Viscous Flow

Sheet : $\nabla^2\mathbf{u}$ — Michaelmas Term 2024

1. Given that in spherical polar coordinates $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, for any differentiable scalar field f and any differentiable vector field $\mathbf{F} = F_r\mathbf{e}_r + F_\theta\mathbf{e}_\theta + F_\phi\mathbf{e}_\phi$,

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\mathbf{e}_\phi,$$

$$\nabla\cdot\mathbf{F} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2F_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(F_\theta\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi},$$

$$\nabla\wedge\mathbf{F} = \frac{1}{r^2\sin\theta}\begin{vmatrix}\mathbf{e_r} & r\mathbf{e}_\theta & r\sin\theta\mathbf{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ F_r & rF_\theta & r\sin\theta F_\phi,\end{vmatrix}$$

and

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2},$$

work out $\boldsymbol{\nabla}^2\mathbf{u}$ in spherical coordinates.

Solution:

We establish the relationship by considering each of the three components. First we calculate

$$\begin{aligned}\nabla\wedge\mathbf{u} &= \frac{1}{r^2\sin\theta}\left[\left(\frac{\partial}{\partial\theta}(r\sin\theta\,u_\phi) - \frac{\partial}{\partial\phi}(ru_\theta)\right)\mathbf{e}_r + r\left(\frac{\partial u_r}{\partial\phi} - \frac{\partial}{\partial r}(r\sin\theta\,u_\phi)\right)\mathbf{e}_\theta\right. \\ &\quad\left. + r\sin\theta\left(\frac{\partial}{\partial r}(ru_\theta) - \frac{\partial u_r}{\partial\theta}\right)\mathbf{e}_\phi\right] \\ &= \left(\frac{1}{r}\frac{\partial u_\phi}{\partial\theta} + \frac{\cos\theta}{r\sin\theta}u_\phi - \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\phi}\right)\mathbf{e}_r + \left(\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\phi} - \frac{1}{r}u_\phi - \frac{\partial u_\phi}{\partial r}\right)\mathbf{e}_\theta \\ &\quad + \left(\frac{\partial u_\theta}{\partial r} + \frac{1}{r}u_\theta - \frac{1}{r}\frac{\partial u_r}{\partial\theta}\right)\mathbf{e}_\phi.\end{aligned}$$

We first consider the components in the $\mathbf{e}_r$ direction, so

$$\begin{aligned}(\nabla\wedge\nabla\wedge\mathbf{u})\cdot\mathbf{e}_r &= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial\theta}\left(r\sin\theta\left(\frac{\partial u_\theta}{\partial r} + \frac{1}{r}u_\theta - \frac{1}{r}\frac{\partial u_r}{\partial\theta}\right)\right)\right. \\ &\quad\left. - \frac{\partial}{\partial\phi}\left(\frac{1}{\sin\theta}\frac{\partial u_r}{\partial\phi} - u_\phi - r\frac{\partial u_\phi}{\partial r}\right)\right] \\ &= \frac{1}{r}\frac{\partial^2 u_\theta}{\partial\theta\partial r} + \frac{1}{r^2}\frac{\partial u_\theta}{\partial\theta} - \frac{1}{r^2}\frac{\partial^2 u_r}{\partial\theta^2} - \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u_r}{\partial\phi^2} + \frac{1}{r^2\sin\theta}\frac{\partial u_\phi}{\partial\phi} \\ &\quad + \frac{1}{r\sin\theta}\frac{\partial^2 u_\phi}{\partial\phi\partial r} + \frac{\cos\theta}{r\sin\theta}\frac{\partial u_\theta}{\partial r} + \frac{\cos\theta}{r^2\sin\theta}u_\theta - \frac{\cos\theta}{r^2\sin\theta}\frac{\partial u_r}{\partial\theta}.\end{aligned}$$

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We also have that

$$\begin{aligned}(\nabla(\nabla\cdot\mathbf{u}))\cdot\mathbf{e}_r &= \frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2u_r\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(u_\theta\sin\theta\right)+\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi}\right)\\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_r}{\partial r}\right)-\frac{2}{r^2}u_r-\frac{1}{r^2}\frac{\partial u_\theta}{\partial\theta}+\frac{1}{r}\frac{\partial^2u_\theta}{\partial r\partial\theta}-\frac{\cos\theta}{r^2\sin\theta}u_\theta\\ &\quad+\frac{\cos\theta}{r\sin\theta}\frac{\partial u_\theta}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial^2u_\phi}{\partial r\partial\phi}-\frac{1}{r^2\sin\theta}\frac{\partial u_\phi}{\partial\phi}.\end{aligned}$$

So

$$\begin{aligned}\left(\nabla^2\mathbf{u}\right)\cdot\mathbf{e}_r &= (\nabla(\nabla\cdot\mathbf{u}))\cdot\mathbf{e}_r-(\nabla\wedge\nabla\wedge\mathbf{u})\cdot\mathbf{e}_r\\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_r}{\partial r}\right)-\frac{2}{r^2}u_r-\frac{1}{r^2}\frac{\partial u_\theta}{\partial\theta}+\cancel{\frac{1}{r}\frac{\partial^2u_\theta}{\partial r\partial\theta}}-\frac{\cos\theta}{r^2\sin\theta}u_\theta\\ &\quad+\cancel{\frac{\cos\theta}{r\sin\theta}\frac{\partial u_\theta}{\partial r}}+\cancel{\frac{1}{r\sin\theta}\frac{\partial^2u_\phi}{\partial r\partial\phi}}-\frac{1}{r^2\sin\theta}\frac{\partial u_\phi}{\partial\phi}\\ &\quad-\cancel{\frac{1}{r}\frac{\partial^2u_\theta}{\partial\theta\partial r}}-\frac{1}{r^2}\frac{\partial u_\theta}{\partial\theta}+\frac{1}{r^2}\frac{\partial^2u_r}{\partial\theta^2}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2u_r}{\partial\phi^2}-\frac{1}{r^2\sin\theta}\frac{\partial u_\phi}{\partial\phi}\\ &\quad-\cancel{\frac{1}{r\sin\theta}\frac{\partial^2u_\phi}{\partial\phi\partial r}}-\cancel{\frac{\cos\theta}{r\sin\theta}\frac{\partial u_\theta}{\partial r}}-\frac{\cos\theta}{r^2\sin\theta}u_\theta+\frac{\cos\theta}{r^2\sin\theta}\frac{\partial u_r}{\partial\theta}\\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_r}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2u_r}{\partial\theta^2}+\frac{\cos\theta}{r^2\sin\theta}\frac{\partial u_r}{\partial\theta}+\frac{1}{r^2\sin\theta}\frac{\partial^2u_r}{\partial\phi^2}\\ &\quad-\frac{2}{r^2}u_r-\frac{2}{r^2\sin\theta}\frac{\partial u_\phi}{\partial\phi}-\frac{2\cos\theta}{r^2\sin\theta}u_\theta-\frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta},\end{aligned}$$

$$\boxed{\therefore\left(\nabla^2\mathbf{u}\right)\cdot\mathbf{e}_r=\nabla^2u_r-\frac{2}{r^2}u_r-\frac{2}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(u_\theta\sin\theta\right)-\frac{2}{r^2\sin\theta}\frac{\partial u_\phi}{\partial\theta},}$$

where we have used the definition of $\nabla^2 f$ to simplify the RHS. Next we look at the $\mathbf{e}_\theta$ components:

$$\begin{aligned}(\nabla\wedge\nabla\wedge\mathbf{u})\cdot\mathbf{e}_\theta &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\phi}\left(\frac{1}{r}\frac{\partial u_\phi}{\partial\theta}+\frac{\cos\theta}{r\sin\theta}u_\phi-\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\phi}\right)\right.\\ &\quad\left.-\frac{\partial}{\partial r}\left(r\sin\theta\frac{\partial u_\theta}{\partial r}+\sin\theta\,u_\theta-\sin\theta\frac{\partial u_r}{\partial\theta}\right)\right]\\ &= \frac{1}{r^2\sin\theta}\frac{\partial^2u_\phi}{\partial\theta\partial\phi}+\frac{\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\phi}{\partial\phi}-\frac{1}{r^2\sin^2\theta}\frac{\partial^2u_\theta}{\partial\phi^2}-\frac{1}{r}\frac{\partial u_\theta}{\partial r}\\ &\quad-\frac{\partial^2u_\theta}{\partial r^2}-\frac{1}{r}\frac{\partial u_\theta}{\partial r}+\frac{1}{r}\frac{\partial^2u_r}{\partial r\partial\theta},\end{aligned}$$

while

$$\begin{aligned}(\nabla(\nabla\cdot\mathbf{u}))\cdot\mathbf{e}_\theta &= \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{\partial u_r}{\partial r}+\frac{2}{r}u_r+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{\cos\theta}{r\sin\theta}u_\theta+\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi}\right)\\ &= \frac{1}{r}\frac{\partial^2u_r}{\partial\theta\partial r}+\frac{2}{r^2}\frac{\partial u_r}{\partial\theta}+\frac{1}{r^2}\frac{\partial^2u_\theta}{\partial\theta^2}+\frac{\cos\theta}{r^2\sin\theta}\frac{\partial u_\theta}{\partial\theta}-\frac{u_\theta}{r^2\sin^2\theta}\\ &\quad+\frac{1}{r^2\sin\theta}\frac{\partial^2u_\phi}{\partial\theta\partial\phi}-\frac{\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\phi}{\partial\phi}.\end{aligned}$$

Thus,

$$\begin{aligned}(\nabla^2\mathbf{u})\cdot\mathbf{e}_\theta &= \frac{1}{r}\frac{\partial^2 u_r}{\partial\theta\partial r} + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial\theta^2} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial u_\theta}{\partial\theta} - \frac{u_\theta}{r^2\sin^2\theta} \\ &\quad + \frac{1}{r^2\sin\theta}\frac{\partial^2 u_\phi}{\partial\theta\partial\phi} - \frac{\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\phi}{\partial\phi} - \frac{1}{r^2\sin\theta}\frac{\partial^2 u_\phi}{\partial\theta\partial\phi} - \frac{\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\phi}{\partial\phi} \\ &\quad + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u_\theta}{\partial\phi^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} + \frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} - \frac{1}{r}\frac{\partial^2 u_r}{\partial r\partial\theta}\end{aligned}$$

$$\boxed{\therefore\ (\nabla^2\mathbf{u})\cdot\mathbf{e}_\theta = \nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r^2\sin^2\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\phi}{\partial\phi}.}$$

Finally, we consider the $\mathbf{e}_\phi$ components:

$$\begin{aligned}(\nabla\wedge\nabla\wedge\mathbf{u})\cdot\mathbf{e}_\phi &= \frac{1}{r}\left[\frac{\partial}{\partial r}\left(\frac{1}{\sin\theta}\frac{\partial u_r}{\partial\phi} - u_\phi - r\frac{\partial u_\phi}{\partial r}\right)\right. \\ &\qquad \left. - \frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial u_\phi}{\partial\theta} + \frac{\cos\theta}{r\sin\theta}u_\phi - \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\phi}\right)\right] \\ &= \frac{1}{r\sin\theta}\frac{\partial^2 u_r}{\partial r\partial\phi} - \frac{2}{r}\frac{\partial u_\phi}{\partial r} - \frac{\partial^2 u_\phi}{\partial r^2} - \frac{1}{r^2}\frac{\partial^2 u_\phi}{\partial\theta^2} - \frac{\cos\theta}{r^2\sin\theta}\frac{\partial u_\phi}{\partial\theta} \\ &\quad + \frac{u_\phi}{r^2\sin^2\theta} + \frac{1}{r^2\sin\theta}\frac{\partial^2 u_\theta}{\partial\theta\partial\phi} - \frac{\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\theta}{\partial\phi}\end{aligned}$$

and

$$\begin{aligned}(\nabla(\nabla\cdot\mathbf{u}))\cdot\mathbf{e}_\phi &= \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{\partial u_r}{\partial r} + \frac{2}{r}u_r + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{\cos\theta}{r\sin\theta}u_\theta + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi}\right) \\ &= \frac{1}{r\sin\theta}\frac{\partial^2 u_r}{\partial\phi\partial r} + \frac{2}{r^2\sin\theta}\frac{\partial u_r}{\partial\phi} + \frac{1}{r^2\sin\theta}\frac{\partial^2 u_\theta}{\partial\phi\partial\theta} \\ &\qquad + \frac{\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\theta}{\partial\phi} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u_\phi}{\partial\phi^2}\end{aligned}$$

Thus,

$$\begin{aligned}(\nabla^2\mathbf{u})\cdot\mathbf{e}_\phi &= \frac{1}{r\sin\theta}\frac{\partial^2 u_r}{\partial\phi\partial r} + \frac{2}{r^2\sin\theta}\frac{\partial u_r}{\partial\phi} + \frac{1}{r^2\sin\theta}\frac{\partial^2 u_\theta}{\partial\phi\partial\theta} + \frac{\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\theta}{\partial\phi} \\ &\quad + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u_\phi}{\partial\phi^2} - \frac{1}{r\sin\theta}\frac{\partial^2 u_r}{\partial r\partial\phi} + \frac{2}{r}\frac{\partial u_\phi}{\partial r} + \frac{\partial^2 u_\phi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u_\phi}{\partial\theta^2} \\ &\quad + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial u_\phi}{\partial\theta} - \frac{u_\phi}{r^2\sin^2\theta} - \frac{1}{r^2\sin\theta}\frac{\partial^2 u_\theta}{\partial\theta\partial\phi} + \frac{\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\theta}{\partial\phi}\end{aligned}$$

$$\boxed{\therefore\ (\nabla^2\mathbf{u})\cdot\mathbf{e}_\phi = \nabla^2 u_\phi + \frac{2}{r^2\sin\theta}\frac{\partial u_r}{\partial\phi} - \frac{u_\phi}{r^2\sin^2\theta} + \frac{2\cos\theta}{r^2\sin\theta}\frac{\partial u_\theta}{\partial\phi}.}$$