of order, and therefore they are called irregular figures.

Definition of Parallel lines.

35 *Parallel or equidiſtant right lines are ſuch, which being in one and the ſelfe ſame ſuperficies, and produced infinitely on both ſydes, do neuer in any part concurre.*

As are the lines *AB*, and *CD*, in the example.

Peticions or requeſtes.

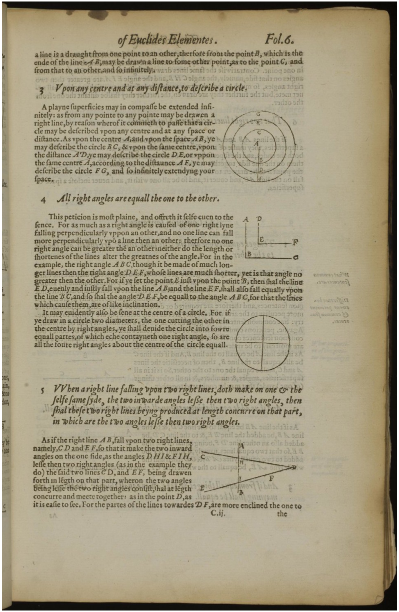
1 *From any point to any point, to draw a right line.*

What Peticions are.

After the definitions, which are the firſt kind of principles, now follow petitions, which are the ſecond kynd of principles: which are certain general ſentences, ſo plain, & ſo perſpicuous, that they are perceiued to be true as ſoone as they are vttered, & no man that hath but common ſence, can, nor will deny them. Of which, the firſt is that, which is here ſet. As from the point *A*, to the point *B*, who wil deny, but eaſily graunt that a right line may be drawn? For two points howſoeuer they be ſet, are imagined to be in one and the ſelfe ſame plaine ſuperficies, wherfore from the one to the other there is ſome ſhorteſt draught, whiche is a right line. Likewiſe any two right lines howſoeuer they be ſet, are imagined to be in one ſuperficies, and therefore from any one line to any one line, may be drawen a ſuperficies.

2 *To produce a right line finite, ſtraight forth continually.*

As to draw in length continually the right line *AB*, who will not graunt? For there is no magnitude ſo great, but that there may be a greter, nor any ſo litle, but that there may be a leſſe. And a line



a line is a draught from one point to an other, therfore from the point *B*, which is the ende of the line *AB*, may be drawn a line to ſome other point, as to the point *C*, and from that to an other, and ſo infinitely.

3 *Vpon any centre and at any diſtance, to deſcribe a circle.*

A playne ſuperficies may in compaſſe be extended infinitely: as from any pointe to any pointe may be drawen a right line, by reaſon wherof it commeth to paſſe that a circle may be deſcribed vpon any centre and at any ſpace or diſtance. As vpon the centre *A*, and vpon the ſpace *AB*, ye may deſcribe the circle *BC*, & vpon the ſame centre, vpon the diſtance *AD*, ye may deſcribe the circle *DE*, or vppon the ſame centre *A*, according to the diſtaunce *AF*, ye may deſcribe the circle *FG*, and ſo infinitely extendyng your ſpace.

4 *All right angles are equall the one to the other.*

This peticion is moſt plaine, and offreth it ſelfe euen to the ſence. For as much as a right angle is cauſed of one right lyne falling perpendicularly vppon an other, and no one line can fall more perpendicularly vpō a line then an other: therfore no one right angle can be greater thē an other: neither do the length or ſhortenes of the lines alter the greatnes of the angle. For in the example, the right angle *ABC*, though it be made of much longer lines then the right angle *DEF*, whoſe lines are much ſhorter, yet is that angle no greater then the other. For if ye ſet the point *E* iuſt vpon the point *B*, then ſhal the line *ED*, euenly and iuſtly fall vpon the line *AB*, and the line *EF*, ſhall alſo fall equally vpon the line *BC*, and ſo ſhal the angle *DEF*, be equall to the angle *ABC*, for that the lines which cauſe them, are of like inclination.

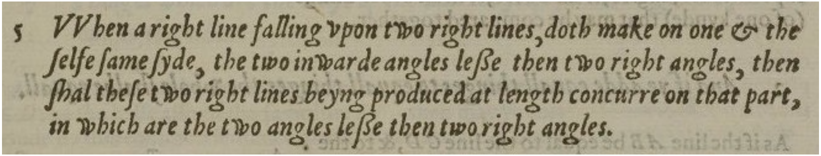
It may euidently alſo be ſene at the centre of a circle. For if ye draw in a circle two diameters, the one cutting the other in the centre by right angles, ye ſhall deuide the circle into fowre equall partes, of which eche contayneth one right angle, ſo are all the foure right angles about the centre of the circle equall.

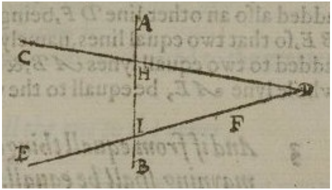
5 *VVhen a right line falling vpon two right lines, doth make on one & the ſelfe ſame ſyde, the two inwarde angles leſſe then two right angles, then ſhal theſe two right lines beyng produced at length concurre on that part, in which are the two angles leſſe then two right angles.*

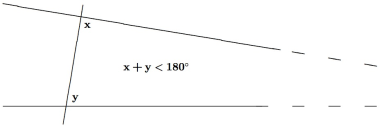
As if the right line *AB*, fall vpon two right lines, namely, *CD* and *EF*, ſo that it make the two inward angles on the one ſide, as the angles *DHI* & *FIH*, leſſe then two right angles (as in the example they do) the ſaid two lines *CD*, and *EF*, being drawen forth in lēgth on that part, wheron the two angles being leſſe thē two right angles conſiſt, ſhal at lēgth concurre and meete together: as in the point *D*, as it is eaſie to ſee. For the partes of the lines towardes *DF*, are more enclined the one to

 the

Postulate 5


5 VVhen a right line falling vpon two right lines, doth make on one & the ſelfe ſame ſyde, the two inwarde angles leſſe then two right angles, then ſhal theſe two right lines beyng produced at length concurre on that part, in which are the two angles leſſe then two right angles.





Equivalent formulation (Proclus, 5th century; John Playfair, 1795): given a straight line L and a point P not on L there is one and only one straight line through P that is parallel to L.

Classical disquiet about the fifth postulate

Original to Euclid? Less 'self-evident' than the other postulates?

Euclid used it (e.g., in the proof of Proposition 29 of Book I), so the property is necessary — but does it in fact follow from the other postulates?

Proclus in commentary on Euclid, 5th century (after citing Ptolemy's attempted proof of the parallel postulate, and discussing the nature of truth, with reference to Aristotle and Plato):

> *It is then clear from this that we must seek a proof of the present theorem, and that it is alien to the special character of postulates.*

Attempted (unsuccessfully) to prove the fifth postulate on the basis of the others

See Heath, pp. 202–220

Mediaeval disquiet about the fifth postulate

In the Islamic world:

Ibn al-Haytham (Alhazen) (965–1039) attempted (unsuccessfully) to prove the parallel postulate by contradiction

Omar Khayyám (1050–1123) attempted to prove the fifth postulate on the basis of the following alternative:

> *two convergent straight lines intersect and it is impossible for two convergent straight lines to diverge in the direction in which they converge*

Described the situations that may occur if the postulate is omitted

Nasir al-Din al-Tusi (1201–1274) criticised Khayyám's attempted proof, offered his own

Al-Tusi's thoughts found their way into Europe via the writings (1298) of his son Sadr al-Tusi

Early modern disquiet about the fifth postulate

After reading al-Tusi, John Wallis showed that the parallel postulate is equivalent to the following:

> *on a given finite straight line it is always possible to construct a triangle similar to a given triangle*

He lectured on this in Oxford in 1663

Attempts to prove the fifth postulate on the basis of Euclid's other axioms had resulted only in equivalent forms — so can we have a consistent geometry in which it the parallel postulate fails?

Early hints of non-Euclidean geometry

Giovanni Girolamo Saccheri (1667–1733): sought to establish the validity of Euclidean geometry — negated the parallel postulate in search of a contradiction; two cases:

- internal angles of a triangle add up to less than two right angles — contradicts Euclid's second postulate
- internal angles of a triangle add up to more than two right angles — leads to non-intuitive ideas

Similar results derived by Johann Heinrich Lambert (1728–1777) in his *Theorie der Parallellinien* (1766)

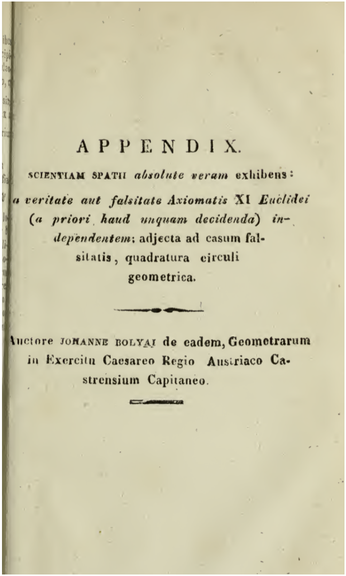
Non-Euclidean geometries

Consistent non-Euclidean geometry probably first constructed (tentatively) by Gauss, c. 1817–1830, but remained unpublished

Problem pursued independently (without success) by Gauss' friend Farkas Bolyai (1775–1856)

Pursued (against paternal advice) and solved by János Bolyai (1802–1860): "I have created a new and different world out of nothing" (1823)

Bolyai's geometry



APPENDIX.

SCIENTIAM SPATII *absolute veram* exhibens:

a veritate aut falsitate Axiomatis XI *Euclidei* (*a priori haud unquam decidenda*) *independentem*; adjecta ad casum falsitatis, quadratura circuli geometrica.

Auctore JOHANNE BOLYAI de eadem, Geometrarum in Exercitu Caesareo Regio Austriaco Castrensium Capitaneo.

Published as appendix 'The science absolute of space: independent of the truth or falsity of Euclid's axiom XI (which can never be decided a priori)' to father's textbook *Tentamen iuventutem studiosam in elementa matheosos introducendi* (1832)

English translation by George Bruce Halstead (1896)

Meanwhile in Russia...

Non-Euclidean geometry developed independently by Nikolai Ivanovich Lobachevskii [Николай Иванович Лобачевский] (1792–1856) using the negation of Playfair's axiom

Lobachevskii's works


Geometrische Untersuchungen

zur

Theorie der Parallellinien

von

Nicolaus Lobatschewsky,
Kaiserl. russ. wirkl. Staatsrathe und ord. Prof. der Mathematik
bei der Universität Kasan.

Berlin. 1840.
In der G. Fincke'schen Buchhandlung

Complicated story of dissemination...

Geometriya [Геометрия] written in 1823 but not published until 1909

Ideas presented in Kazan in 1826, published there 1829 — but rejected by St Petersburg Academy

Other works in Russian, French and German, including *Geometrische Untersuchungen zur Theorie der Parallellinien* (1840), *Pangéométrie* (1855)

(See Tom Lehrer for an unfair characterisation of Lobachevskii: https://youtu.be/IL4vWJbwmqM)

Acceptance and impact of non-Euclidean geometries

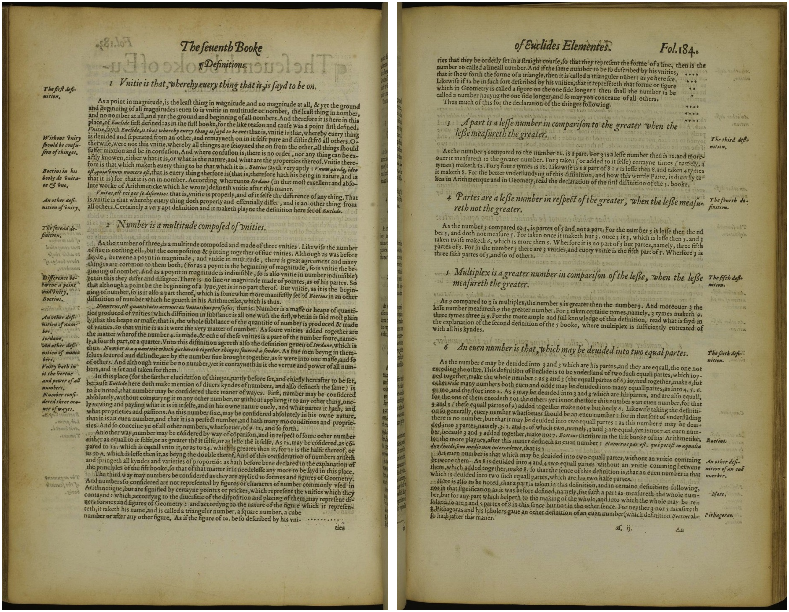
Slow to gain acceptance due to

- obscurity of publications
- lack of intuitive understanding

But non-Euclidean geometries

- overturned old ideas of mathematical certainty
- introduced new ideas about space
- helped drive the late 19th-century move towards axiomatisation

Euclid on numbers (positive integers)



The ſeuenth Booke

¶Definitions.

The firſt definition.

1 *Vnitie is that, whereby euery thing that is, is ſayd to be on.*

As a point in magnitude, is the leaſt thing in magnitude, and no magnitude at all, & yet the ground and beginning of all magnitudes: euen ſo is vnitie in multitude or nomber, the leaſt thing in nomber, and no nomber at all, and yet the ground and beginning of all nombers. And therefore it is here in this place, of *Euclide* firſt defined: as in the firſt booke, for the like reaſon and cauſe was a point firſt defined. *Vnitie*, ſayth *Euclide, is that whereby euery thing is ſayd to be one:* that is, vnitie is that, whereby euery thing is deuided and ſeperated from an other, and remayneth on in it ſelfe pure and diſtinct frō all others. Otherwiſe, were not this vnitie, whereby all thinges are ſeioyned the on from the other, all things ſhould ſuffer mixtion and be in confuſion. And where confuſion is, there is no order, nor any thing can be exactly knowen, either what it is, or what is the nature, and what are the properties thereof. Vnitie therefore is that which maketh euery thing to be that which it is. *Boetius* ſayth very aptly: *Vnum quodq; ideo eſt, quia vnum numero eſt*, that is euery thing therfore is (that is, therefore hath his being in nature, and is that it is) for that it is on in nomber. According whereunto *Iordane* (in that moſt excellent and abſolute worke of Arithmeticke which he wrote) defineth vnitie after this maner.

Without vnity ſhould be confuſion of thinges.

Boetius in his booke de vnitate & vno.

Vnitas, eſt rei per ſe diſcretio: that is, vnitie is properly, and of it ſelfe the difference of any thing. That is, vnitie is that whereby euery thing doth properly and eſſentially differ, and is an other thing from all others. Certainely a very apt definition and it maketh playne the definition here ſet of *Euclide*.

An other definition of vnity.

The ſecond definition.

2 *Number is a multitude compoſed of vnities.*

As the number of three, is a multitude compoſed and made of three vnities. Likewiſe the number of fiue is nothing elſ, but the compoſition & putting together of fiue vnities. Although as was before ſayde, betwene a poynt in magnitude, and vnitie in multitude, there is great agreement and many thinges are common to them both, (for as a poynt is the beginning of magnitude, ſo is vnitie the beginning of nomber. And as a poynt in magnitude is indiuiſible, ſo is alſo vnitie in number indiuiſible) yet in this they differ and diſagree. There is no line or magnitude made of pointes, as of his partes. So that although a point be the beginning of a lyne, yet is it no part therof. But vnitie, as it is the beginning of number, ſo is it alſo a part therof, which is ſomewhat more manifeſtly ſet of *Boetius* in an other diffinition of number which he geueth in his Arithmetike, which is thus.

Difference betwene a point and vnity.

Boetius.

Numerus, eſt quantitatis aceruus ex vnitatibus profuſus, that is. Number is a maſſe or heape of quantities produced of vnities: which diffinition in ſubſtance is all one with the firſt, wherin is ſaid moſt plainly, that the heape or maſſe, that is, the whole ſubſtance of the quantitie of number is produced & made of vnities. So that vnitie is as it were the very matter of number. As foure vnities added together are the matter wherof the number 4. is made, & eche of theſe vnities is a part of the number foure, namely, a fourth part, or a quarter. Vnto this diffinition agreeth alſo the definition geuen of *Iordane*, which is thus. *Number is a quantitie which gathereth together thinges ſeuered a ſonder*. As fiue men beyng in themſelues ſeuered and diſtincte, are by the number fiue brought together, as it were into one maſſe, and ſo of others. And although vnitie be no number, yet it contayneth in it the vertue and power of all numbers, and is ſet and taken for them.

An other definition of number.

Iordane.

An other definition of numbers.

Vnity hath in it the vertue and power of all numbers.

In this place (for the farther elucidation of thinges, partly before ſet, and chiefly hereafter to be ſet, becauſe *Euclide* here doth make mention of diuers kyndes of numbers, and alſo defineth the ſame) is to be noted, that number may be conſidered three maner of wayes. Firſt, number may be conſidered abſolutely, without comparyng it to any other number, or without applieng it to any other thing, onely vewing and payſing what it is in it ſelfe, and in his owne nature onely, and what partes it hath, and what proprieties and paſſions. As this number ſixe, may be conſidered abſolutely in his owne nature, that it is an euen number, and that it is a perfect number, and hath many mo conditions and proprieties. And ſo conceiue ye of all other numbers, whatſoeuer, of 9. 12. and ſo forth.

Number conſidered three maner of wayes.

An other way, number may be cōſidered by way of cōpariſon, and in reſpect of ſome other number either as equall to it ſelfe, or as greater thē it ſelfe, or as leſſe thē it ſelfe. As 12. may be cōſidered, as cōpared to 12. which is equall vnto it, or as to 24. which is greater then it, for 12 is the halfe thereof, or as to 6. which is leſſe then it, as beyng the double therof. And of this conſideration of numbers ariſeth and ſpringeth all kyndes and varieties of proportiō: as hath before bene declared in the explanation of the principles of the fift booke, ſo that of that matter it is needeleſſe any more to be ſayd in this place.

The third way may numbers be conſidered as they are applied to formes and figures of Geometry. And numbers ſo conſidered are not repreſented by figures or characters of number commonly vſed in Arithmetique, but are ſignified by certayne pointes or prickes, which repreſent the vnities which they contayne: which, accordyng to the diuerſitie of the diſpoſition and placing of them, may repreſent diuers formes and figures of Geometry: and accordyng to the nature of the figure which it repreſenteth, it taketh his name, and is called a trianguler number, a ſquare number, a cube number or after any other figure. As if the figure of 10. be ſo deſcribed by his vni-

ties

of Euclides Elementes. Fol. 184.

ties that they be orderly ſet in a ſtraight courſe, ſo that they repreſent the forme of a line, then is the number 10 called a lineall number. And if the ſame number 10 be ſo deſcribed by his vnities, that it ſhew forth the forme of a triangle, then it is called a trianguler nūber: as ye here ſee. Likewiſe if 12 be in ſuch ſort deſcribed by his vnities, that it repreſēteth that forme or figure which in Geometry is called a figure on the one ſide longer: then ſhall the number 12 be called a number hauyng the one ſide longer, and ſo may you conceaue of all others.

Thus much of this for the declaration of the thinges following.

The third definition.

3 *A part is a leſſe number in compariſon to the greater when the leſſe meaſureth the greater.*

As the number 3 compared to the number 12. is a part. For 3 is a leſſe number then is 12. and moreouer it meaſureth 12 the greater number. For 3 taken (or added to it ſelfe) certayne times (namely, 4 tymes) maketh 12. For 3 foure tymes is 12. Likewiſe is 2 a part of 8: 2 is leſſe then 8, and taken 4 tymes it maketh 8. For the better vnderſtandyng of this diffinition, and how this worde Parte, is diuerſly taken in Arithmetique and in Geomety, read the declaration of the firſt diffinition of the 5. booke.

The fourth definition.

4 *Partes are a leſſe number in reſpect of the greater, when the leſſe meaſureth not the greater.*

As the number 3 compared to 5, is partes of 5 and not a part. For the number 3 is leſſe then the nūber 5, and doth not meaſure 5. For taken once it maketh but 3. once 3 is 3, which is leſſe then 5. and 3 taken twiſe maketh 6, which is more then 5. Wherfore it is no part of 5 but partes, namely, three fifth partes of 5. For in the number 3 there are 3 vnities, and euery vnitie is the fifth part of 5. Wherfore 3 is three fifth partes of 5, and ſo of others.

The fifth definition.

5 *Multiplex is a greater number in compariſon of the leſſe, when the leſſe meaſureth the greater.*

As 9 compared to 3 is multiplex, the number 9 is greater then the number 3. And moreouer 3 the leſſe number meaſureth 9 the greater number. For 3 taken certaine tymes, namely, 3 tymes maketh 9. three tymes three is 9. For the more ample and full knowledge of this definition, read what is ſayd in the explanation of the ſecond definition of the 5 booke, where multiplex is ſufficiently entreated of with all his kyndes.

The ſixth definition.

6 *An euen number is that, which may be deuided into two equal partes.*

As the number 6 may be deuided into 3 and 3 which are his partes, and they are equall, the one not exceding the other. This definition of Euclide is to be vnderſtand of two ſuch equall partes, which ioyned together, make the whole number: as 3 and 3 (the equall partes of 6) ioyned together, make 6, for otherwiſe many numbers both euen and odde may be deuided into many equall partes, as into 4. 5. 6. or mo, and therfore into 2. As 9 may be deuided into 3 and 3 which are his partes, and are alſo equall, for the one of them exceedeth not the other: yet is not therfore this number 9 an euen number, for that 3 and 3 (theſe equall partes of 9) added together make not 9 but onely 6. Likewiſe taking the definition ſo generally, euery number whatſoeuer ſhould be an euen number: for in that ſort of vnderſtāding there is no number, but that it may be deuided into two equall partes: as this number 7 may be deuided into 3 partes, namely, 3. 1. and 3. of which two, namely, 3 and 3 are equal, yet is not 7 an euen number, becauſe 3 and 3 added together, make not 7. *Boetius* therfore in the firſt booke of his Arithmetike, for the more playnes, after this maner defineth an euen number: *Numerus par eſt, qui poteſt in aequalia duo diuidi, vno medio non intercedente*, that is:

Boetius.

An euen number is that which may be deuided into two equall partes, without an vnitie comming betwene them. As 8 is deuided into 4 and 4 two equall partes without an vnitie comming betwene them, which added together, make 8, ſo that the ſence of this definition is, that an euen number is that which is deuided into two ſuch equall partes, which are his two halfe partes.

An other definition of an euē number.

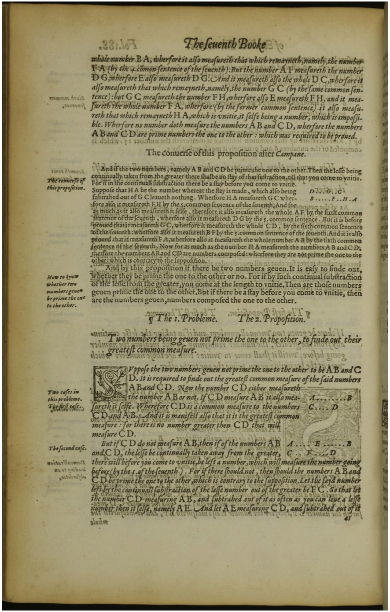
Here is alſo to be noted, that a part is taken in this definition, and in certaine definitions following, not in that ſignification as it was before defined, namely, for ſuch a part as meaſureth the whole number, but for any part which helpeth to the making of the whole, and into which the whole may be reſolued, ſo are 3 and 5 partes of 8 in this ſence but not in the other ſence. For neyther 3 nor 5 meaſureth 8. Pithagoras and his ſcholers gaue an other definition of an euen number (which definition *Boetius* alſo hath) after this maner.

Note.

Pithagoras.

ſſ. ij. An

The Euclidean algorithm (Proposition VII.2)



The ſeuenth Booke

whole number B A, wherfore it alſo meaſureth that which remayneth, namely, the number F A (by the 4. cōmon ſentence of the ſeuenth). But the number A F meaſureth the number D G, wherfore E alſo meaſureth D G. And it meaſureth alſo the whole D C, wherfore it alſo meaſureth that which remayneth, namely, the number G C (by the ſame common ſentence): but G C meaſureth the number F H, wherfore alſo E meaſureth F H, and it meaſureth the whole number F A, wherfore (by the former common ſentence) it alſo meaſureth that which remayneth H A, which is vnitie, it ſelfe being a number, which is impoſſible. Wherfore no number doth meaſure the numbers A B and C D, wherfore the numbers A B and C D are prime numbers the one to the other: which was required to be proued.

The conuerſe of this propoſition after *Campane*.

The conuerſe of this propoſition.

And if the two numbers, namely A B and C D be prime the one to the other. Then the leſſe being continually taken from the greater there ſhalbe no ſtay of that ſuſtraction, till that you come to vnitie. For if in the continuall ſubſtraction there be a ſtay before you come to vnitie. Suppoſe that H A be the number whereat the ſtay is made, which alſo being ſubtrahed out of G C leaueth nothing. Wherfore H A meaſureth G C wherfore alſo it meaſureth F H by the 5. common ſentence of the ſeuenth. And for as much as it alſo meaſureth it ſelfe, therefore it alſo meaſureth the whole A F by the ſixth common ſentence of the ſeuenth, wherfore alſo it meaſureth D G by the 5. common ſentence. But it is before proued that it meaſureth G C, wherfore it meaſureth the whole C D, by the ſixth common ſentence of the ſeuenth: wherfore alſo it meaſureth B F by the 5. common ſentence of the ſeuenth. And it is alſo proued that it meaſureth F A, wherfore alſo it meaſureth the whole number A B by the ſixth common ſentence of the ſeuenth. Now for as much as the number H A meaſureth the numbers A B and C D, therfore the numbers AB and CD are numbers compoſed: wherfore they are not prime the one to the other: which is contrary to the ſuppoſition.

D . . . G . . C
B . . . . . F . . H . A

How to know whether two numbers geuen be prime the one to the other.

And by this propoſition if there be two numbers geuen. It is eaſy to finde out, whether they be prime the one to the other or no. For if by ſuch continual ſubſtraction of the leſſe from the greater, you come at the length to vnitie. Then are thoſe numbers geuen prime the one to the other. But if there be a ſtay before you come to vnitie, then are the numbers geuen, numbers compoſed the one to the other.

¶ The 1. Probleme. The 2. Propoſition.

Two numbers being geuen not prime the one to the other, to finde out their greateſt common meaſure.

Two caſes in this probleme. The firſt caſe.

Suppoſe the two numbers geuen not prime the one to the other to be A B *and* C D. *It is required to finde out the greateſt common meaſure of the ſaid numbers* A B *and* C D. *Now the number* C D *either meaſureth the number* A B *or not. If* C D *meaſure* A B *it alſo meaſureth it ſelfe. Wherefore* C D *is a common meaſure to the numbers* C D *and* A B. *And it is manifeſt alſo that it is the greateſt common meaſure: for there is no number greater then* C D *that will meaſure* C D.

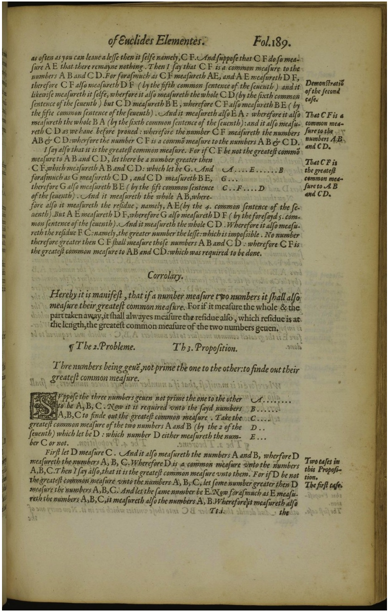
A . . . . B
C . . . . D

The ſecond caſe.

But if C D *do not meaſure* A B, *then if of the numbers* A B *and* C D, *the leſſe be continually taken away from the greater, there will before you come to vnitie, be left a number, which will meaſure the number going before (by the 1. of the ſeuenth). For if there ſhould not, then ſhould the numbers* A B *and* C D *be prime the one to the other, which is contrary to the ſuppoſition. Let the ſayd number left by the continuall ſubſtraction of the leſſe number out of the greater be* F C. *So that let the number* C D *meaſuring* A B, *and ſubtrahed out of it as often as you can leue a leſſe number then it ſelfe, namely* A E. *And let* A E *meaſuring* C D, *and ſubtrahed out of it*

A . . . . E . . . . . . B
C . . F . . . . D

45



of Euclides Elementes. Fol. 189.

as often as you can leaue a leſſe then it ſelfe namely, C F. *And ſuppoſe that* C F *do ſo meaſure* A E *that there remayne nothing. Then I ſay that* C F *is a common meaſure to the numbers* A B *and* C D. *For foraſmuch as* C F *meaſureth* AE, *and* A E *meaſureth* D F, *therefore* C F *alſo meaſureth* D F *(by the fifth common ſentence of the ſeuenth) and it likewiſe meaſureth it ſelfe, wherfore it alſo meaſureth the whole* C D *(by the ſixth common ſentence of the ſeuenth) but* C D *meaſureth* B E, *wherefore* C F *alſo meaſureth* B E *(by the fifte common ſentence of the ſeuenth). And it meaſureth alſo* E A: *wherefore it alſo meaſureth the whole* B A *(by the ſixth common ſentence of the ſeuenth): and it alſo meaſureth* C D *as we haue before proued: wherefore the number* C F *meaſureth the numbers* AB *&* C D: *wherfore the number* C F *is a cōmō meaſure to the numbers* A B *&* C D.

Demonſtratiō of the ſecond caſe.

That C F is a common meaſure to the numbers A B and C D.

I ſay alſo that it is the greateſt common meaſure. For if C F *be not the greateſt cōmō meaſure to* A B *and* C D, *let there be a number greater then* C F, *which meaſureth* A B *and* C D: *which let be* G. *And foraſmuch as* G *meaſureth* C D, *and* C D *meaſureth* B E, *therefore* G *alſo meaſureth* B E *(by the fift common ſentence of the ſeuenth). And it meaſureth the whole* A B, *wherefore alſo it meaſureth the reſidue, namely,* A E *(by the 4. common ſentence of the ſeuenth). But* A E *meaſureth* D F, *wherefore* G *alſo meaſureth* D F *(by the foreſayd 5. common ſentence of the ſeuenth). And it meaſureth the whole* C D. *Wherefore it alſo meaſureth the reſidue* F C: *namely, the greater number the leſſe: which is impoſſible. No number therefore greater then* C F *ſhall meaſure thoſe numbers* A B *and* C D: *wherefore* C F *is the greateſt common meaſure to* AB *and* CD: *which was required to be done.*

A . . . . E . . . . . . B
G . . .
C . . F . . . . D

That C F is the greateſt common meaſure to A B and C D.

Corrolary.

Hereby it is manifeſt, that if a number meaſure two numbers it ſhall alſo meaſure their greateſt common meaſure. For if it meaſure the whole & the part taken away, it ſhall alwayes meaſure the reſidue alſo, which reſidue is at the length, the greateſt common meaſure of the two numbers geuen.

¶ The 2. Probleme. Th 3. Propoſition.

Thre numbers being geuē, not prime the one to the other: to finde out their greateſt common meaſure.

Suppoſe the three numbers geuen not prime the one to the other to be A, B, C. *Now it is required vnto the ſayd numbers* A, B, C *to finde out the greateſt common meaſure. Take the greateſt common meaſure of the two numbers* A *and* B *(by the 2 of the ſeuenth) which let be* D: *which number* D *either meaſureth the number* C *or not.*

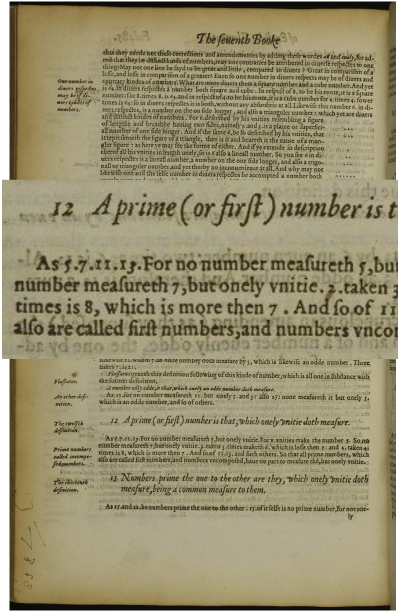
A . . . .
B . . . . . .
C . . . .
D . .
E . . .

Firſt let D *meaſure* C. *And it alſo meaſureth the numbers* A *and* B, *wherfore* D *meaſureth the numbers* A, B, C. *Wherefore* D *is a common meaſure vnto the numbers* A, B, C. *Then I ſay alſo, that it is the greateſt common meaſure vnto them. For if* D *be not the greateſt common meaſure vnto the numbers* A, B, C, *let ſome number greater then* D *meaſure the numbers* A, B, C. *And let the ſame number be* E. *Now foraſmuch as* E *meaſureth the numbers* A, B, C, *it meaſureth alſo the numbers* A, B. *Wherefore it meaſureth alſo the*

Two caſes in this Propoſition. The firſt caſe.

Tt.i.

Euclid on prime numbers



The seuenth Booke

that they neede not these corrections and amendementes by adding these wordes *all* and *onely*, for admit that they be distinct kinds of numbers, may not contraries be attributed in diuerse respectes to one thing? May not one line be sayd to be great and little, compared to diuers? Great in comparison of a lesse, and lesse in comparison of a greater? Euen so one number in diuers respects may be of diuers and contrary kindes of numbers. What are more diuers them a square number and a cube number. And yet is 64. in diuers respectes a number both square and cube. In respect of 8. to be his roote, it is a square number: for 8. times 8. is 64. and in respect of 4. to be his roote, it is a cube number for 4. times 4. fower times is 64: so in diuers respectes it is both, without any absurditie at all. Likewise this number 6. in diuers respectes, is a number on the on side longer, and also a trianguler number: which yet are diuers and distinct kindes of numbers. For 6. described by his vnities resembling a figure of lengthe and breadthe hauing two sides, namely 2. and 3. is a plaine or superficiall number of one side longer. And if the same 6, be so described by his vnities, that it representeth the figure of a triangle, then is it and beareth it the name of a triangler figure: as here ye may see the forme of either. And if ye extende in description therof all his vnities in length onely, so is 6 also a lineall number. So you see 6 in diuers respectes is a lineall number, a number on the one side longer, and also a trigonall or trianguler number, and yet therby no inconuenience at all. And why may not likewise one and the selfe number in diuers respectes be accompted a number both

One number in diuers respectes may be of diuers kindes of numbers.

Likewise 21. whom 7. an odde number doth measure by 3, which is likewise an odde number. Three times 7. is 21.

Flussatus geueth this definition following of this kinde of number, which is all one in substance with the former definition.

Flussates.

A number odly odde, is that, which onely an odde number doth measure.

As 15. for no number measureth 15. but onely 5. and 3: also 25: none measureth it but onely 5. which is an odde number, and so of others.

An other definition.

The twelfth definition.

12 A prime (or first) number is that, which onely vnitie doth measure.

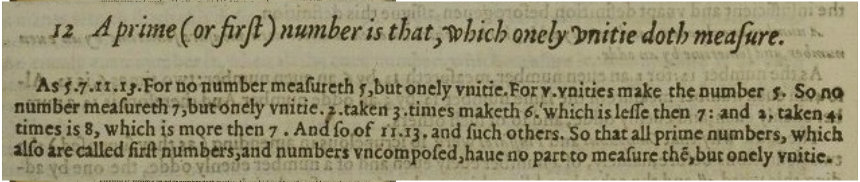
As 5.7.11.13. For no number measureth 5, but onely vnitie. For v. vnities make the number 5. So no number measureth 7, but onely vnitie. 2. taken 3. times maketh 6. which is lesse then 7: and 2. taken 4. times is 8, which is more then 7. And so of 11.13. and such others. So that all prime numbers, which also are called first numbers, and numbers vncomposed, haue no part to measure thē, but onely vnitie.

Prime numbers called incomposed numbers.

The thirtenth definition.

13 Numbers prime the one to the other are they, which onely vnitie doth measure, being a common measure to them.

As 15. and 22. be numbers prime the one to the other. 15. of it selfe is no prime number, for not onely



12 A prime (or first) number is that, which onely vnitie doth measure.

As 5.7.11.13. For no number measureth 5, but onely vnitie. For v. vnities make the number 5. So no number measureth 7, but onely vnitie. 2. taken 3. times maketh 6. which is lesse then 7: and 2. taken 4. times is 8, which is more then 7. And so of 11.13. and such others. So that all prime numbers, which also are called first numbers, and numbers vncomposed, haue no part to measure thē, but onely vnitie.

Euclid on prime numbers (Proposition IX.20)

But now ſuppoſe that A do not meaſure D. Then I ſay that it is not poßible to finde out a fourth number proportionall with theſe numbers A, B, C. For if it be poßible, let there be found ſuch a number, and let the ſame be E. Wherfore that which is produced of A into E is equall to that which is produced of B into C. But that which is produced of B into C is D. Wherfore that which is produced of A into E is equall vnto D. Wherefore A multiplieng E produced D, wherfore A meaſureth D, but it alſo meaſureth it not, which is impoßible. Wherfore it is impoßible to finde out a fourth number proportionall, with theſe numbers A, B, C whenſoeuer A meaſureth not D.

But now ſuppoſe that A, B, C be neither in continuall proportiō, neither alſo their extremes be prime the one to the other. And let B multiplieng C produce D. And in like ſorte may we proue that if A do meaſure D, it is poſſible to finde out a fourth number proportionall with them. But if it do not meaſure D, thē is it impoſſible: which was required to be proued.

A ....
B ....
C ....
E ————————————————
D 1350

¶ The 20. Theoreme. The 20. Propoſition.

Prime numbers being geuen how many ſoeuer, there may be geuen more prime numbers.

Suppoſe that the prime numbers geuen be A, B, C. Then I ſay, that there are yet more prime numbers beſides A, B, C. Take (by the 38. of the ſeuenth) the leſt number whom theſe numbers A, B, C do meaſure, and let the ſame be D E. And vnto D E adde vnitie D F. Now E F is either a prime number or not. Firſt let it be a prime number, then are there found theſe prime numbers A, B, C, and E F more in multitude then the prime numbers firſt geuen A, B, C.

But now ſuppoſe that E F be not prime. Wherefore ſome prime number meaſureth it (by the 24. of the ſeuenth). Let a prime number meaſure it, namely, G. Then I ſay, that G is none of theſe numbers A, B, C. For if G be one and the ſame with any of theſe A, B, C. But A, B, C, meaſure the nūber D E: wherfore G alſo meaſureth D E: and it alſo meaſureth the whole E F. Wherefore G being a number ſhall meaſure the reſidue D F being vnitie: which is impoßible. Wherefore G is not one and the ſame with any of theſe prime numbers A, B, C: and it is alſo ſuppoſed to be a prime number. Wherefore there are found theſe prime numbers A, B, C, G, being more in multitude then the prime numbers geuen A, B, C: which was required to be demonſtrated.

A ..
B ...
C ....
E 114 D . F
G ....

✻ A Corollary.

By thys Propoſition it is manifeſt, that the multitude of prime numbers is infinite.

¶ The 21. Theoreme. The 21. Propoſition.

If euen nūbers how many ſoeuer be added together: the whole ſhall be euē.

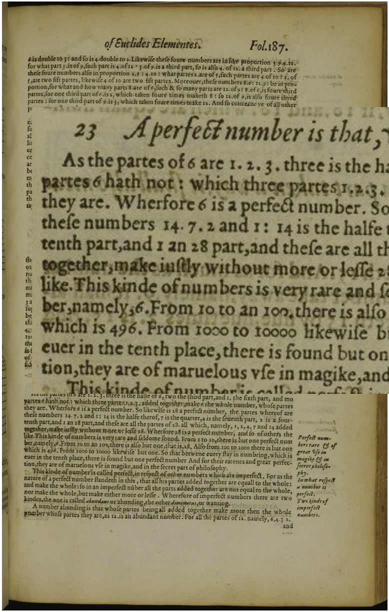
Prime numbers being geuen how many ſoeuer, there may be geuen more prime numbers.

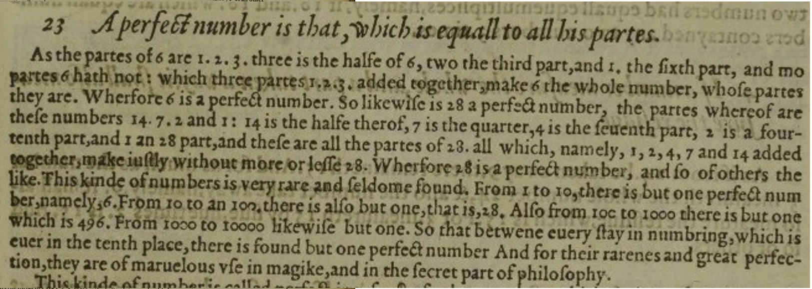
Suppoſe that the prime numbers geuen be A, B, C. Then I ſay, that there are yet more prime numbers beſides A, B, C. Take (by the 38. of the ſeuenth) the leſt number whom theſe numbers A, B, C do meaſure, and let the ſame be D E. And vnto D E adde vnitie D F. Now E F is either a prime number or not. Firſt let it be a prime number, then are there found theſe prime numbers A, B, C, and E F more in multitude then the prime numbers firſt geuen A, B, C.

But now ſuppoſe that E F be not prime. Wherefore ſome prime number meaſureth it (by the 24. of the ſeuenth). Let a prime number meaſure it, namely, G. Then I ſay, that G is none of theſe numbers A, B, C. For if G be one and the ſame with any of theſe A, B, C. But A, B, C, meaſure the nūber D E: wherfore G alſo meaſureth D E: and it alſo meaſureth the whole E F. Wherefore G being a number ſhall meaſure the reſidue D F being vnitie: which is impoßible. Wherefore G is not one and the ſame with any of theſe prime numbers A, B, C: and it is alſo ſuppoſed to be a prime number. Wherefore there are found theſe prime numbers A, B, C, G, being more in multitude then the prime numbers geuen A, B, C: which was required to be demonſtrated.

A ..
B ...
C ....
E 114 D . F
G ....

Euclid on perfect numbers

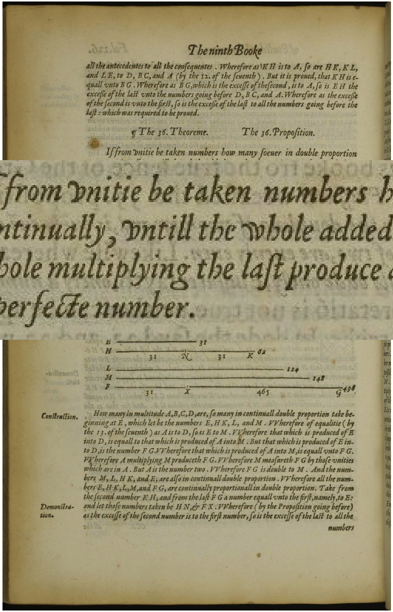




23 *A perfect number is that, which is equall to all his partes.*

As the partes of 6 are 1. 2. 3. three is the halfe of 6, two the third part, and 1. the sixth part, and mo partes 6 hath not: which three partes 1,2,3. added together, make 6 the whole number, whose partes they are. Wherfore 6 is a perfect number. So likewise is 28 a perfect number, the partes whereof are these numbers 14. 7. 2 and 1: 14 is the halfe therof, 7 is the quarter, 4 is the seuenth part, 2 is a fourtenth part, and 1 an 28 part, and these are all the partes of 28. all which, namely, 1, 2, 4, 7 and 14 added together, make iustly without more or lesse 28. Wherfore 28 is a perfect number, and so of others the like. This kinde of numbers is very rare and seldome found. From 1 to 10, there is but one perfect number, namely, 6. From 10 to an 100, there is also but one, that is, 28. Also from 10c to 1000 there is but one which is 496. From 1000 to 10000 likewise but one. So that betwene euery stay in numbring, which is euer in the tenth place, there is found but one perfect number And for their rarenes and great perfection, they are of maruelous vse in magike, and in the secret part of philosophy.

Euclid on perfect numbers (Proposition IX.36)



The ninth Booke

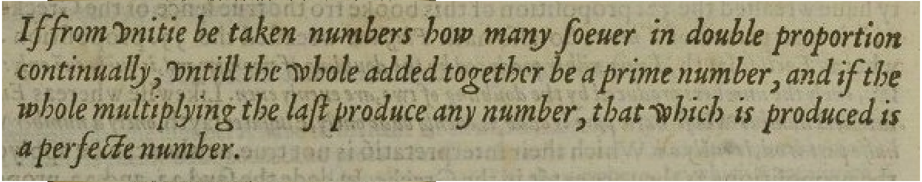
all the antecedentes to all the consequentes. Wherefore as K H is to A, so are H K, K L, and L E, to D, B C, and A (by the 12. of the seuenth). But it is proued, that K H is equall vnto B G. Wherefore as B G, which is the excesse of the second, is to A, so is E H the excesse of the last vnto the numbers going before D, B C, and A. Wherefore as the excesse of the second is vnto the first, so is the excesse of the last to all the numbers going before the last: which was required to be proued.

¶ The 36. Theoreme. The 36. Proposition.

If from vnitie be taken numbers how many soeuer in double proportion

Construction. How many in multitude A, B, C, D, are, so many in continuall double proportion take beginning at E, which let be the numbers E, H K, L, and M. VVherefore of equalitie (by the 13. of the seuenth) as A is to D, so is E to M. VVherefore that which is produced of E into D, is equall to that which is produced of A into M. But that which is produced of E into D, is the number F G. VVherefore that which is produced of A into M, is equall vnto F G. VVherefore A multiplying M produceth F G. VVherefore M measureth F G by those vnities which are in A. But A is the number two. VVherefore F G is double to M. And the numbers M, L, H K, and E, are also in continuall double proportion. VVherefore all the numbers E, H K, L, M, and F G, are continually proportionall in double proportion. Take from the second number K H, and from the last F G a number equall vnto the first, namely, to E: and let those numbers taken be H N, & F X. VVherefore (by the Proposition going before) as the excesse of the second number is to the first number, so is the excesse of the last to all the numbers

Demonstration.



If from vnitie be taken numbers how many soeuer in double proportion continually, vntill the whole added together be a prime number, and if the whole multiplying the last produce any number, that which is produced is a perfecte number.

In modern terms: if $2^n - 1$ is prime, then $2^{n-1}(2^n - 1)$ is perfect

Number theory after Euclid

Very little for many centuries...

Recall that Diophantus' *Arithmetica* (13 books, c. AD 250) featured number problems; for example [from Lecture IX]:

> Problem I.27: *Find two numbers such that their sum and product are given numbers*

The *Arithmetica* also features problems and ideas that we would now classify as number-theoretic; for example:

> Problem III.19: *To find four numbers such that the square of their sum plus or minus any one singly gives a square*
>
> Problem V.9: *To divide unity into two parts such that, if a given number is added to either part, the result will be a square*

Restrictions on the permitted form of solutions to problems eventually gave rise to the notion of Diophantine equations

Number theory outside Europe

Sūnzĭ Suànjīng 孙子算经 (*The Mathematical Classic of Master Sun*) (3rd–5th century BC) contains a statement, but no proof, of the Chinese Remainder Theorem for the solution of simultaneous congruences

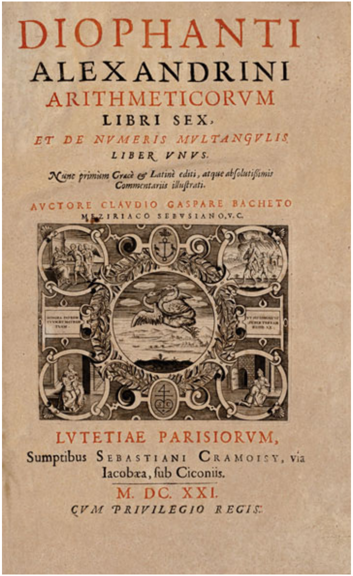
An algorithm for the solution was provided by Aryabhata in 6th-century India

In 7th-century India, Brahmagupta studied Diophantine equations (including Pell's equation — see later, and also: Toke Knudsen and Keith Jones, 'The Pell Equation in India', 2017)

These works were unknown in Europe until the 19th century

See: Eva Caianiello, 'Indeterminate linear problems from Asia to Europe', *Lettera Matematica* 6 (2018), 233–243

17th-century number theory



Bachet's Latin edition of Diophantus' *Arithmetica* (1621)

Pierre de Fermat owned a 1637 edition, which he studied and annotated

Fermat on number theory

Fermat's Little Theorem: if a is any integer and p is prime then p divides $a^p - a$

Studies of 'Pell's Equation' $x^2 - Dy^2 = 1$

Conjectures on perfect numbers [more in a moment]

Studies of diophantine problems leading to 'Fermat's Last Theorem' [more in a moment]

Published nothing — had to be exhorted to write his ideas down

(See *Mathematics emerging*, §§6.1–6.3)

The 'Last Theorem'

Arithmetica Problem II.8 concerns the splitting of a given square number into two other squares

Fermat's marginal note:

> *It is impossible to separate a cube into two cubes, or a fourth power into two fourth powers, or in general, any power higher than the second, into two like powers. I have discovered a truly marvelous proof of this, which this margin is too narrow to contain.*

(See: Simon Singh, *Fermat's Last Theorem*, Fourth Estate, 1998)

Perfect numbers

Euclid's Theorem: if $2^n - 1$ is prime then $2^{n-1}(2^n - 1)$ is perfect

Fermat to Mersenne (1640): if $2^n - 1$ is prime then n must be prime

Mersenne (1644): if $p \leq 257$ and $2^p - 1$ is prime then p is one of 2, 3, 5, 7, 13, 17, 67 (a misprint for 61 perhaps?), 127, 257. Not quite right: $2^{89} - 1$, $2^{107} - 1$ are prime and $2^{257} - 1$ is composite.

Euler: proof that all even perfect numbers are of Euclid's form (proved 1749, but published posthumously)

(See *Mathematics emerging*, §6.1.2)

NB. 52 Mersenne primes are currently known, the largest being $2^{136,279,841} - 1$ (found in October 2024)

17th-century attitudes to number theory

Fermat failed to spark an interest in number theory in his contemporaries

Pascal to Fermat (1655):

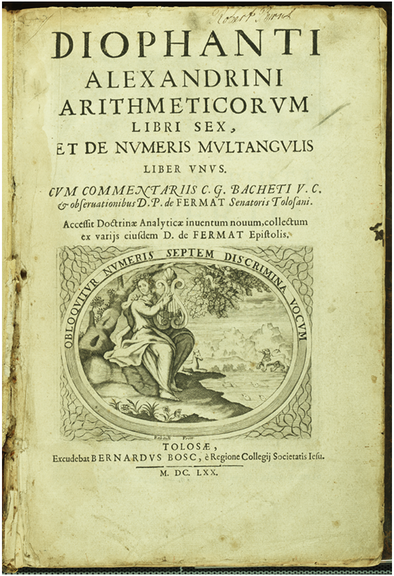
> *... seek elsewhere those who can follow you in your numerical discoveries ... I confess to you that this goes far beyond me ...*

Number-theoretic investigations were widely regarded as trivial and uninteresting

Huygens to Wallis:

> *There is no lack of better topics for us to spend our time on ...*

The ‘rebirth’ of number theory



1670 edition of Bachet, published by Samuel Fermat, including his father’s notes

The ‘Last Theorem’ was not the only result for which Fermat failed to provide a proof

Number theory was ‘reborn’ from the attempts of Euler (and later Lagrange and Legendre) to fill the gaps left by Fermat

Euler on number theory

Euler (1747):

> *Nor is the author disturbed by the authority of the greatest mathematicians when they sometimes pronounce that number theory is altogether useless and does not deserve investigation. In the first place, knowledge is always good in itself, even when it seems to be far removed from common use. Secondly, all the aspects of the truth which are accessible to our mind are so closely related to one another that we dare not reject any of them as being altogether useless. ...*
>
> *Consequently, the present author considers that he has by no means wasted his time and effort in attempting to prove various theorems concerning integers and their divisors. ... Moreover, there is little doubt that the method used here by the author will turn out to be of no small value in other investigations of greater import.*

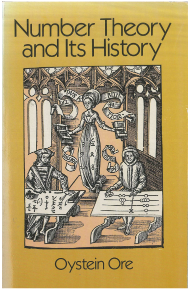
19th-century number theory

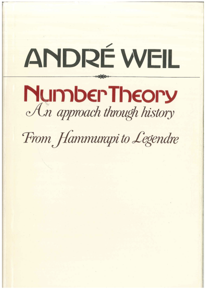
Gauss's *Disquisitiones arithmeticae* (1801) became a key text for many years to come: modular arithmetic, quadratic forms, cyclotomy, ...

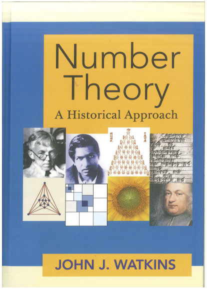
Number-theoretic problems (especially attempts to prove Fermat's Last Theorem) led to the development of ideal theory, and the linking of number theory and abstract algebra in algebraic number theory

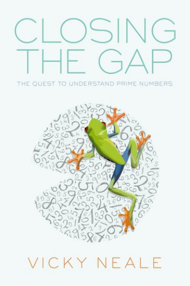
By the end of the 19th century, a new branch, analytic number theory, had also emerged (e.g., Riemann hypothesis, Prime Number Theory $\pi(x) \sim \frac{x}{\log x}, \ldots$)

The history of number theory









Leonard Eugene Dickson, *History of the theory of numbers*, 3 vols., Carnegie Institution of Washington, 1919–1923: I, II, III